Shortlist 2008 C5

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Episode 8

Problem

Let $S = \{x_1, x_2, \dots, x_{k+\ell}\}$ be a subset of $[0, 1]$. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k+\ell}{2k\ell}$$

Prove that the number of nice subsets is at least $\frac{2}{k+\ell}\binom{k+\ell}{k}$.

Video

https://youtu.be/ikvgjL-82tE

Solution

We let $\Delta(A)$ denote the quantity in the absolute value as a function of A. Let $n = k + \ell$ for brevity.

Consider a random permutation π of $\{1, 2, \ldots, n\}$. Look at the following $k + \ell$ sets, the "cyclic shifts":

$$\begin{aligned} A_1 &= \left\{x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}\right\} \\ A_2 &= \left\{x_{\pi(2)}, x_{\pi(3)}, \ldots, x_{\pi(k+1)}\right\} \\ A_3 &= \left\{x_{\pi(3)}, x_{\pi(4)}, \ldots, x_{\pi(k+2)}\right\} \\ &\vdots \\ A_{n-1} &= \left\{x_{\pi(n-1)}, x_{\pi(n)}, \ldots, x_{\pi(n-2)}\right\} \\ A_n &= \left\{x_{\pi(n)}, x_{\pi(1)}, \ldots, x_{\pi(n-1)}\right\}. \end{aligned}$$

We evidently have

$$\sum_1^n \Delta(A_i) = 0.$$

However, any two consecutive $\Delta(A_i)$ are going to differ by at most $\delta \overset{\text{def}}{=} \frac{1}{k} + \frac{1}{\ell}$. However we note that:

Lemma. *If we have (cyclic) sequence of $n \ge 2$ numbers with mean zero, and whose consecutive terms differ by at most δ, then there will always be at least two numbers with absolute value at most $\delta/2$.*

In other words, among these n sets, there must be at least two nice ones, regardless of π.

However, if p is the fraction of k-element subsets which are nice, then the expected number of nice sets is exactly $n \cdot p$. So we conclude $p \ge \frac{2}{n}$ as desired.