Shortlist 2004 N2

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Twitch Solves ISL

Episode 7

Problem

Define $f\colon \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \sum_{k=1}^{n} \gcd(k, n).$$

For each positive integer a, show that the equation $f(x) = ax$ has at least one solution and determine whether or not that solution is unique.

External Link

https://aops.com/community/p191396

Solution

Let $g(n) = f(n)/n$, so we are interested in the outputs of g. We start with:

Claim. The function g is multiplicative and satisfies

$$g(p^e) = \frac{p-1}{p} \cdot e + 1$$

for any prime power p^e.

Proof. First, write

$$f(n) = \sum_{d \mid n} d\varphi(n/d) = \mathrm{id} * \varphi$$

to get that f is multiplicative (as the Dirichlet convolution of two multiplicative functions). Thus $g(n) = f(n)/n$ is multiplicative too.

Now note that for any prime power p^e, we have

$$g(p^e) = \frac{f(p^e)}{p^e} = \frac{p^e \cdot 1 + p^{e-1}(p-1) + \cdots + 1 \cdot (p^e - p^{e-1})}{p^e} = e + 1 - \frac{e}{p}$$

so the second part is true too. □

In particular, we have

$$g(2^{2a-2}) = a$$

so we already know every a has the solution $x = 2^{2a-2}$.

We now show that this is the only solution if and only if a is a power of 2.

First, if $q > 1$ is any odd divisor of a, then writing $a = q \cdot b$, one can note that

$$g(2^{2b-2}) = b$$
$$g(3^{\frac{3}{2}(q-1)}) = q$$

and in this way we generate a new solution to the given equation. This shows the solution we found is not unique when a is not a power of 2.

Conversely, suppose $a = 2^\ell$ is a power of 2 and x is an integer with

$$g(x) = a = 2^\ell.$$

Note that if y is an odd prime power, then

- $\nu_2(g(y)) = 0$, and
- $g(y) > 1$.

So by measuring ν_2, we get $\nu_2(g(x)) = \ell \implies \nu_2(x) = 2a - 2$ matching the solution we found before. But then for size reasons, we must have $x = 2^{2a-2}$, as desired.