

USEMO 2019/5

Evan Chen

TWITCH SOLVES ISL

Episode 11

Problem

Let \mathcal{P} be a regular polygon, and let \mathcal{V} be the set of its vertices. Each point in \mathcal{V} is colored red, white, or blue. A subset of \mathcal{V} is *patriotic* if it contains an equal number of points of each color, and a side of \mathcal{P} is *dazzling* if its endpoints are of different colors.

Suppose that \mathcal{V} is patriotic and the number of dazzling edges of \mathcal{P} is even. Prove that there exists a line, not passing through any point of \mathcal{V} , dividing \mathcal{V} into two nonempty patriotic subsets.

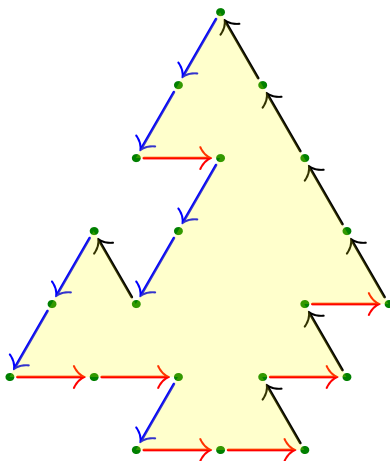
Video

https://youtu.be/Ordn01T_q4Y

Solution

We prove the contrapositive: if there is no way to split \mathcal{V} into two patriotic sets, then the number of dazzling edges is odd.

Let $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be a root of unity. Read the n vertices of the polygon in order starting from any point. In the complex plane, start from the origin and, corresponding to red, white, or blue, move by 1 , ζ , or ζ^2 , respectively, to get a path. The diagram below shows an example (where black stands in for white, for legibility reasons).



Note that:

- The path we get is actually a closed loop, since \mathcal{V} was assumed to be patriotic.
- Because there is no nontrivial patriotic subset, this closed loop does not intersect itself, so it corresponds to some polygon \mathcal{Q} .

We have to show the number m of vertices of \mathcal{Q} (corresponding to dazzling edges) is odd. Let x and y denote the number of 60° and 300° angles, so $60x + 300y = 180(x + y - 2)$. This gives $x - y = 3$ so $x + y$ is odd.