Shortlist 2012 A2

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Twitch Solves ISL

Episode 9

Problem

Decide whether there exists a partition of S into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint, where (a) $S = \mathbb{Z}$, (b) $S = \mathbb{Q}$.

Video

https://youtu.be/pG4Ea3O88aA

External Link

https://aops.com/community/p3160551

Solution

For (a), just take $A = \{0 \bmod 3\}$, $B = \{1 \bmod 3\}$, $C = \{2 \bmod 3\}$.

Actually, let's prove this construction for (a) is unique, up to relabeling the sets. We prove that:

Claim. We have $A + B - C \subseteq C$.

Proof. The hypothesis states that since if $a + b - c = a'$ for some $a, a' \in A$, $b \in B$, $c \in C$, we get a contradiction, and similarly. Similarly, we have $B + C - A \subseteq A$, $C + A - B \subseteq B$. $\square$

Let a, a' be any two elements of A. Let b_0 be any fixed but arbitrary element of B. Consider any $c \in C$. Then

$$a + b_0 - c \in C \implies a' + b_0 - (a + b_0 - c) = \boxed{c + (a' - a) \in C}.$$

This, together with symmetric variants, implies that A, B, C must be arithmetic progressions with the same common difference (by considering the smallest gap between two consecutive elements in the same set). Hence that common difference must be 3 and the uniqueness is proved.

We show this implies the answer to (b) is negative. Assume for contradiction such a partition exists. Let a_0, b_0, c_0 be arbitrary elements of A, B, C. Choose a large integer n such that na_0, nb_0, nc_0 are all integers divisible by 3. Then consider $nA \cap \mathbb{Z}$, $nB \cap \mathbb{Z}$, $nC \cap \mathbb{Z}$. It is a partition satisfying (a) and with each set nonempty, but it is not the one we expected (because all three sets have multiples of 3). This is a contradiction.