# Shortlist 2004 C5 <br> Evan Chen 

Twitch Solves ISL
Episode 9

## Problem

Let $N$ be a positive integer. Alice and Bob play a game. First Alice writes down 1 first. Then every player (starting with Bob) sees the last number written. If it is $n$, then that player may write either $n+1$ or $2 n$, but the written number cannot exceed $N$. The player who writes $N$ wins. For which $N$ does Bob win?

## External Link

https://aops.com/community/p251895

## Solution

Call this game the $N$-game (as we will induct on $N$ ). The answer is that Bob wins the $N$-game are exactly those that, when expressed in binary, have no $1^{\prime}$ 's in the $2^{0}, 2^{2}, 2^{4}$, ...positions.

Setup. Define the $N$-table as follows. It has $N$ entries indexed by $\{1, \ldots, N\}$.

- The $n$th entry is losing if and only if a player who sees this number on their turn wins; otherwise, it is winning. Hence by definition the last entry is always $L$ (because a player who is faced with $N$ has just lost!).
- Equivalently, here is a recursive description. We say $N$ is losing. Then, a given $n$ is winning if either $n+1$ or $2 n$ is losing (ignoring $2 n$ if $n>N / 2$ ); else, it is winning.
- Bob wins the $N$-game if and only if 1 is $W$.

The tables are shown below for $N=12$, respectively.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $L$ | $W$ | $L$ | $W$ | $W$ | $W$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ |  |  |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |  |
| $L$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ |  |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $L$ | $W$ | $L$ | $W$ | $W$ | $W$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ | $W$ | $L$ |

Proof. We start with the following claim.
Claim. If $N$ is odd, then Alice wins the $N$-game.
Proof. Alice can force Bob to stay on odd numbers.
We now address the case where $N$ is even.
Claim. Suppose $N \equiv 2(\bmod 4)$ and $N>2$. Then Bob wins the $N$-game if and only if he wins the $(N-2)$-game.

Proof. In the $N$-table, we have $N$ is losing, $N-1$ is winning, $N-2$ is losing, and so on; until $N / 2+1$ is losing. Thus $N / 2$ is winning.

Now note that we may delete $N$ and $N-1$ (the last two columns) from the table without affecting any other entries; indeed $N / 2$ was winning anyways as $N / 2+1$ is losing. Thus the $N$-table is the $(N-2)$-table with two extra columns at the end.

Claim. Suppose $N \equiv 0(\bmod 4)$ and $N>4$. Then Bob wins the $N$-game if and only if he wins the $N / 4$-game.

Proof. In the $N$-table, we have $N$ is losing, $N-1$ is winning, $N-2$ is losing, and so on; until $N / 2+1$ is winning. Now from this it follows $N / 2, N / 2-1, \ldots, N / 4+1$ are all winning.

Since this entire range of numbers is winning and thus don't affect any later numbers, we wind up finding the first $N / 4$ numbers are just a copy of the $(N / 4)$-table.

These three claims imply the answer directly by induction.

