USEMO 2019/3

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Twitch Solves ISL

Episode 10

Problem

Consider an infinite grid $\mathcal{G}$ of unit square cells. A *chessboard polygon* is a simple polygon (i.e. not self-intersecting) whose sides lie along the gridlines of $\mathcal{G}$.

Nikolai chooses a chessboard polygon F and challenges you to paint some cells of $\mathcal{G}$ green, such that any chessboard polygon congruent to F has at least 1 green cell but at most 2020 green cells. Can Nikolai choose F to make your job impossible?

Video

https://youtu.be/V2TNgUwbs6A

External Link

https://aops.com/community/p15412083

Solution

The answer is YES, the task can be made impossible.

The solution is split into three parts. First, we describe a "polygon with holes" F. In the second part we prove that this F works. Finally, we show how to take care of the holes to obtain a true polygon.

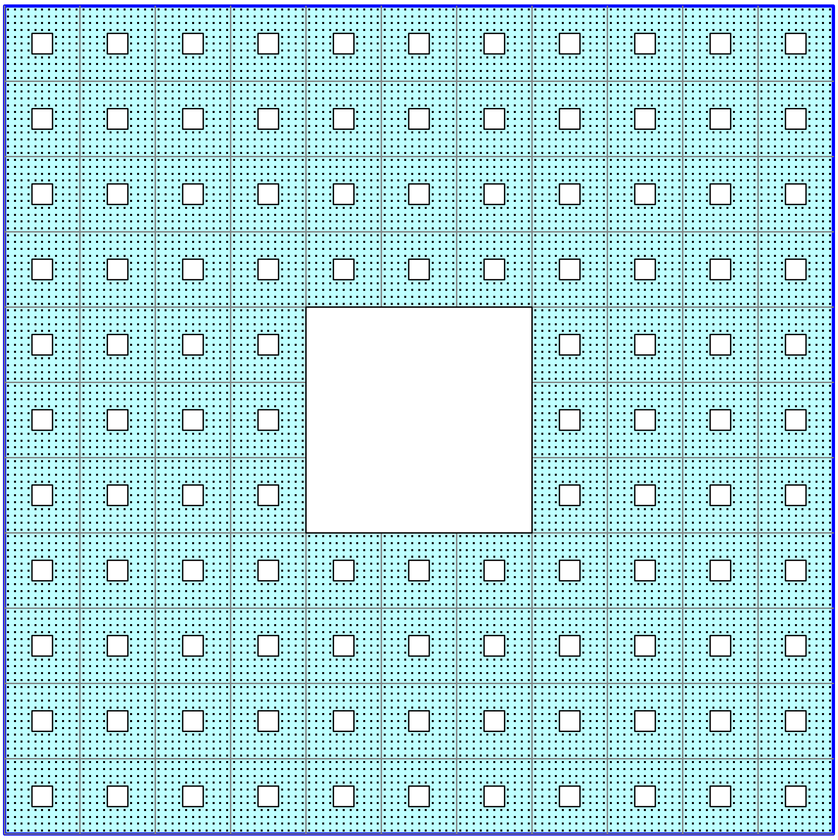
Part 1. Construction. Choose large integers $m \geq 5$, $n \geq 1$ with m odd. We will let

$$s = m^n$$

throughout the solution.

Let F_0 be a square of side length s. Starting from F_0, iterate the following procedure. Divide F_i into squares of side s/m^i and poke a square hole of side $3s/m^{i+1}$ (a *phase-i* hole) in the center of each such square to obtain F_{i+1}. Finally, let $F = F_n$.

The output is shown below for $m = 11$ and $n = 3$. The claim is that for a suitable choice of (m, n) this will serve as the desired example.



Part 2. Proof of this example. Suppose we have a green coloring as described. For every green cell, the *standard copy* of F is a copy of F centered at the green cell.

Claim. The standard copies of F completely cover the plane.

Proof. This is equivalent to every copy of F having at least one green cell, owing to symmetry of F. $\square$

Claim. In any square C with side length $5s$, there are at least $n + 1$ standard copies of F.

Proof. Let C_0 be a square of side $3s$ cocentric with C. Starting from C_0, iterate the following procedure for $1 \le i \le n+1$:

- Let S_{i-1} be one of the standard copies of F that cover the center of C_{i-1},
- If $i \neq n+1$, let C_i be a phase-i hole in S_{i-1} that lies in the interior of C_{i-1}.

Since C_0 and the holes $C_1, C_2, \dots, C_n$ are nested, the standard copies $S_0, S_1, S_2, \dots, S_n$ of F are distinct. It follows that C contains at least $n+1$ standard copies of F. □

However, the area of F is $s^2\left(1 - \frac{9}{m^2}\right)^n$. Therefore, at least one cell c within C is covered by at least

$$k = \frac{n+1}{25}\left(1 - \frac{9}{m^2}\right)^n$$

standard copies of F. The copy of F centered at c then contains at least k green cells.

When $n = 25 \cdot 2020$ and m is sufficiently large, however, we get $k > 2020$.

Part 3. Handling the holes. We are left to show how to repair the above construction so that F becomes a true polygon.

Let D be any sufficiently large positive integer. Consider a homothetic copy F_D of F scaled by a factor of D. Cut several canals of unit width into F_D so that F_D continues to be connected and every hole in F_D is joined by a canal to the boundary of F_D. (Canals do not need to be straight; they may go around holes.) When F_D is repaired in this way, it becomes a true polygon F'_D.

Since the total area of all canals is proportional to D and the total area of F_D is proportional to D^2, when D becomes arbitrarily large the ratio of the area of F'_D to the area of F_D becomes arbitrarily close to one. Therefore, for all sufficiently large D our proof that F is in fact a counterexample goes through for F'_D as well, with straightforward adjustments.

The solution is complete.

Remark (Author comments)**.** Some time after I came up with this problem, Ilya Bogdanov pointed out to me that it is similar to problem 3.6 in Ilya Bogdanov and Grigory Chelnokov, Pokritiya Kletchatimi Figurkami, Summer Conference of the Tournament of Towns, 2007, https://www.turgor.ru/lktg/2007/3/index.php. The USEMO directors agreed that the two problems are different enough that mine was suitable for the contest.