# Shortlist 2004 A2 <br> Evan Chen <br> Twitch Solves ISL <br> Episode 7 

## Problem

Let $a_{0}, a_{1}, a_{2}, \ldots$ be an infinite sequence of real numbers satisfying the equation

$$
a_{n}=\left|a_{n+1}-a_{n+2}\right|
$$

for all $n \geq 0$, where $a_{0}$ and $a_{1}$ are two different positive reals. Can this sequence $a_{0}, a_{1}$, $a_{2}, \ldots$ be bounded?

## Video

https://youtu.be/ESnvvXn0B6Q

## External Link

https://aops.com/community/p152740

## Solution

Answer: the sequence cannot be bounded.
By induction one may verify easily that:

- all terms of the sequence are positive;
- no two adjacent terms are equal;

Call an index $n$ nice if $a_{n}<a_{n+1}$.
Claim. If $n$ is not nice, then $n+1$ must be.
Proof. If $a_{n}>a_{n+1}$ then $a_{n+2}=a_{n}+a_{n+1}>a_{n+1}$.
If $n$ is a nice index, then we define $\psi(n)=\left(a_{n}, a_{n+1}-a_{n}\right)$, a pair of two positive real numbers. Equivalently, by $\psi(n)=(x, y)$ we mean that $a_{n}=x$ and $a_{n+1}=x+y$.

Claim. If $n$ is nice and $\psi(n)=(x, y)$ then either

- $n+1$ is nice and $\psi(n)=(x+y, x)$; or
- $n+2$ is nice and $\psi(n)=(y, x+y)$.

Proof. This follows by computation. Let $a_{n}=x$ and $a_{n+1}=x+y$. Then $a_{n+2}=$ $a_{n+1} \pm a_{n}$.

- If the sign is + , we get $a_{n+2}=2 x+y=(x+y)+x$.
- If the sign is - , we get $a_{n+2}=y$, and then $a_{n+3}=(x+y)+y$.

Thus the sequence of $\psi$-pairs obviously grows without bound starting from the first nice index.

