Shortlist 2008 G6

Evan Chen

Twitch Solves ISL

Episode 8

Problem

Let $ABCD$ be a given convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that

$$\begin{aligned} \angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA \\ = \angle PDA + \angle PCB = 90^\circ \end{aligned}$$

if and only if the diagonals AC and BD are perpendicular.

Video

https://youtu.be/okxnxCnAAas

External Link

https://aops.com/community/p1555924

Solution

The problem is killed by quoting the following theorem:

Theorem 1. If X is a point such that $\angle AXD + \angle BXC = 180^\circ$, then the isogonal conjugate of X exists.

Proof, for completeness only. It's enough for the projections of X to the sides to be cyclic, by considering the six-point circle. Let them be X_1, X_2, X_3, X_4.

$$\measuredangle XX_4X_1 = \measuredangle XAX_1$$
$$\vdots$$

and this means $\measuredangle X_1X_2X_3 = \measuredangle X_1X_4X_3$. $\square$

Now in one direction, if the diagonals are perpendicular and meet at Q, then its isogonal conjugate exists, and is seen to have the desired property.

Conversely, given such a point P, it has an isogonal conjugate Q which satisfies $\angle AQB = \angle BQC = \angle CQD = \angle DQA = 90^\circ$, which implies that Q is the perpendicular intersection of the diagonals.