IMO 1993/3

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Twitch Solves ISL

Episode 13

Problem

On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

Video

https://youtu.be/9YfK2HmNkEs

External Link

https://aops.com/community/p372309

Solution

The answer is all n not divisible by 3.

First, we show the task is impossible when $3 \mid n$. Color the board periodically red, green, blue as shown:

$$\begin{bmatrix} R & G & B & R & G & \dots \\ G & B & R & G & B & \dots \\ B & R & G & B & R & \dots \\ R & G & B & R & G & \dots \\ G & B & R & G & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Note that every move changes the parity of the number of tokens of each color. When $3 \mid n$, there are an equal number of tokens of each color, so it is impossible to arrive at one token of one color and none of the others.

We now devise an algorithm for $3 \nmid n$. This requires two parts:

Claim. Suppose we have the following configuration, where a $*$ represents a cell with a token and a $\cdot$ represents an empty cell:

$$\begin{bmatrix} & * & \\ & * & \\ \cdot & * & * \end{bmatrix}$$

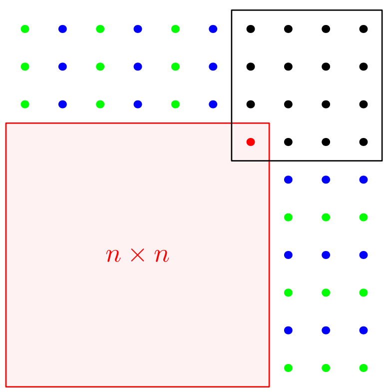
Then we can remove the three cells in the center column.

Proof. Jump the lower-right corner, then the upper counter, then the lower-left counter. $\square$

Using this, we can show $n = 4$ is possible in the following way:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \mapsto \begin{bmatrix} * & & & \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \mapsto \begin{bmatrix} * & & & \\ * & & & \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$
$$\mapsto \begin{bmatrix} & & & \\ & & & \\ & * & * & * \\ * & * & * & * \end{bmatrix} \mapsto \begin{bmatrix} & & & \\ & & & \\ & & & \\ * & * & * & * \end{bmatrix} \mapsto \begin{bmatrix} & & & \\ & & & \\ & & & \\ * & & & \end{bmatrix}$$

This lets us to reduce an $(n + 3) \times (n + 3)$ square to an $n \times n$ square in the following way:



Remove the green and blue dots shown in alternating waves, then repeat the strategy with $n = 4$ to remove all black dots.