Shortlist 2007 N5

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Twitch Solves ISL

Episode 13

Problem

Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(1) = 1$ and for all positive integers m and n and primes p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

Video

External Link

Solution

The answer is that f must be the identity (which works). We let $\operatorname{rad} n$ denote the product of distinct prime factors of n. We also let $P(m,n)$ denote the given statement.

Remark. The original problem gave the hypothesis f was surjective in place of $f(1) = 1$. Our version is better: to see that f surjective implies $f(1) = 1$ anyways, consider $P(1, N-1)$ if $f(N) = 1$ held for any $N \geq 2$.

Claim. We have $\operatorname{rad} f(2^e) = 2$ for every positive integer e; that is, $f(2)$, $f(4)$, ...are all powers of 2.

Proof. Follows by taking $P(2^{e-1}, 2^{e-1})$ and induction. $\qquad\square$

Claim. We have $f(2) = 2$, $f(3) = 3$, $f(4) = 4$.

Proof. First, $P(3,1)$ gives that $f(3) = 2^y - 1$ for some y. Now if $f(2) = 2^x$ then $P(2,1)$ gives
$$\operatorname{rad}(2^x + 1) = \operatorname{rad}(2^y - 1).$$
We now consider two cases.

- If $x = 3$ then this forces $y = 2$. So $f(2) = 8$ and $f(3) = 3$. But $\operatorname{rad}(1 + f(4)) = \operatorname{rad}(f(2) + f(3)) = 11$, so there is no possible value of $f(4)$, contradiction.

- Otherwise, let p be a primitive prime divisor of $2^{2x} - 1$ which exists by Zsigmondy. It also divides the right-hand side, so $2x \mid y$, and $2^{2x} - 1 \mid 2^y - 1$. This could only happen if $2^x - 1 = 1$, so $x = 1$. This gives $y = 2$ (again by Zsigmondy) and we're done.

Hence $f(2) = 2$ and $f(3) = 3$. Now $\operatorname{rad}(1 + f(4)) = \operatorname{rad}(f(2) + f(3)) = 5$, and since $f(4)$ is a power of 2, Zsigmondy implies $f(4) = 4$. $\qquad\square$

Moving forward, we apply the following simple consequence of Zsigmondy:

Lemma. If $\operatorname{rad}(2^x - 1) = \operatorname{rad}(2^y - 1)$ then $x = y$. Similarly if $\operatorname{rad}(2^x + 1) = \operatorname{rad}(2^y + 1)$ then $x = y$.

We now prove by induction on $e \geq 1$ the statement that
$$f(n) = n \qquad \text{for } 2^e \leq n \leq 2^{e+1}.$$
The base case is already set for us. For the inductive step, suppose f is the identity for $n \leq 2^e$. We proceed in three steps:

- First, we have
$$\operatorname{rad}\left(1 + f\left(2^{e+1} - 1\right)\right) = 2$$
$$\operatorname{rad}\left(f\left(2^{e+1} - 1\right)\right) = \operatorname{rad}\left(f(2^e - 1) + f(2^e)\right) = \operatorname{rad}(2^{e+1} - 1).$$
The first equation just says $f(2^{e+1} - 1) = 2^t - 1$ for some t. So the latter equation, together with the lemma, gives $f(2^{e+1} - 1) = 2^{e+1} - 1$.

- Next, we have
$$\operatorname{rad}\left(f(2^{e+1}) + 1\right) = \operatorname{rad} f(2^{e+1} + 1) = \operatorname{rad}\left((2^{e+1} - 1) + 2\right)$$
$$= \operatorname{rad}\left(f(2^{e+1} - 1) + f(2)\right) = \operatorname{rad}\left(2^{e+1} + 1\right)$$
which gives $f(2^{e+1}) = 2^{e+1}$.

- Now assume $2^e + 1 \le n \le 2^{e+1} - 2$. Let $1 \le k \le 2^e - 2$ be such than $n + k = 2^{e+1} - 1$. Then

$$\operatorname{rad}\left(f(n) + k\right) = \operatorname{rad}\left(f(n) + f(k)\right) = \operatorname{rad}\left(f(n+k)\right) = \operatorname{rad}\left(2^{e+1} - 1\right)$$
$$\operatorname{rad}\left(f(n) + k + 1\right) = \operatorname{rad}\left(f(n) + f(k+1)\right) = \operatorname{rad}\left(f(n+k+1)\right) = 2.$$

So we have an integer x (depending on k) such that

$$f(n) + k + 1 = 2^x \implies \operatorname{rad}(2^x - 1) = \operatorname{rad}\left(f(n) + k\right) = \operatorname{rad}(2^{e+1} - 1).$$

Hence $x = e + 1$. And so $f(n) = n$.

This completes the induction and hence the problem is solved.