USEMO 2019/5

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Twitch Solves ISL

Episode 11

Problem

Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be the set of its vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is *patriotic* if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is *dazzling* if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point of $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

Video

https://youtu.be/0rdn01T_q4Y

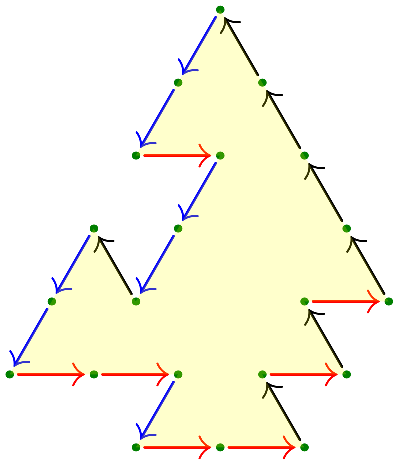
External Link

https://aops.com/community/p15425728

Solution

We prove the contrapositive: if there is no way to split $\mathcal{V}$ into two patriotic sets, then the number of dazzling edges is odd.

Let $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be a root of unity. Read the n vertices of the polygon in order starting from any point. In the complex plane, start from the origin and, corresponding to red, white, or blue, move by 1, ζ, or ζ^2, respectively, to get a path. The diagram below shows an example (where black stands in for white, for legibility reasons).



Note that:

- The path we get is actually a closed loop, since $\mathcal{V}$ was assumed to be patriotic.
- Because there is no nontrivial patriotic subset, this closed loop does not intersect itself, so it corresponds to some polygon $\mathcal{Q}$.

We have to show the number m of vertices of $\mathcal{Q}$ (corresponding to dazzling edges) is odd. Let x and y denote the number of 60° and 300° angles, so $60x + 300y = 180(x + y - 2)$. This gives $x - y = 3$ so $x + y$ is odd.