Shortlist 2004 A2

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Twitch Solves ISL

Episode 7

Problem

Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying

$$a_n = |a_{n+1} - a_{n+2}|$$

for all $n \geq 0$, where a_0 and a_1 are two different positive real numbers. Can this sequence a_0, a_1, a_2, ... be bounded?

Video

https://youtu.be/ESnvvXn0B6Q

External Link

https://aops.com/community/p152740

Solution

Answer: the sequence cannot be bounded.

By induction one may verify easily that:

- all terms of the sequence are positive;
- no two adjacent terms are equal;

Call an index n *nice* if $a_n < a_{n+1}$.

Claim. If n is not nice, then $n + 1$ must be.

Proof. If $a_n > a_{n+1}$ then $a_{n+2} = a_n + a_{n+1} > a_{n+1}$. □

If n is a nice index, then we define $\psi(n) = (a_n, a_{n+1} - a_n)$, a pair of two positive real numbers. Equivalently, by $\psi(n) = (x, y)$ we mean that $a_n = x$ and $a_{n+1} = x + y$.

Claim. If n is nice and $\psi(n) = (x, y)$ then either

- $n + 1$ is nice and $\psi(n) = (x + y, x)$; or
- $n + 2$ is nice and $\psi(n) = (y, x + y)$.

Proof. This follows by computation. Let $a_n = x$ and $a_{n+1} = x + y$. Then $a_{n+2} = a_{n+1} \pm a_n$.

- If the sign is $+$, we get $a_{n+2} = 2x + y = (x + y) + x$.
- If the sign is $-$, we get $a_{n+2} = y$, and then $a_{n+3} = (x + y) + y$. □

Thus the sequence of ψ-pairs obviously grows without bound starting from the first nice index.