# USEMO 2019/6 

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## Twitch Solves ISL

Episode 11

## Problem

Let $A B C$ be an acute scalene triangle with circumcenter $O$ and altitudes $\overline{A D}, \overline{B E}, \overline{C F}$. Let $X, Y, Z$ be the midpoints of $\overline{A D}, \overline{B E}, \overline{C F}$. Lines $A D$ and $Y Z$ intersect at $P$, lines $B E$ and $Z X$ intersect at $Q$, and lines $C F$ and $X Y$ intersect at $R$.

Suppose that lines $Y Z$ and $B C$ intersect at $A^{\prime}$, and lines $Q R$ and $E F$ intersect at $D^{\prime}$. Prove that the perpendiculars from $A, B, C, O$ to the lines $Q R, R P, P Q, A^{\prime} D^{\prime}$, respectively, are concurrent.

## Video

https://youtu.be/Ordn01T_q4Y

## External Link

https://aops.com/community/p15425714

## Solution

We present two solutions.
Radical axis approach (author's solution). The main idea is to show that ( $D E F$ ) and $(X Y Z)$ has radical axis $\overline{A^{\prime} D^{\prime}}$.
Let $H$ be the orthocenter of $\triangle A B C$. We'll let $(A H),(B H),(C H)$ denote the circles with diameters $\overline{A H}, \overline{B H}, \overline{C H}$.


Claim. Points $H, D, Y, Z$ are cyclic.
Proof. Let $M$ be the midpoint of $\overline{B C}$. We claim they lie on a circle with $\overline{H M}$.
Clearly $\angle H D M=90^{\circ}$. The segment $\overline{Y M}$ is the $B$-midline of $\triangle B E C$, so $\overline{Y M} \| \overline{E C} \perp$ $\overline{H Y}$ : thus $\angle H Y M=90^{\circ}$. Similarly $\angle H Z M=90^{\circ}$.

Claim. The point $P$ is the radical center of $(H B),(H C),(X Y Z),(H Y Z D)$. Also, $Q R$ is the radical axis of $(H A)$ and $(X Y Z)$.

Proof. First part since $P H \cdot P D=P Y \cdot P Z$; second part by symmetric claims.
We are now ready for the key claim.
Claim (Key claim). The points $A^{\prime}$ and $D^{\prime}$ lie on the radical axis of ( $D E F$ ) and ( $X Y Z$ ).
Proof. The radical center of $(D E F),(X Y Z),(H Y Z D)$ is $A^{\prime}=\overline{Y Z} \cap \overline{B C}$, and the radical center of $(D E F),(X Y Z),(H A)$ is $D^{\prime}=\overline{E F} \cap \overline{Q R}$, so we're done.

Let $S$ be the center of $(X Y Z)$ and $T$ the reflection of $H$ over $S$. Let $N$ denote the nine-point center.

Claim (Concurrence). The point $T$ is the concurrency point in the problem.
Proof. The line through the centers of $(H A)$ and $(X Y Z)$ is perpendicular to the radical axis $\overline{Q R}$. Now, a homothety with center $H$ and scale 2 sends these centers to $A$ and $T$, so $\overline{A T} \perp \overline{Q R}$. Similarly, $\overline{B T} \perp \overline{R P}$ and $\overline{C T} \perp \overline{P Q}$.

Similarly from $\overline{N S} \perp \overline{A^{\prime} D^{\prime}}$, a dilation at $H$ by a factor of 2 shows $\overline{O T} \perp \overline{A^{\prime} D^{\prime}}$, as desired.

Remark (Author comments on problem creation). The main goal was to create a problem to showcase the midpoints of the altitudes: while they arise due to the midpoint of altitude lemma (Lemma 4.14 in EGMO), I have rarely seen them studied in their own right. This problem strives to be a synthesis of properties relating to the midpoints of altitudes.

Remark. An original, more long-winded version of the problem asks to show that if $B^{\prime}$, $C^{\prime}, E^{\prime}, F^{\prime}$ are defined similarly, then all six points are collinear and perpendicular to $\overline{O T}$. The second approach below proves this.

Orthology approach (from contestants). Define $B^{\prime}, C^{\prime}, E^{\prime}, F^{\prime}$ in an analogous fashion,
Claim. Points $A^{\prime}, B^{\prime}, C^{\prime}, D^{\prime}, E^{\prime}, F^{\prime}$ are collinear.
Proof. Three applications of Desargue:

- $A B C$ and $X Y Z$ are perspective at $H$ so $A^{\prime}, B^{\prime}, C^{\prime}$ are collinear.
- $D E F$ and $P Q R$ are perspective at $H$ so $D^{\prime}, E^{\prime}, F^{\prime}$ are collinear.
- $C^{\prime} F R$ and $B^{\prime} E Q$ are perspective through $A$-altitude so $B^{\prime} C^{\prime}, E F, Q R$ are concurrent (at $D^{\prime}$ ).

Claim. The perpendiculars from $A, B, C$ to $\overline{Q R}, \overline{R P}, \overline{P Q}$ are concurrent.
Proof. This follows from the fact that $\triangle A B C$ and $\triangle P Q R$ are orthologic with one orthology center at $O$.

Claim. The perpendiculars from $A, O, C$ to $\overline{Q R}, \overline{D^{\prime} F^{\prime}}, \overline{P Q}$ are concurrent.
Proof. This follows from the fact that $\triangle D^{\prime} F^{\prime} Q$ and $\triangle A O C$ are orthologic with one orthology center at $E$ (note that $\overline{A O} \perp \overline{E D^{\prime} F}$ ).

Remark. This solution does not even use the fact that $X, Y, Z$ were the midpoints of the altitudes!

