

Shortlist 2012 G3

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TWITCH SOLVES ISL

Episode 6

Problem

In an acute triangle ABC the points D , E and F are the feet of the altitudes through A , B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Video

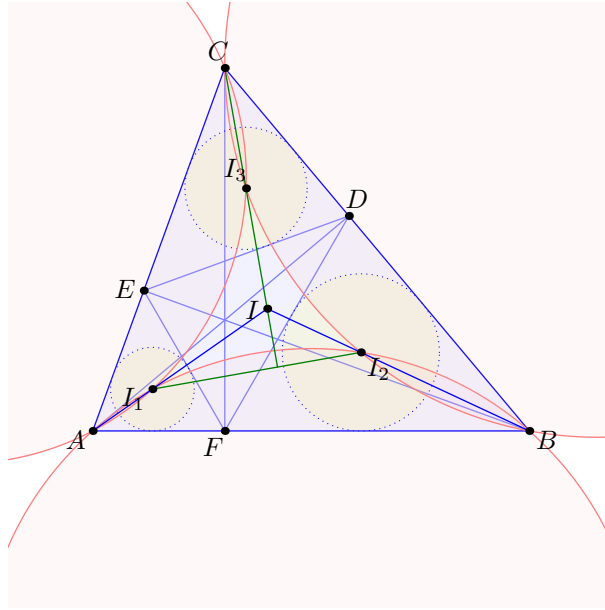
<https://youtu.be/DpUbRGjqTiI>

External Link

<https://aops.com/community/p3160579>

Solution

We let I_3 denote the incenter of $\triangle CDE$.



Claim. The points A, B, I_1, I_2 are concyclic. Similarly for the other two pairs.

Proof. By power of a point from I . Note that $\triangle AEF \sim \triangle ABC$, so $\frac{AI_1}{AI} = \frac{AE}{AB}$. The identity we wish to prove thus may be written as

$$AI^2 \cdot \left(1 - \frac{AE}{AB}\right) = BI^2 \left(1 - \frac{BD}{AB}\right).$$

However, $AE/AB = \cos A$, $BD/AB = \cos B$, and $AI/BI = \frac{\sin(B/2)}{\sin(A/2)}$, so this is immediate. \square

Apply the claim twice to obtain that $\overline{CI_3}$ (which is the $\angle C$ -bisector) is the radical axis of (ACI_1) and (BCI_2) . Now note that line IC also passes through the circumcenter of $\triangle ABI$ (which is the arc midpoint of \widehat{AB}), so since $\overline{I_1I_2}$ and \overline{AB} are anti-parallel, the desired perpendicularity follows.

Since $\overline{O_1O_2}$ is perpendicular to the radical axis, we're all set.