# EGMO 2012/4 

Evan Chen
Twitch Solves ISL

Episode 12

## Problem

A set $A$ of integers is called sum-full if $A \subseteq A+A$. A set $A$ of integers is said to be zero-sum-free if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of $A$. Does there exist a sum-full zero-sum-free set of integers?

## External Link

https://aops.com/community/p2658986

## Solution

The answer is YES, such a set exists, and is given by

$$
A=\{1,-2,3,-5,8,-13,21,-34,55, \ldots\} .
$$

To prove every number can be expressed as a (possibly empty) subset of $A$, we have the following stronger claim.

Claim. Let $N$ be an integer (possibly zero or negative). Let $F$ denote the smallest Fibonacci number which is strictly greater than $|N|$. Then there exists a representation of $N$ as the sum of a (possibly empty) subset of $A$, where no element used has absolute value greater than $N$.

Proof. By induction on $|N|$. Omitted.
On the other hand, the Zeckendorf representation theorem implies that it's not possible to express 0 as a nonempty subset; such a representation would amount to an identity of the form

$$
F_{i_{1}}+F_{i_{2}}+\cdots=F_{j_{1}}+F_{j_{2}}+\ldots
$$

where no terms on either side are consecutive, and both sides have no common terms; and this is impossible.

