Iberoamerican 2001

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Twitch Solves ISL

Episode 13

Problem

Let X be a set with n elements. Given $k > 2$ subsets of X, each with at least r elements, show that we can find two of them whose intersection has at least $r - \frac{nk}{4(k-1)}$ elements.

Video

https://youtu.be/sqcw8uB2M-8

Solution

Assume for contradiction this is not the case. We double-count the number N of triples of the form

$$(x, A_i, A_j) \qquad \text{where} \quad x \in A_i,\ x \in A_j.$$

On the one hand, counting by (A_i, A_j) first, we have

$$N \le \binom{k}{2} \cdot \left[r - \frac{nk}{4(k-1)} \right].$$

On other hand, summing by $x \in X$, we have

$$N = \sum_x \binom{\#\text{ times } x \text{ appears}}{2} \ge n \cdot \binom{\frac{rk}{n}}{2}$$

by Jensen. Therefore we obtain

$$\frac{1}{2} n \cdot \frac{rk}{n} \cdot \left(\frac{rk}{n} - 1 \right) < \frac{k(k-1)}{2} \cdot \left[r - \frac{nk}{4(k-1)} \right]$$

which rearranges to

$$r \cdot \left(\frac{rk}{n} - 1 \right) < (k-1)r - \frac{n}{4} \cdot k \iff \frac{1}{n} \left(r - \frac{n}{2} \right)^2 < 0$$

and this is a contradiction.