

Shortlist 2009 A1

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TWITCH SOLVES ISL

Episode 8

Problem

Find the largest possible integer k , such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides to obtain

$b_1 \leq b_2 \leq \dots \leq b_{2009}$, the lengths of the blue sides,

$r_1 \leq r_2 \leq \dots \leq r_{2009}$, the lengths of the red sides,

and $w_1 \leq w_2 \leq \dots \leq w_{2009}$, the lengths of the white sides.

Then there are at least k indices j with the property that b_j, r_j, w_j form the sides of a nondegenerate triangle.

Solution

The answer is $k = 1$.

To see at least one triangle must be formed, simply note that the longest sides b_{2009} , r_{2009} , w_{2009} of each color necessarily form a triangle; for if $w_{2009} > b_{2009} + r_{2009}$ (say) then no triangle can have side w_{2009} .

Conversely, we give a construction where only $j = 2009$ works:

$\triangle\#1$	$\triangle\#2$	$\triangle\#3$	$\triangle\#4$	\dots	$\triangle\#2008$	$\triangle\#2009$
$b_1 = 10000$	$b_2 = 10001$	$b_3 = 10002$	$b_4 = 10003$	\dots	$b_{2008} = 12007$	$b_{2009} = 12008$
$w_{2009} = 14200$	$w_1 = 10000$	$w_2 = 10001$	$w_3 = 10002$	\dots	$b_{2007} = 12006$	$b_{2008} = 12007$
$r_{2009} = 24199$	$r_1 = 20000$	$r_2 = 20002$	$r_3 = 20004$	\dots	$b_{2007} = 24012$	$b_{2008} = 24014$

This counterexample shows $k = 1$ is the largest integer for which the statement is true.