USEMO 2019/1

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Twitch Solves ISL

Episode 10

Problem

Let $ABCD$ be a cyclic quadrilateral. A circle centered at O passes through B and D and meets lines BA and BC again at points E and F (distinct from A, B, C). Let H denote the orthocenter of triangle DEF. Prove that if lines AC, DO, EF are concurrent, then triangles ABC and EHF are similar.

Video

https://youtu.be/V2TNgUwbs6A

External Link

https://aops.com/community/p15412066

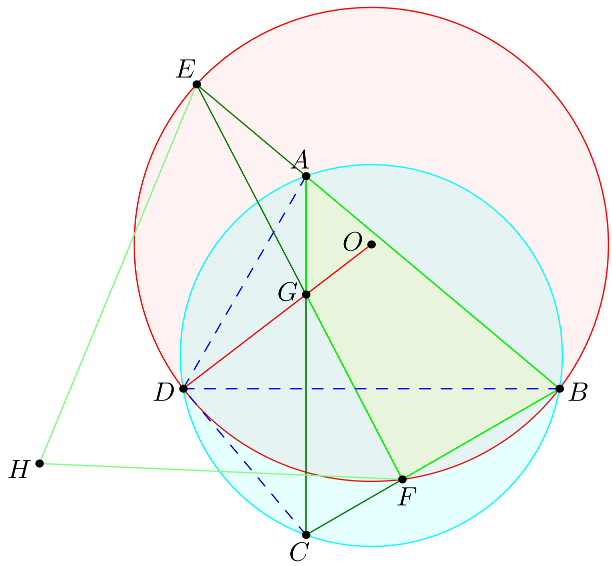
Solution

Define G as the intersection of $\overline{AC}$ and $\overline{EF}$.

Claim. Quadrilateral $DCGF$ is cyclic.

First proof. Because $\measuredangle DCG = \measuredangle DCA = \measuredangle DBA = \measuredangle DBE = \measuredangle DFE = \measuredangle DFG$ □

Second proof. Follows since D is Miquel point of $GABF$. □



Claim. If G lies on line $\overline{DO}$, we have $\overline{AC} \perp \overline{BD}$.

Proof. We have

$$\begin{aligned} \measuredangle BDG = \measuredangle BDO &= 90^\circ - \measuredangle DEB = 90^\circ - \measuredangle DFB \\ &= 90^\circ - \measuredangle DFC = 90^\circ - \measuredangle DGC. \end{aligned}$$ □

To finish,

$$\measuredangle HEF = 90^\circ - \measuredangle EFD = 90^\circ - \measuredangle EBD = 90^\circ - \measuredangle ABD = \measuredangle CAB.$$

Similarly $\measuredangle HFE = \measuredangle ACB$ and the proof is done.

Remark. The original version of this problem was in the converse direction: showing that $\overline{AC} \perp \overline{BD}$ implied the concurrence. Unfortunately, this turns out to be susceptible to Cartesian coordinates by setting the x and y axes along these lines, as well as complex methods.

Interestingly, it does not appear to be easy to show directly that the converse of the problem implies the original statement (other than actually solving the problem, and adapting the proof). Note in particular that the case where $E = A$ and $F = C$ is a counterexample to the converse direction as stated.