Shortlist 2012 G3

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Twitch Solves ISL

Episode 6

Problem

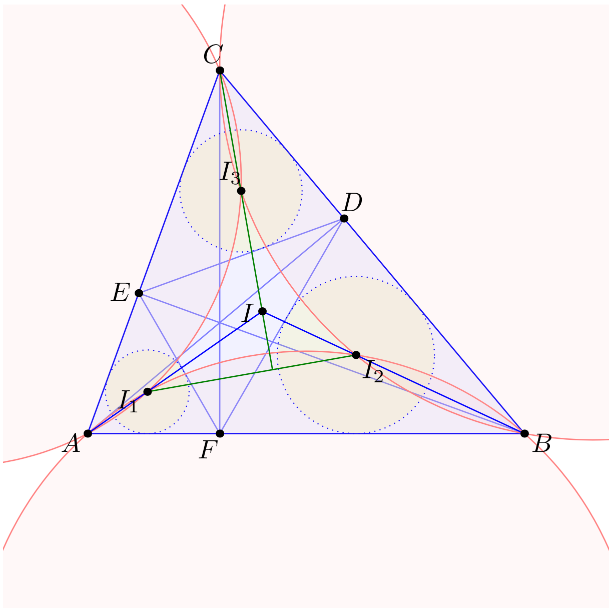
In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Video

https://youtu.be/DpUbRGjqTiI

Solution

We let I_3 denote the incenter of $\triangle CDE$.



Claim. The points A, B, I_1, I_2 are concyclic. Similarly for the other two pairs.

Proof. By power of a point from I. Note that $\triangle AEF \overset{-}{\sim} \triangle ABC$, so $\frac{AI_1}{AI} = \frac{AE}{AB}$. The identity we wish to prove thus may be written as

$$AI^2 \cdot \left(1 - \frac{AE}{AB}\right) = BI^2 \left(1 - \frac{BD}{AB}\right).$$

However, $AE/AB = \cos A$, $BD/AB = \cos B$, and $AI/BI = \frac{\sin(B/2)}{\sin(A/2)}$, so this is immediate. $\square$

Apply the claim twice to obtain that $\overline{CI_3}$ (which is the $\angle C$-bisector,) is the radical axis of (ACI_1) and (BCI_2). Now note that line IC also passes through the circumcenter of $\triangle ABI$ (which is the arc midpoint of $\widehat{AB}$), so since $\overline{I_1 I_2}$ and $\overline{AB}$ are anti-parallel, the desired perpendicularity follows.

Since $\overline{O_1 O_2}$ is perpendicular to the radical axis, we're all set.