Shortlist 2004 C3

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Episode 7

Problem

The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices.

Video

https://youtu.be/0f4qSCsWQjI

External Link

https://aops.com/community/p251391

Solution

The answer is n. As the case $n = 4$ is easily done by hand, we will assume $n \geq 5$.

To show that at least n edges always remain, note that

- The operation preserves connectedness of the graph;
- The operation preserves the fact that the graph is *not* bipartite.

This implies the graph will always have n edges leftover (since a connected graph always has at least $n-1$ edges, but with equality only when it is a tree, and trees are necessarily bipartite).

Conversely, for any $n \geq 5$ one can reach the graph consisting of a K_5 plus a path of length $n-5$ hanging off one vertex. Then one can prune the K_5 down to C_5 to achieve the desired construction.