# EGMO 2012/4

Evan Chen

Twitch Solves ISL

Episode 12

## Problem

A set $A$ of integers is called *sum-full* if $A \subseteq A + A$. A set $A$ of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of $A$. Does there exist a sum-full zero-sum-free set of integers?

## Solution

The answer is YES, such a set exists, and is given by

$$A = \{1, -2, 3, -5, 8, -13, 21, -34, 55, \dots\}.$$

To prove every number can be expressed as a (possibly empty) subset of $A$, we have the following stronger claim.

**Claim.** Let $N$ be an integer (possibly zero or negative). Let $F$ denote the smallest Fibonacci number which is strictly greater than $|N|$. Then there exists a representation of $N$ as the sum of a (possibly empty) subset of $A$, where no element used has absolute value greater than $N$.

*Proof.* By induction on $|N|$. Omitted. □

On the other hand, the *Zeckendorf representation theorem* implies that it's not possible to express 0 as a nonempty subset; such a representation would amount to an identity of the form

$$F_{i_1} + F_{i_2} + \dots = F_{j_1} + F_{j_2} + \dots$$

where no terms on either side are consecutive, and both sides have no common terms; and this is impossible.