Twitch 006.1

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Twitch Solves ISL

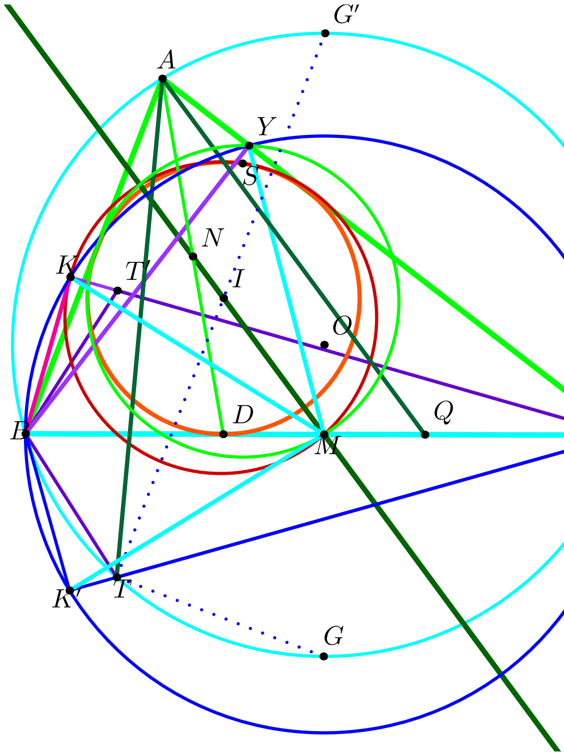
Episode 6

Problem

Let ABC be a triangle and let T be the contact point of the A-mixtilinear incircle with the circumcircle, and let T' be the reflection of T over BC. Prove that the nine-point circle of $T'BC$ is tangent to the incircle.

Solution

The following solution is typeset thanks to Srijon Sarkar. Points defined as in diagram shown.



Claim. The nine-point circle of $T'BC$ coincides with the reflection of the nine-point circle of ABC over $\overline{MI}$.

Proof. It suffices to show that the foot K from B to CT', when reflected over $\overline{MI}$, is the foot Y from B to CA, i.e. we want $IK = IY$.

- $\angle BKC = \angle BYC = 90^\circ \implies MB = MC = MY = MK \implies \{K, Y\} \in \odot(BC)$.
- Since AQ and AT are Isogonals w.r.t A-angle bisector, by Mixtilinear copying (property of T) and angle chase we have:

$$\angle BCT = \angle BAT = \angle QAC = \angle T'CB = \angle KCB \implies \angle QAC = \angle KCB.$$

- Since D and Q are the A-intouch point and A-extouch point respectively, hence by **Diameter of the Incircle lemma** $BD = QC \implies MD = MQ$. In $\triangle ADQ$, we had N as the midpoint of AD and now M as the midpoint of DQ, so $NM \parallel AQ$.

  If D' is the antipode of D w.r.t $\odot(I)$, then A, D', Q are collinear by **Diameter of the Incircle lemma**. Now, by homothety at D with a ratio of $-\frac{1}{2}$ we get A, D' and Q mapped to N, I and M respectively, thus N, I, M are collinear.

  Hence, $IM \parallel AQ$. Now, we reflect K over BC, the reflected point K' falls on $\odot(BKYC)$.

- Next, we note that:

$$\angle BMI = \angle BQA$$

$$\begin{aligned} &= \angle QAC + \angle QCA \\ &= \angle KCB + \angle BCA \\ &= \angle K'CB + \angle BCA \\ &= \angle TCA \\ \implies &\angle BMI = \angle TCA. \end{aligned}$$

- As $\angle KMK' = 2\angle KCT$ and $\angle KMY = 2\angle KCY$ we get

$$\implies \frac{1}{2}\angle K'MY = \angle TCA \implies \angle BMI = \frac{1}{2}\angle K'MY.$$

- Now, since $\angle KMB = \angle K'MB$, using the above result we get $\angle YMI = \angle KMI \implies \overline{MI}$ is the bisector of $\angle YMK$. We also had $MK = MY$, so $IK = IY$.

$\square$

And thus, the nine-point circle of $\triangle T'BC$ is tangent to the incircle of $\triangle ABC$ (by appealing to Feuerbach theorem).