Japan 2019/4

Evan Chen

Twitch Solves ISL

Episode 12

Problem

Let ABC be a triangle with its incenter I, incircle ω, and let M be a midpoint of the $\overline{BC}$. The line through A perpendicular to $\overline{BC}$ and the line through M perpendicular to $\overline{AI}$ meet at K. Show that the circle with diameter $\overline{AK}$ is tangent to ω.

Video

https://youtu.be/cm8svlDoPfs

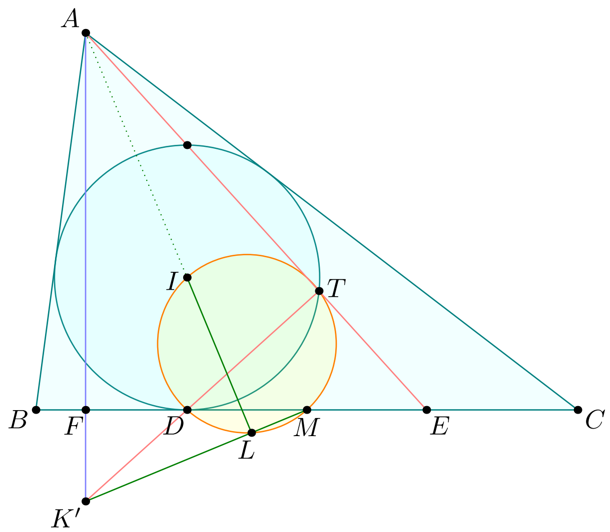
External Link

https://aops.com/community/p11754885

Solution

We make the following preliminary setup.

- Let E be the tangency point of the A-excircle, so $\overline{AE}$ passes through the antipode of D on ω.
- Thus $\overline{AE}$ intersects the incircle again at T, the foot from D to $\overline{AE}$. Since $DM = ME$ and $\measuredangle DTE = 90^\circ$, it follows that $MD = MT = ME$ so in fact $\overline{MT}$ is also tangent to ω.
- Let $\overline{AF}$ be the A-altitude.



Let K' be the intersection of $\overline{TD}$ with A-altitude. By homothety, the circle with diameter $\overline{AK'}$ is certainly tangent to ω. We are going to prove $K' = K$.

Let L denote the second intersection of $\overline{KM}$ with $(IDMT)$, so $\measuredangle MLI = 90^\circ$.

Claim. Quadrilateral $ATLK'$ is cyclic.

Proof. Since $\measuredangle K'LT = \measuredangle MLT = \measuredangle MDT = \measuredangle FDT = \measuredangle FAT = \measuredangle K'AT$. $\square$

Thus $\measuredangle MLA = \measuredangle ALK' = \measuredangle ATK' = 90^\circ$ as well. Thus we conclude $\overline{AI}$ and $\overline{MK'}$ are perpendicular at L as desired.