# Shortlist 2012 G3

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TWITCH SOLVES ISL

Episode 6

### **Problem**

In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are  $I_1$  and  $I_2$  respectively; the circumcenters of the triangles  $ACI_1$  and  $BCI_2$  are  $O_1$  and  $O_2$  respectively. Prove that  $I_1I_2$  and  $O_1O_2$  are parallel.

## Video

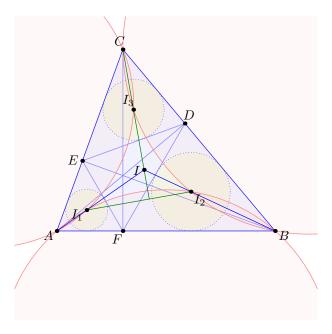
https://youtu.be/DpUbRGjqTiI

### **External Link**

https://aops.com/community/p3160579

### Solution

We let  $I_3$  denote the incenter of  $\triangle CDE$ .



Claim. The points  $A, B, I_1, I_2$  are concyclic. Similarly for the other two pairs.

*Proof.* By power of a point from I. Note that  $\triangle AEF \sim \triangle ABC$ , so  $\frac{AI_1}{AI} = \frac{AE}{AB}$ . The identity we wish to prove thus may be written as

$$AI^2 \cdot \left(1 - \frac{AE}{AB}\right) = BI^2 \left(1 - \frac{BD}{AB}\right).$$

However,  $AE/AB = \cos A$ ,  $BD/AB = \cos B$ , and  $AI/BI = \frac{\sin(B/2)}{\sin(A/2)}$ , so this is immediate.

Apply the claim twice to obtain that  $\overline{CI_3}$  (which is the  $\angle C$ -bisector) is the radical axis of  $(ACI_1)$  and  $(BCI_2)$ . Now note that line IC also passes through the circumcenter of  $\triangle ABI$  (which is the arc midpoint of  $\widehat{AB}$ ), so since  $\overline{I_1I_2}$  and  $\overline{AB}$  are anti-parallel, the desired perpendicularity follows.

Since  $\overline{O_1O_2}$  is perpendicular to the radical axis, we're all set.