JMO 2020/4

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Twitch Solves ISL

Episode 16

Problem

Let $ABCD$ be a convex quadrilateral inscribed in a circle and satisfying

$$DA < AB = BC < CD.$$

Points E and F are chosen on sides CD and AB such that $\overline{BE} \perp \overline{AC}$ and $\overline{EF} \parallel \overline{BC}$. Prove that $FB = FD$.

Video

https://youtu.be/r7j0oRtpErA

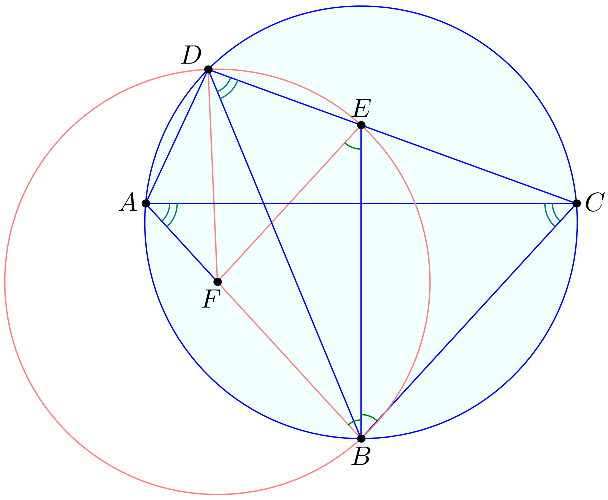
External Link

https://aops.com/community/p15952890

Solution

We present three approaches. We note that in the second two approaches, the result remains valid even if $AB \neq BC$, as long E is replaced by the point on $\overline{AC}$ satisfying $EA = EC$. So the result is actually somewhat more general.

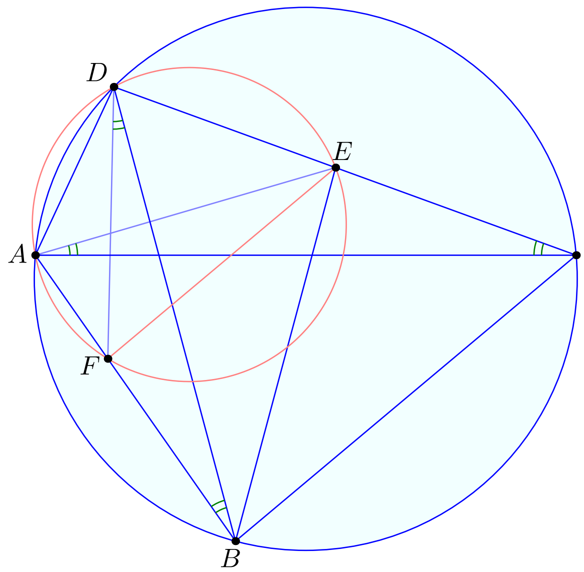
First solution by inscribed angle theorem. Since $\overline{EF} \parallel \overline{BC}$ we may set $\theta = \angle FEB = \angle CBE = \angle EBF$. This already implies $FE = FB$, so we will in fact prove that F is the circumcenter of $\triangle BED$.



Note that $\angle BDC = \angle BAC = 90^\circ - \theta$. However, $\angle BFE = 180^\circ - 2\theta$. So by the inscribed angle theorem, D lies on the circle centered at F with radius $FE = FB$, as desired.

Remark. Another approach to the given problem is to show that B is the D-excenter of $\triangle DAE$, and F is the arc midpoint of $\widehat{DAE}$ of the circumcircle of $\triangle DAE$. In my opinion, this approach is much clumsier.

Second general solution by angle chasing. By Reim's theorem, $AFED$ is cyclic.



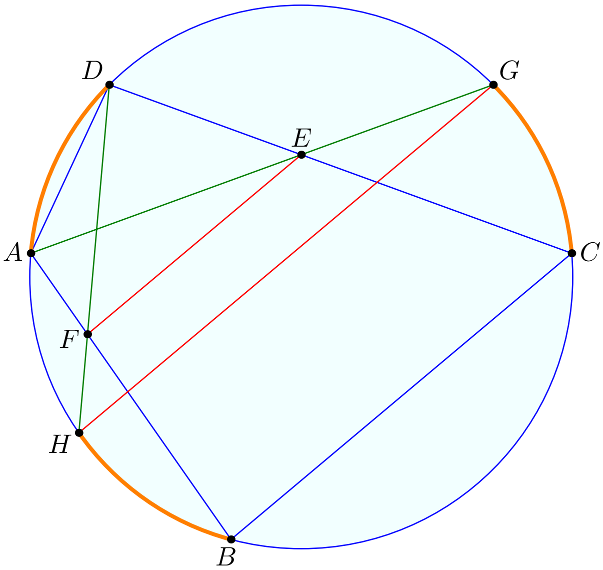
Hence

$$\measuredangle FDB = \measuredangle FDC - \measuredangle BDC = \measuredangle FAE - \measuredangle FAC$$

$$= \measuredangle CAE = \measuredangle ECA = \measuredangle DCA = \measuredangle DBA = \measuredangle DBF$$

as desired.

Third general solution by Pascal. Extend rays AE and DF to meet the circumcircle again at G and H. By Pascal's theorem on $HDCBAG$, it follows that E, F, and $GH \cap BC$ are collinear, which means that $\overline{EF} \parallel \overline{GH} \parallel \overline{BC}$.



Since $EA = EC$, it follows $DAGC$ in isosceles trapezoid. But also $GHBC$ is an isosceles trapezoid. Thus $\mathrm{m}\widehat{DA} = \mathrm{m}\widehat{GC} = \mathrm{m}\widehat{BH}$, so $DAHB$ is an isosceles trapezoid. Thus $FD = FB$.

Remark. Addicts of projective geometry can use Pascal on $DBCAHG$ to finish rather than noting the equal arcs.