JMO 2020/2

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Twitch Solves ISL

Episode 16

Problem

Let ω be the incircle of a fixed equilateral triangle ABC. Let ℓ be a variable line that is tangent to ω and meets the interior of segments BC and CA at points P and Q, respectively. A point R is chosen such that $PR = PA$ and $QR = QB$. Find all possible locations of the point R, over all choices of ℓ.

Video

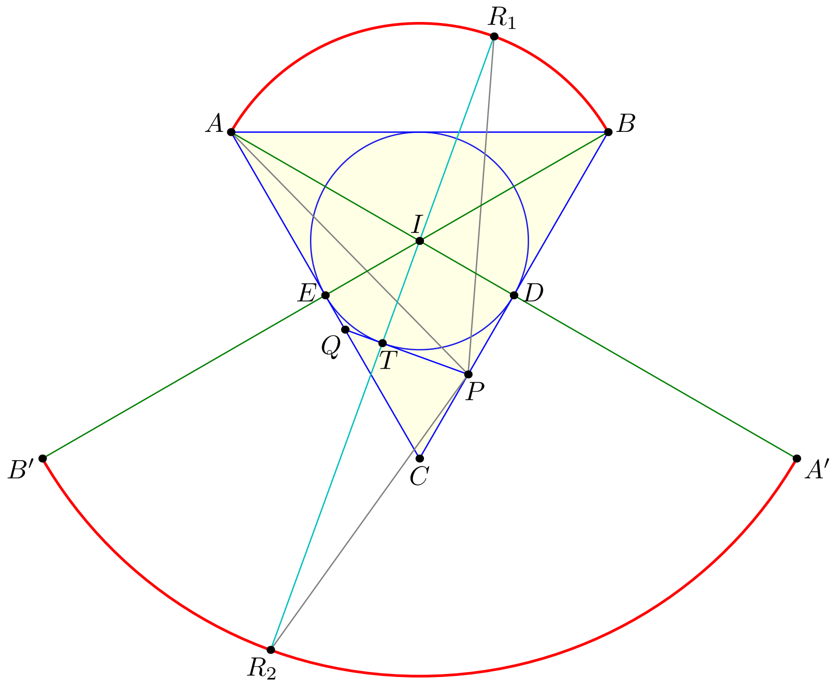
https://youtu.be/r7j0oRtpErA

External Link

https://aops.com/community/p15952801

Solution

Let r be the inradius. Let T be the tangency point of $\overline{PQ}$ on arc $\widehat{DE}$ of the incircle, which we consider varying. We define R_1 and R_2 to be the two intersections of the circle centered at P with radius PA, and the circle centered at Q with radius QB. We choose R_1 to lie on the opposite side of C as line PQ.



Claim. The point R_1 is the unique point on ray TI with $R_1I = 2r$.

Proof. Define S to be the point on ray TI with $SI = 2r$. Note that there is a homothety at I which maps $\triangle DTE$ to $\triangle ASB$, for some point S.

Note that since $TASD$ is an isosceles trapezoid, it follows $PA = PS$. Similarly, $QB = QS$. So it follows that $S = R_1$. $\square$

Since T can be any point on the open arc $\widehat{DE}$, it follows that the locus of R_1 is exactly the open 120° arc of $\widehat{AB}$ of the circle centered at I with radius $2r$ (i.e. the circumcircle of ABC).

It remains to characterize R_2. Since $TI = r$, $IR_1 = 2r$, it follows $TR_2 = 3r$ and $IR_2 = 4r$. Define A' on ray DI such that $A'I = 4r$, and B' on ray IE such that $B'I = 4r$. Then it follows, again by homothety, that the locus of R_2 is the 120° arc $\widehat{A'B'}$ of the circle centered at I with radius $4r$.

In conclusion, the locus of R is the two open 120° arcs we identified.