Grant Yu

Evan Chen

Twitch Solves ISL

Episode 13

Problem

There are n pairwise intersecting circles in the plane, no three passing through the same point. The n circles dissect the plane into some number of regions bounded by circular arcs. We assign a direction (clockwise or counterclockwise) to each of the n circles and count C, the number of regions R such that the boundary of R can be traveled along the directions of edges. Prove that if n is sufficiently large, then this assignment could be done such that $C > \frac{4}{11\pi}n^2$.

Video

https://youtu.be/6DxiStGJQb0

Solution

Take any intersection point as a vertex. We use

$$V - E + F = 2$$

in the usual way. Here $V = n(n-1)$ and $E = n(2n-2)$; hence the number of faces is

$$F = n(2n-2) - n(n-1) + 2 = n^2 - n + 2$$

We now do the assignment randomly by coin flip. Suppose the regions have $a_1, a_2, \dots, a_m$ sides, $m = n^2 - n + 2$ (including unbounded). Then the expected number of good regions from this (possibly including the unbounded face) is

$$\sum_{i=1}^{m} \frac{1}{2^{a_i - 1}}$$

where

$$\mathbb{E}[a_i] = \frac{2E}{F} = \frac{4n(n-1)}{n^2 - n + 1} \approx 4$$

Thus by Jensen inequality, one can get at least $(1/8 - o(1))n^2$ good regions.