Shortlist 2004 G3

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Twitch Solves ISL

Episode 7

Problem

Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$ and let $D = \overline{AO} \cap \overline{BC}$. Let E and F denote the circumcenters of triangles ABD and ACD. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Video

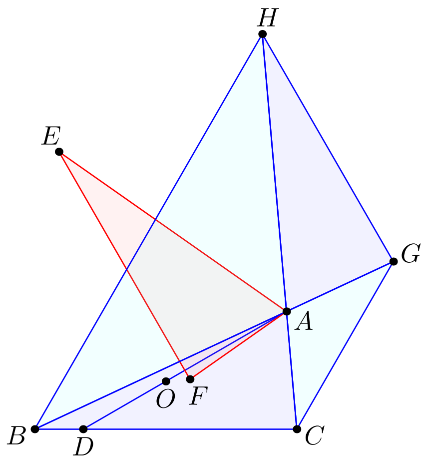
https://youtu.be/3W4kkTPJup0

Solution

We start with a few observations which are always true regardless of the condition.

- Quadrilateral $HGCB$ is always an isosceles trapezoid, and in particular $BC = GH$.
- By angle chasing $\overline{AO} \perp \overline{GH}$ always holds. (One clean way to see this is to note that $\overline{GH}$ and $\overline{BC}$ are antiparallel through $\angle A$.) This implies $\overline{EF} \parallel \overline{GH}$.
- By Salmon theorem, we always have

$$\triangle AEF \overset{+}{\sim} \triangle ABC.$$



We begin now with:

Claim. We have $\triangle AEF \cong \triangle ABC$ if and only if $\angle C - \angle B = 60^\circ$.

Proof. The congruence just means $FA = AC$. Since $FA = FC$ always, triangle AFC is equilateral if and only if $\angle AFC = 60^\circ \iff \angle ADC = 30^\circ$. As $\angle ADC = (90^\circ - \angle C) + \angle B$, and the result follows. □

Claim. We have $EFGH$ is a parallelogram if and only if $\triangle AEF \cong \triangle ABC$.

Proof. Since we already know $\overline{EF} \parallel \overline{GH}$, the parallelogram condition is equivalent to $EF = GH$, but as $GH = BC$ we get the earlier congruence. □

It remains only to show that if $EFGH$ is a parallelogram then it is also a rectangle. In the situation of the claims, note that $EG = FH$ by symmetry through the oppositely congruent triangles $\triangle AEF$ and $\triangle AHG$ as needed.