Taiwan TST 2019/2J/2

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Twitch Solves ISL

Episode 13

Problem

Let ABC be a triangle with incircle ω and circumcircle Ω. Assume that ω is tangent to sides AB, AC at F, E, respectively. Line EF meets Ω at points P and Q. Let M be the midpoint of $\overline{BC}$.

Take a point R on the circumcircle of $\triangle MPQ$, say Γ, such that $\overline{MR} \perp \overline{EF}$. Prove that line AR, ω and Γ intersect at one point.

Video

https://youtu.be/KcHiYXOJi_Q

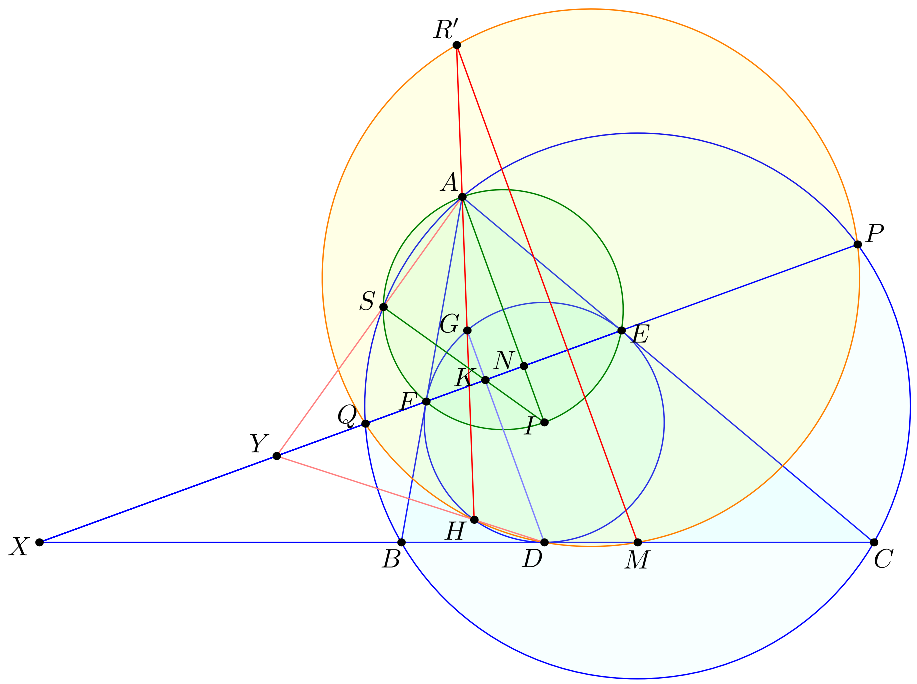
Solution

Let D be the foot from I to $\overline{BC}$.

Claim. Quadrilateral $MDPQ$ is cyclic.

Proof. Let $X = PQ \cap BC$. Then $(XD; BC) = -1$ by Ceva/Menelaus with the Gergonne point. Hence, $XD \cdot XM = XB \cdot XC = XP \cdot XQ$ as needed. □

We let G be the intersection of the D-altitude to $\overline{EF}$ with the incircle. Let line AG meet the incircle again at H.



Claim. Quadrilateral $HDPQ$ is cyclic.

Proof HDPQ cyclic. For this proof, we need to introduce several points:

- Let K be the foot from D to $\overline{EF}$.
- Let N be the midpoint of $\overline{EF}$.
- Let S be the inverse of K with respect to the incircle, which is known to satisfy $\angle AIS = 90°$.
- Let $Y = \overline{HD} \cap \overline{EF}$.

We have $-1 = (EF; GH) \overset{D}{=} (EF; KY)$. Since $\overline{SK}$ bisects $\angle FSE$, this implies Y also lies on line AS. We can then calculate

$$YH \cdot YD = YF \cdot YE = YK \cdot YN = YS \cdot YA = YP \cdot YQ$$

which implies the concyclic condition. □

We let R' be a point on line $\overline{AGH}$ such that $\overline{R'M} \perp \overline{PQ}$. Then

$$\measuredangle HR'M = \measuredangle HGD = \measuredangle HDB = \measuredangle HDM$$

so $R' = R$ and the problem is solved (the concurrence point in the problem is H).