

# JMO 2020/6

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TWITCH SOLVES ISL

Episode 16

## Problem

Let  $n \geq 2$  be an integer. Let  $P(x_1, x_2, \dots, x_n)$  be a nonconstant  $n$ -variable polynomial with real coefficients. Assuming that  $P$  vanishes whenever two of its arguments are equal, prove that  $P$  has at least  $n!$  terms.

## Video

<https://youtu.be/r7j0oRtpErA>

## External Link

<https://aops.com/community/p15952921>

## Solution

We present two solutions.

**First solution using induction (by Ankan).** Begin with the following observation:

**Claim.** Let  $1 \leq i < j \leq n$ . There is no term of  $P$  which omits both  $x_i$  and  $x_j$ .

*Proof.* Note that  $P$  ought to become identically zero if we set  $x_i = x_j = 0$ , since it is zero for any choice of the remaining  $n - 2$  variables, and the base field  $\mathbb{R}$  is infinite.  $\square$

**Remark** (Technical warning for experts). The fact we used is not true if  $\mathbb{R}$  is replaced by a field with finitely many elements, such as  $\mathbb{F}_p$ , even with one variable. For example the one-variable polynomial  $X^p - X$  vanishes on every element of  $\mathbb{F}_p$ , by Fermat's little theorem.

We proceed by induction on  $n \geq 2$  with the base case  $n = 2$  being clear. Assume WLOG  $P$  is not divisible by any of  $x_1, \dots, x_n$ , since otherwise we may simply divide out this factor. Now for the inductive step, note that

- The polynomial  $P(0, x_2, x_3, \dots, x_n)$  obviously satisfies the inductive hypothesis and is not identically zero since  $x_1 \nmid P$ , so it has at least  $(n - 1)!$  terms.
- Similarly,  $P(x_1, 0, x_3, \dots, x_n)$  also has at least  $(n - 1)!$  terms.
- Similarly,  $P(x_1, x_2, 0, \dots, x_n)$  also has at least  $(n - 1)!$  terms.
- ...and so on.

By the claim, all the terms obtained in this way came from different terms of the original polynomial  $P$ . Therefore,  $P$  itself has at least  $n \cdot (n - 1)! = n!$  terms.

**Remark.** Equality is achieved by the Vandermonde polynomial  $P = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ .

**Second solution using Vandermonde polynomial (by Yang Liu).** Since  $x_i - x_j$  divides  $P$  for any  $i \neq j$ , it follows that  $P$  should be divisible by the Vandermonde polynomial

$$V = \prod_{i < j} (x_j - x_i) = \sum_{\sigma} \text{sgn}(\sigma) x_1^{\sigma(0)} x_2^{\sigma(1)} \dots x_n^{\sigma(n-1)}$$

where the sum runs over all permutations  $\sigma$  on  $\{0, \dots, n - 1\}$ .

Consequently, we may write

$$P = \sum_{\sigma} \text{sgn}(\sigma) x_1^{\sigma(0)} x_2^{\sigma(1)} \dots x_n^{\sigma(n-1)} Q$$

The main idea is that each of the  $n!$  terms of the above sum has a monomial not appearing in any of the other terms.

As an example, consider  $x_1^{n-1} x_2^{n-2} \dots x_{n-1}^1 x_n^0$ . Among all monomial in  $Q$ , consider the monomial  $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$  with the largest  $e_1$ , then largest  $e_2$ ,  $\dots$  (In other words, take the lexicographically largest  $(e_1, \dots, e_n)$ .) This term

$$x_1^{e_1 + (n-1)} x_2^{e_2 + (n-2)} \dots x_n^{e_n}$$

can't appear anywhere else because it is strictly lexicographically larger than any other term appearing in any other expansion.

Repeating this argument with every  $\sigma$  gives the conclusion.