USEMO 2019/6

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Twitch Solves ISL

Episode 11

Problem

Let ABC be an acute scalene triangle with circumcenter O and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let X, Y, Z be the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Lines AD and YZ intersect at P, lines BE and ZX intersect at Q, and lines CF and XY intersect at R.

Suppose that lines YZ and BC intersect at A', and lines QR and EF intersect at D'. Prove that the perpendiculars from A, B, C, O to the lines QR, RP, PQ, $A'D'$, respectively, are concurrent.

Video

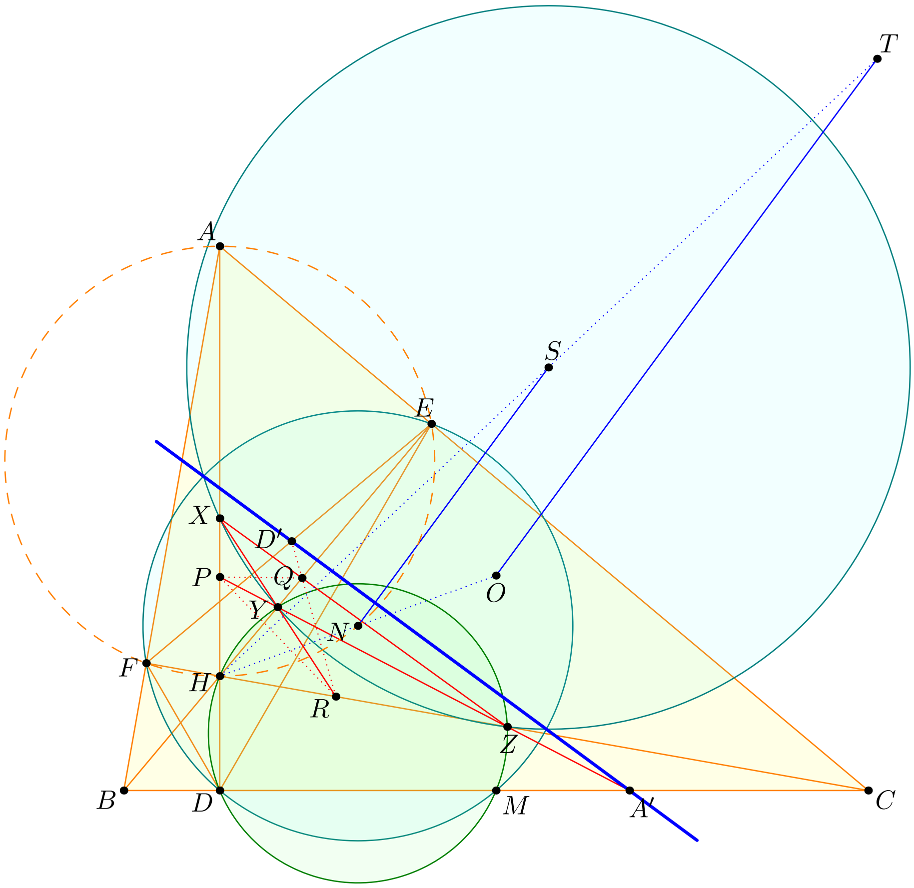
https://youtu.be/0rdn01T_q4Y

Solution

We present two solutions.

Radical axis approach (author's solution) The main idea is to show that (DEF) and (XYZ) has radical axis $\overline{A'D'}$.

Let H be the orthocenter of $\triangle ABC$. We'll let (AH), (BH), (CH) denote the circles with diameters $\overline{AH}$, $\overline{BH}$, $\overline{CH}$.



Claim. Points H, D, Y, Z are cyclic.

Proof. Let M be the midpoint of $\overline{BC}$. We claim they lie on a circle with $\overline{HM}$.

Clearly $\angle HDM = 90^\circ$. The segment $\overline{YM}$ is the B-midline of $\triangle BEC$, so $\overline{YM} \parallel \overline{EC} \perp \overline{HY}$: thus $\angle HYM = 90^\circ$. Similarly $\angle HZM = 90^\circ$. $\square$

Claim. The point P is the radical center of (HB), (HC), (XYZ), $(HYZD)$. Also, QR is the radical axis of (HA) and (XYZ).

Proof. First part since $PH \cdot PD = PY \cdot PZ$; second part by symmetric claims. $\square$

We are now ready for the key claim.

Claim (Key claim)**.** The points A' and D' lie on the radical axis of (DEF) and (XYZ).

Proof. The radical center of (DEF), (XYZ), $(HYZD)$ is $A' = \overline{YZ} \cap \overline{BC}$, and the radical center of (DEF), (XYZ), (HA) is $D' = \overline{EF} \cap \overline{QR}$, so we're done. $\square$

Let S be the center of (XYZ) and T the reflection of H over S. Let N denote the nine-point center.

Claim (Concurrence)**.** The point T is the concurrency point in the problem.

Proof. The line through the centers of (HA) and (XYZ) is perpendicular to the radical axis $\overline{QR}$. Now, a homothety with center H and scale 2 sends these centers to A and T, so $\overline{AT} \perp \overline{QR}$. Similarly, $\overline{BT} \perp \overline{RP}$ and $\overline{CT} \perp \overline{PQ}$.

Similarly from $\overline{NS} \perp \overline{A'D'}$, a dilation at H by a factor of 2 shows $\overline{OT} \perp \overline{A'D'}$, as desired. □

Remark (Author comments on problem creation)**.** The main goal was to create a problem to showcase the midpoints of the altitudes: while they arise due to the midpoint of altitude lemma (Lemma 4.14 in EGMO), I have rarely seen them studied in their own right. This problem strives to be a synthesis of properties relating to the midpoints of altitudes.

Remark. An original, more long-winded version of the problem asks to show that if B', C', E', F' are defined similarly, then all six points are collinear and perpendicular to $\overline{OT}$. The second approach below proves this.

Orthology approach (from contestants) Define B', C', E', F' in an analogous fashion,

Claim. Points A', B', C', D', E', F' are collinear.

Proof. Three applications of Desargue:

- ABC and XYZ are perspective at H so A', B', C' are collinear.
- DEF and PQR are perspective at H so D', E', F' are collinear.
- $C'FR$ and $B'EQ$ are perspective through A-altitude so $B'C'$, EF, QR are concurrent (at D').

□

Claim. The perpendiculars from A, B, C to $\overline{QR}$, $\overline{RP}$, $\overline{PQ}$ are concurrent.

Proof. This follows from the fact that $\triangle ABC$ and $\triangle PQR$ are orthologic with one orthology center at O. □

Claim. The perpendiculars from A, O, C to $\overline{QR}$, $\overline{D'F'}$, $\overline{PQ}$ are concurrent.

Proof. This follows from the fact that $\triangle D'F'Q$ and $\triangle AOC$ are orthologic with one orthology center at E (note that $\overline{AO} \perp \overline{ED'F}$). □

Remark. This solution does not even use the fact that X, Y, Z were the midpoints of the altitudes!