Shortlist 2006 C6

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Episode 15

Problem

A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

Video

https://youtu.be/n7Z5I5I0R-c

Solution

The idea is to show that the condition in the problem is equivalent to the condition in Hall's Marriage Lemma, wherein we try to match down-triangles to non-hole up-triangles.

To this end, let's say two holes are *adjacent* if they share a vertex, and a set of holes is *connected* if they form a connected component with respect to the adjacency condition. Finally, the *bounding box* of a set of holes is the smallest upwards equilateral triangle of some side length (by inclusion) that contains all the triangles.

Claim. Consider a connected set D of down-triangles which has a bounding box of side length k. Then at least $|D| + k$ up-triangles are adjacent to at least one of the down-triangles in D.

Outline of proof. The idea is to go by induction on $|D|$, by deleting the uppermost triangle which touches the left boundary of the bounding box of D. (The catch is that doing this deletion may disconnect D, so this case needs to be handled as well.) $\square$

Suppose the problem condition holds; we prove Hall's condition. Consider a general set $\mathcal{D}$ of down-triangles. The $\mathcal{D}$ is the disjoint union $D_1 \sqcup \cdots \sqcup D_k$ of k connected components. The problem condition implies that now that there are at least $|D_i|$ up-holes neighboring some down-hole in D_i, and any up-hole is adjacent to down-triangles in at most one D_i; so Hall's condition holds for $\mathcal{D}$.

For the converse direction, if Hall's condition holds, taking $\mathcal{D}$ to be all the down-triangles in some equilateral triangle of side length k implies the result. (This does not need the claim; it is the easy direction.)