

Japan 2019/4

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TWITCH SOLVES ISL

Episode 12

Problem

Let ABC be a triangle with its incenter I , incircle ω , and let M be a midpoint of the \overline{BC} . The line through A perpendicular to \overline{BC} and the line through M perpendicular to \overline{AI} meet at K . Show that the circle with diameter \overline{AK} is tangent to ω .

Video

<https://youtu.be/cm8sv1DoPfs>

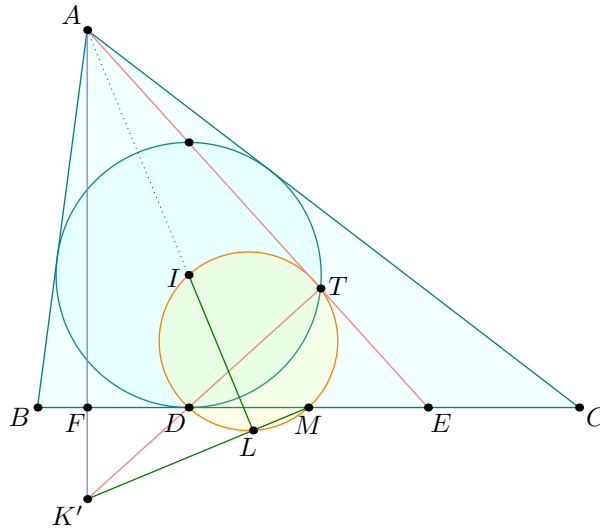
External Link

<https://aops.com/community/p11754885>

Solution

We make the following preliminary setup.

- Let E be the tangency point of the A -excircle, so \overline{AE} passes through the antipode of D on ω .
- Thus \overline{AE} intersects the incircle again at T , the foot from D to \overline{AE} . Since $DM = ME$ and $\angle DTE = 90^\circ$, it follows that $MD = MT = ME$ so in fact \overline{MT} is also tangent to ω .
- Let \overline{AF} be the A -altitude.



Let K' be the intersection of \overline{TD} with A -altitude. By homothety, the circle with diameter $\overline{AK'}$ is certainly tangent to ω . We are going to prove $K' = K$.

Let L denote the second intersection of \overline{KM} with $(IDMT)$, so $\angle MLI = 90^\circ$.

Claim. Quadrilateral $ATLK'$ is cyclic.

Proof. Since $\angle K'LT = \angle MLT = \angle MDT = \angle FDT = \angle FAT = \angle K'AT$. □

Thus $\angle MLA = \angle ALK' = \angle ATK' = 90^\circ$ as well. Thus we conclude \overline{AI} and $\overline{MK'}$ are perpendicular at L as desired.