# JMO 2020/4 

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Twitch Solves ISL
Episode 16

## Problem

Let $A B C D$ be a convex quadrilateral inscribed in a circle and satisfying

$$
D A<A B=B C<C D
$$

Points $E$ and $F$ are chosen on sides $C D$ and $A B$ such that $\overline{B E} \perp \overline{A C}$ and $\overline{E F} \| \overline{B C}$. Prove that $F B=F D$.

## Video

https://youtu.be/r7j0oRtpErA

## External Link

https://aops.com/community/p15952890

## Solution

We present three approaches. We note that in the second two approaches, the result remains valid even if $A B \neq B C$, as long $E$ is replaced by the point on $\overline{A C}$ satisfying $E A=E C$. So the result is actually somewhat more general.

First solution by inscribed angle theorem. Since $\overline{E F} \| \overline{B C}$ we may set $\theta=\angle F E B=$ $\angle C B E=\angle E B F$. This already implies $F E=F B$, so we will in fact prove that $F$ is the circumcenter of $\triangle B E D$.


Note that $\angle B D C=\angle B A C=90^{\circ}-\theta$. However, $\angle B F E=180^{\circ}-2 \theta$. So by the inscribed angle theorem, $D$ lies on the circle centered at $F$ with radius $F E=F B$, as desired.

Remark. Another approach to the given problem is to show that $B$ is the $D$-excenter of $\triangle D A E$, and $F$ is the arc midpoint of $\widehat{D A E}$ of the circumcircle of $\triangle D A E$. In my opinion, this approach is much clumsier.

Second general solution by angle chasing. By Reim's theorem, $A F E D$ is cyclic.


Hence

$$
\measuredangle F D B=\measuredangle F D C-\measuredangle B D C=\measuredangle F A E-\measuredangle F A C
$$

$$
=\measuredangle C A E=\measuredangle E C A=\measuredangle D C A=\measuredangle D B A=\measuredangle D B F
$$

as desired.

Third general solution by Pascal. Extend rays $A E$ and $D F$ to meet the circumcircle again at $G$ and $H$. By Pascal's theorem on $H D C B A G$, it follows that $E, F$, and $G H \cap B C$ are collinear, which means that $\overline{E F}\|\overline{G H}\| \overline{B C}$.


Since $E A=E C$, it follows $D A G C$ in isosceles trapezoid. But also $G H B C$ is an isosceles trapezoid. Thus $\mathrm{m} \widehat{D A}=\mathrm{m} \widehat{G C}=\mathrm{m} \widehat{B H}$, so $D A H B$ is an isosceles trapezoid. Thus $F D=F B$.

Remark. Addicts of projective geometry can use Pascal on $D B C A H G$ to finish rather than noting the equal arcs.

