# Shortlist 2012 A2 <br> Evan Chen <br> Twitch Solves ISL <br> Episode 9 

## Problem

Decide whether there exists a partition of $S$ into three non-empty subsets $A, B, C$ such that the sets $A+B, B+C, C+A$ are disjoint, where (a) $S=\mathbb{Z}$, (b) $S=\mathbb{Q}$.

## Video

https://youtu.be/pG4Ea3088aA

## External Link

https://aops.com/community/p3160551

## Solution

For (a), just take $A=\{0 \bmod 3\}, B=\{1 \bmod 3\}, C=\{2 \bmod 3\}$.
Actually, let's prove this construction for (a) is unique, up to relabeling the sets. We prove that:

Claim. We have $A+B-C \subseteq C$.
Proof. The hypothesis states that since if $a+b-c=a^{\prime}$ for some $a, a^{\prime} \in A, b \in B$, $c \in C$, we get a contradiction, and similarly. Similarly, we have $B+C-A \subseteq A$, $C+A-B \subseteq B$.

Let $a, a^{\prime}$ be any two elements of $A$. Let $b_{0}$ be any fixed but arbitrary element of $B$. Consider any $c \in C$. Then

$$
a+b_{0}-c \in C \Longrightarrow a^{\prime}+b_{0}-\left(a+b_{0}-c\right)=c+\left(a^{\prime}-a\right) \in C .
$$

This, together with symmetric variants, implies that $A, B, C$ must be arithmetic progressions with the same common difference (by considering the smallest gap between two consecutive elements in the same set). Hence that common difference must be 3 and the uniqueness is proved.

We show this implies the answer to (b) is negative. Assume for contradiction such a partition exists. Let $a_{0}, b_{0}, c_{0}$ be arbitrary elements of $A, B, C$. Choose a large integer $n$ such that $n a_{0}, n b_{0}, n c_{0}$ are all integers divisible by 3 . Then consider $n A \cap \mathbb{Z}, n B \cap \mathbb{Z}$, $n C \cap \mathbb{Z}$. It is a partition satisfying (a) and with each set nonempty, but it is not the one we expected (because all three sets have multiples of 3 ). This is a contradiction.

