Shortlist 2004 C5

Evan Chen

Twitch Solves ISL

Episode 9

Problem

Let N be a positive integer. Alice and Bob play a game. First Alice writes down 1 first. Then every player (starting with Bob) sees the last number written. If it is n, then that player may write either $n+1$ or $2n$, but the written number cannot exceed N. The player who writes N wins. For which N does Bob win?

External Link

https://aops.com/community/p251895

Solution

Call this game the N-game (as we will induct on N). The answer is that Bob wins the N-game are exactly those that, when expressed in binary, have no 1's in the 2^0, 2^2, 2^4, ...positions.

Setup. Define the N*-table* as follows. It has N entries indexed by $\{1, \dots, N\}$.

- The nth entry is **losing** if and only if a player who sees this number on their turn wins; otherwise, it is **winning**. Hence by definition the last entry is always L (because a player who is faced with N has just lost!).
- Equivalently, here is a recursive description. We say N is losing. Then, a given n is winning if either $n+1$ or $2n$ is losing (ignoring $2n$ if $n > N/2$); else, it is winning.
- Bob wins the N-game if and only if 1 is W.

The tables are shown below for $N = 12$, respectively.

1	2	3	4	5	6	7	8	9	10	11	12		
L	W	L	W	W	W	W	L	W	L	W	L		
1	2	3	4	5	6	7	8	9	10	11	12	13	
L	W	L	W	L	W	L	W	L	W	L	W	L	
1	2	3	4	5	6	7	8	9	10	11	12	13	14
L	W	L	W	W	W	W	L	W	L	W	L	W	L

Proof. We start with the following claim.

Claim. If N is odd, then Alice wins the N-game.

Proof. Alice can force Bob to stay on odd numbers. $\square$

We now address the case where N is even.

Claim. Suppose $N \equiv 2 \pmod 4$ and $N > 2$. Then Bob wins the N-game if and only if he wins the $(N-2)$-game.

Proof. In the N-table, we have N is losing, $N-1$ is winning, $N-2$ is losing, and so on; until $N/2+1$ is losing. Thus $N/2$ is winning.

Now note that we may delete N and $N-1$ (the last two columns) from the table without affecting any other entries; indeed $N/2$ was winning anyways as $N/2+1$ is losing. Thus the N-table is the $(N-2)$-table with two extra columns at the end. $\square$

Claim. Suppose $N \equiv 0 \pmod 4$ and $N > 4$. Then Bob wins the N-game if and only if he wins the $N/4$-game.

Proof. In the N-table, we have N is losing, $N-1$ is winning, $N-2$ is losing, and so on; until $N/2+1$ is winning. Now from this it follows $N/2$, $N/2-1$, ..., $N/4+1$ are all winning.

Since this entire range of numbers is winning and thus don't affect any later numbers, we wind up finding the first $N/4$ numbers are just a copy of the $(N/4)$-table. $\square$

These three claims imply the answer directly by induction.