# Japan 2019/4

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Twitch Solves ISL

Episode 12

## Problem

Let $ABC$ be a triangle with its incenter $I$, incircle $\omega$, and let $M$ be a midpoint of the $\overline{BC}$. A line through $A$ perpendicular to $\overline{BC}$ and a line through $M$ perpendicular to $\overline{AI}$ meet at $K$. Show that the circle with diameter $\overline{AK}$ is tangent to $\omega$.

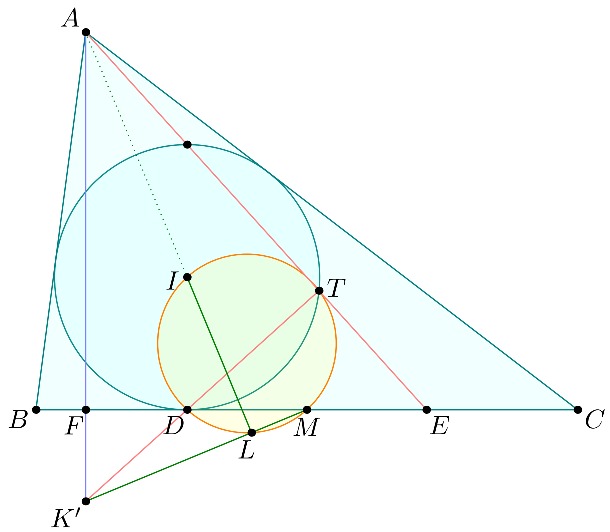
## Video

https://youtu.be/cm8svlDoPfs

## Solution

We make the following preliminary setup.

- Let $E$ be the tangency point of the $A$-excircle, so $\overline{AE}$ passes through the antipode of $D$ on $\omega$.
- Thus $\overline{AE}$ intersects the incircle again at $T$, the foot from $D$ to $\overline{AE}$. Since $DM = ME$ and $\measuredangle DTE = 90^\circ$, it follows that $MD = MT = ME$ so in fact $\overline{MT}$ is also tangent to $\omega$.
- Let $\overline{AF}$ be the $A$-altitude.




Let $K'$ be the intersection of $\overline{TD}$ with $A$-altitude. By homothety, the circle with diameter $\overline{AK'}$ is certainly tangent to $\omega$. We are going to prove $K' = K$.

Let $L$ denote the second intersection of $\overline{KM}$ with $(IDMT)$, so $\measuredangle MLI = 90^\circ$.

**Claim.** Quadrilateral $ATLK'$ is cyclic.

*Proof.* Since $\measuredangle K'LT = \measuredangle MLT / \measuredangle MDT = \measuredangle FDT = \measuredangle FAT = \measuredangle K'AT$. □

Thus $\measuredangle MLA = \measuredangle ALK' = \measuredangle ATK' = 90^\circ$ as well. Thus we conclude $\overline{AI}$ and $\overline{MK'}$ are perpendicular at $L$ as desired.