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Episode 12

Problem

Show that there do not exist natural numbers $a_1, a_2, \cdots, a_{2018}$ such that the numbers

$$(a_1)^{2018} + a_2, (a_2)^{2018} + a_3, \cdots, (a_{2018})^{2018} + a_1$$

are all powers of 5.

Solution

Assume the contrary.

Claim. None of the numbers are divisible by 5.

Proof. If $\nu_5(a_1)$ is maximal and positive, then $\nu_5(a_1^{2018}) > \nu_5(a_2)$, so $a_1^{2018} + a_2$ cannot be a power of 5. $\square$

Let 5^e denote the smallest of the powers of 5, so all elements are divisible by 5^e. Evidently, $e \geq 2$, and for every index i,

$$a_{i+2} \equiv -a_i^{2018^2} \pmod{5^e}$$

Claim. We have $a_i \equiv 4 \pmod 5$ for all i.

Proof. This follows immediately form the previous displayed equation. $\square$

Now let $x = -a_1 \pmod{5^e}$. But if we iterate this 1009 times we obtain

$$x \equiv x^{2018^{2018}} \pmod{5^e}$$

Thus $x \bmod 5^e$ has order dividing $N = 2018^{2018} - 1$. But since $N \equiv 3 \pmod 5$, we find $\gcd(N, \varphi(5^e)) = 1$, So this forces $x \equiv 1$.

In other words, $a_1 \equiv -1 \pmod{5^e}$, and the same is true for any other term. But $(5^e - 1)^{2018} > 5^e$ and we have a size contradiction.