USEMO 2019/4

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Twitch Solves ISL

Episode 11

Problem

Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

Video

https://youtu.be/0rdn01T_q4Y

Solution

The idea is to pick $n = c \cdot p \cdot (p-1)$ for suitable integer c. In what follows, everything is written modulo p.

Claim. When $n = c \cdot p \cdot (p-1)$, the left-hand side is equal to

$$\begin{aligned} c \cdot \sum_{a=0}^{p-2} \sum_{b=1}^{p-1} b^a = c \cdot \big[& 1^0 + 2^0 + \dots + (p-1)^0 \\ & + 1^1 + 2^1 + \dots + (p-1)^1 \\ & + 1^2 + 2^2 + \dots + (p-1)^2 \\ & + \dots \\ & + 1^{p-2} + 2^{p-2} + \dots + (p-1)^{p-2} \big]. \end{aligned}$$

Proof. In the original sum, we discard all the terms divisible by p, reduce all the bases modulo p, and reduce all the exponents modulo $p-1$ (by Fermat's little theorem). Then each block of $p(p-1)$ terms equals

$$\begin{aligned} & 1^0 + 2^{p-2} + 3^{p-3} + \dots + (p-1)^1 \\ & + 1^{p-2} + 2^{p-3} + 3^{p-4} + \dots + (p-1)^0 \\ & + 1^{p-3} + 2^{p-4} + 3^{p-5} + \dots + (p-1)^{p-2} \\ & + \dots \\ & + 1^1 + 2^0 + 3^{p-2} + \dots + (p-1)^2 \end{aligned}$$

which rearranges to the desired sum. $\square$

Claim. We have

$$\sum_{a=0}^{p-2} \sum_{b=1}^{p-1} b^a \equiv -1 \pmod{p}.$$

First proof. By the geometric series formula

$$\sum_{a=0}^{p-2} b^a = \frac{b^{p-1}-1}{b-1} = 0 \qquad \forall\, b = 2, 3, \dots p-1.$$

The terms with $b = 1$ contribute $1^0 + 1^1 + \dots + 1^{p-2} = p-1$ and done. $\square$

Second proof. In fact, it's a classical lemma (proved in the same way, using primitive roots) that

$$\sum_{b=1}^{p-1} b^a \equiv \begin{cases} -1 & p-1 \mid a \\ 0 & p-1 \nmid a \end{cases} \pmod{p}$$

so this is immediate. $\square$

Thus we simply need to select $c \equiv -2020 \pmod{p}$ and win (and $c > 0$).