

# Shortlist 2008 C5

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Episode 8

## Problem

Let  $S = \{x_1, x_2, \dots, x_{k+\ell}\}$  be a subset of  $[0, 1]$ . A  $k$ -element subset  $A \subset S$  is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k + \ell}{2k\ell}$$

Prove that the number of nice subsets is at least  $\frac{2}{k+\ell} \binom{k+\ell}{k}$ .

## Video

<https://youtu.be/ikvgjL-82tE>

## External Link

<https://aops.com/community/p1555908>

## Solution

We let  $\Delta(A)$  denote the quantity in the absolute value as a function of  $A$ . Let  $n = k + \ell$  for brevity.

Consider a random permutation  $\pi$  of  $\{1, 2, \dots, n\}$ . Look at the following  $k + \ell$  sets, the “cyclic shifts”:

$$\begin{aligned} A_1 &= \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}\} \\ A_2 &= \{x_{\pi(2)}, x_{\pi(3)}, \dots, x_{\pi(k+1)}\} \\ A_3 &= \{x_{\pi(3)}, x_{\pi(4)}, \dots, x_{\pi(k+2)}\} \\ &\vdots \\ A_{n-1} &= \{x_{\pi(n-1)}, x_{\pi(n)}, \dots, x_{\pi(n-2)}\} \\ A_n &= \{x_{\pi(n)}, x_{\pi(1)}, \dots, x_{\pi(n-1)}\}. \end{aligned}$$

We evidently have

$$\sum_1^n \Delta(A_i) = 0.$$

However, any two consecutive  $\Delta(A_i)$  are going to differ by at most  $\delta := \frac{1}{k} + \frac{1}{\ell}$ . However we note that:

**Lemma.** If we have (cyclic) sequence of  $n \geq 2$  numbers with mean zero, and whose consecutive terms differ by at most  $\delta$ , then there will always be at least two numbers with absolute value at most  $\delta/2$ .

In other words, among these  $n$  sets, there must be at least two nice ones, regardless of  $\pi$ .

However, if  $p$  is the fraction of  $k$ -element subsets which are nice, then the expected number of nice sets is exactly  $n \cdot p$ . So we conclude  $p \geq \frac{2}{n}$  as desired.