Shortlist 2007 N5

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Episode 13

Problem

Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(1) = 1$ and for all positive integers m and n and primes p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

Video

https://youtu.be/OExdK27KHx8

Solution

The answer is that f must be the identity (which works). We let $\operatorname{rad} n$ denote the product of distinct prime factors of n. We also let $P(m,n)$ denote the given statement.

Remark. The original problem gave the hypothesis f was surjective in place of $f(1)=1$. Our version is better: to see that f surjective implies $f(1)=1$ anyways, consider $P(1,N-1)$ if $f(N)=1$ held for any $N\ge 2$.

Claim. We have $\operatorname{rad} f(2^e)=2$ for every positive integer e; that is, $f(2)$, $f(4)$, ... are all powers of 2.

Proof. Follows by taking $P(2^{e-1},2^{e-1})$ and induction. $\square$

Claim. We have $f(2)=2$, $f(3)=3$, $f(4)=4$.

Proof. First, $P(3,1)$ gives that $f(3)=2^y-1$ for some y. Now if $f(2)=2^x$ then $P(2,1)$ gives

$$\operatorname{rad}(2^x+1)=\operatorname{rad}(2^y-1).$$

We now consider two cases.

- If $x=3$ then this forces $y=2$. So $f(2)=8$ and $f(3)=3$. But $\operatorname{rad}(1+f(4))=\operatorname{rad}(f(2)+f(3))=11$, so there is no possible value of $f(4)$, contradiction.
- Otherwise, let p be a primitive prime divisor of $2^{2x}-1$ which exists by Zsigmondy. It also divides the right-hand side, so $2x\mid y$, and $2^{2x}-1\mid 2^y-1$. This could only happen if $2^x-1=1$, so $x=1$. This gives $y=2$ (again by Zsigmondy) and we're done.

Hence $f(2)=2$ and $f(3)=3$. Now $\operatorname{rad}(1+f(4))=\operatorname{rad}(f(2)+f(3))=5$, and since $f(4)$ is a power of 2, Zsigmondy implies $f(4)=4$. $\square$

Moving forward, we apply the following simple consequence of Zsigmondy:

Lemma. *If* $\operatorname{rad}(2^x-1)=\operatorname{rad}(2^y-1)$ *then* $x=y$. *Similarly if* $\operatorname{rad}(2^x+1)=\operatorname{rad}(2^y+1)$ *then* $x=y$.

We now prove by induction on $e\ge 1$ the statement that

$$f(n)=n \qquad \text{for } 2^e\le n\le 2^{e+1}.$$

The base case is already set for us. For the inductive step, suppose f is the identity for $n\le 2^e$. We proceed in three steps:

- First, we have

$$\begin{aligned}\operatorname{rad}\left(1+f\left(2^{e+1}-1\right)\right)&=2\\ \operatorname{rad}\left(f\left(2^{e+1}-1\right)\right)&=\operatorname{rad}\left(f(2^e-1)+f(2^e)\right)=\operatorname{rad}(2^{e+1}-1).\end{aligned}$$

  The first equation just says $f(2^{e+1}-1)=2^x-1$ for some x.

- Next, we have

$$\begin{aligned}\operatorname{rad}\left(f(2^{e+1})+1\right)&=\operatorname{rad} f(2^{e+1}+1)=\operatorname{rad}\left((2^{e+1}-1)+2\right)\\ &=\operatorname{rad}\left(f(2^{e+1}-1)+f(2)\right)=\operatorname{rad}\left(2^{e+1}+1\right)\end{aligned}$$

  which gives $f(2^{e+1})=2^{e+1}$.

- Now assume $2^e+1 \le n \le 2^{e+1}-2$. Let $1 \le k \le 2^e-2$ be such than $n+k = 2^{e+1}-1$. Then

$$\begin{aligned} \operatorname{rad}(f(n)+k) &= \operatorname{rad}(f(n)+f(k)) = \operatorname{rad}(f(n+k)) = \operatorname{rad}\left(2^{e+1}-1\right) \\ \operatorname{rad}(f(n)+k+1) &= \operatorname{rad}(f(n)+f(k+1)) = \operatorname{rad}(f(n+k+1)) = 2. \end{aligned}$$

  So we have an integer x such that

$$f(n)+k+1 = 2^x \implies \operatorname{rad}(2^x-1) = \operatorname{rad}(f(n)+k) = \operatorname{rad}(2^{e+1}-1).$$

  Hence $x = e+1$. And so $f(n) = n$.

This completes the induction and hence the problem is solved.