

Taiwan TST 2019/2J/2

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TWITCH SOLVES ISL

Episode 13

Problem

Let ABC be a triangle with incircle ω and circumcircle Ω . Assume that ω is tangent to sides AB, AC at F, E , respectively. Line EF meets Ω at points P and Q . Let M be the midpoint of \overline{BC} .

Take a point R on the circumcircle of $\triangle MPQ$, say Γ , such that $\overline{MR} \perp \overline{EF}$. Prove that line AR , ω and Γ intersect at one point.

Video

https://youtu.be/KcHiYX0Ji_Q

External Link

<https://aops.com/community/p14533530>

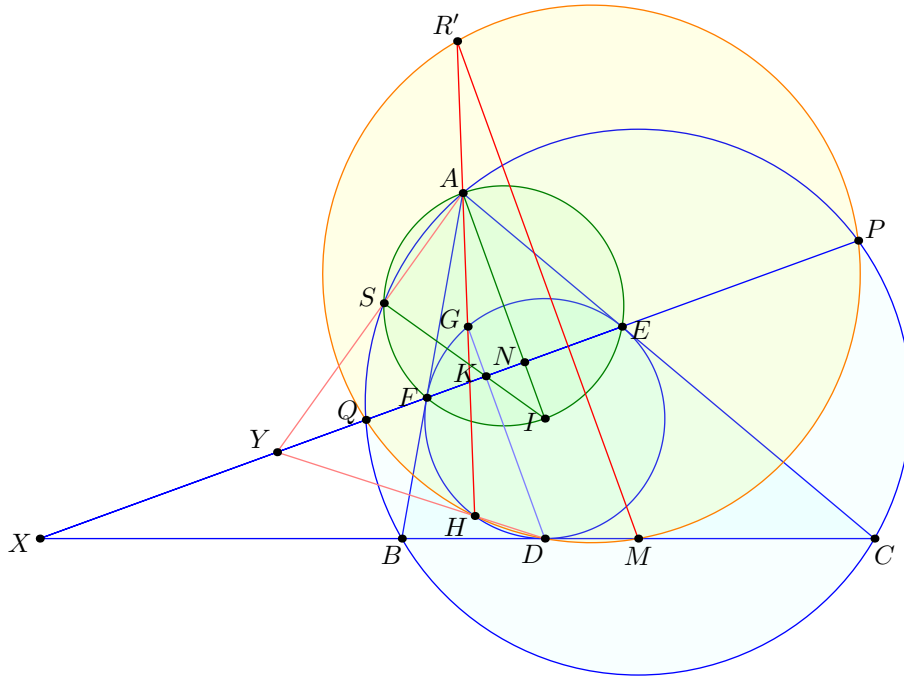
Solution

Let D be the foot from I to \overline{BC} .

Claim. Quadrilateral $MDPQ$ is cyclic.

Proof. Let $X = PQ \cap BC$. Then $(XD; BC) = -1$ by Ceva/Menelaus with the Gergonne point. Hence, $XD \cdot XM = XB \cdot XC = XP \cdot XQ$ as needed. \square

We let G be the intersection of the D -altitude to \overline{EF} with the incircle. Let line AG meet the incircle again at H .



Claim. Quadrilateral $HDPQ$ is cyclic.

Proof HDPQ cyclic. For this proof, we need to introduce several points:

- Let K be the foot from D to \overline{EF} .
- Let N be the midpoint of \overline{EF} .
- Let S be the inverse of K with respect to the incircle, which is known to satisfy $\angle AIS = 90^\circ$.
- Let $Y = \overline{HD} \cap \overline{EF}$.

We have $-1 = (EF; GH) \stackrel{D}{=} (EF; KY)$. Since \overline{SK} bisects $\angle FSE$, this implies Y also lies on line AS . We can then calculate

$$YH \cdot YD = YF \cdot YE = YK \cdot YN = YS \cdot YA = YP \cdot YQ$$

which implies the concyclic condition. \square

We let R' be a point on line \overline{AGH} such that $\overline{R'M} \perp \overline{PQ}$. Then

$$\angle HR'M = \angle HGD = \angle HDB = \angle HDM$$

so $R' = R$ and the problem is solved (the concurrence point in the problem is H).