# Shortlist 2012 G3 <br> Evan Chen 

## Twitch Solves ISL

Episode 6

## Problem

In an acute triangle $A B C$ the points $D, E$ and $F$ are the feet of the altitudes through $A, B$ and $C$ respectively. The incenters of the triangles $A E F$ and $B D F$ are $I_{1}$ and $I_{2}$ respectively; the circumcenters of the triangles $A C I_{1}$ and $B C I_{2}$ are $O_{1}$ and $O_{2}$ respectively. Prove that $I_{1} I_{2}$ and $O_{1} O_{2}$ are parallel.

## Video

https://youtu.be/DpUbRGjqTiI

## External Link

https://aops.com/community/p3160579

## Solution

We let $I_{3}$ denote the incenter of $\triangle C D E$.


Claim. The points $A, B, I_{1}, I_{2}$ are concyclic. Similarly for the other two pairs.
Proof. By power of a point from $I$. Note that $\triangle A E F \sim \triangle A B C$, so $\frac{A I_{1}}{A I}=\frac{A E}{A B}$. The identity we wish to prove thus may be written as

$$
A I^{2} \cdot\left(1-\frac{A E}{A B}\right)=B I^{2}\left(1-\frac{B D}{A B}\right) .
$$

However, $A E / A B=\cos A, B D / A B=\cos B$, and $A I / B I=\frac{\sin (B / 2)}{\sin (A / 2)}$, so this is immediate.

Apply the claim twice to obtain that $\overline{C I_{3}}$ (which is the $\angle C$-bisector) is the radical axis of $\left(A C I_{1}\right)$ and $\left(B C I_{2}\right)$. Now note that line $I C$ also passes through the circumcenter of $\triangle A B I$ (which is the arc midpoint of $\widehat{A B}$ ), so since $\overline{I_{1} I_{2}}$ and $\overline{A B}$ are anti-parallel, the desired perpendicularity follows.

Since $\overline{O_{1} O_{2}}$ is perpendicular to the radical axis, we're all set.

