

# Shortlist 2006 A5

Evan Chen

TWITCH SOLVES ISL

Episode 6

## Problem

If  $a, b, c$  are the sides of a triangle, prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

## Video

<https://youtu.be/DGLLEYuvHVA>

## External Link

<https://aops.com/community/p741368>

## Solution

To deal with the long denominators, let  $x = \sqrt{b} + \sqrt{c} - \sqrt{a}$ , etc. for brevity. Note that  $x > 0$ , and similarly for the others. Then we have that

$$0 < b + c - a = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x+z}{2}\right)^2 - \left(\frac{y+z}{2}\right)^2 = x^2 + xy + xz - yz$$

Thus the problem asks us to prove that

$$\sum_{\text{cyc}} \frac{\sqrt{x^2 + xy + xz - yz}}{x} \leq 3\sqrt{2}$$

for positive real numbers  $x, y, z$  satisfying the extra condition that all radicals are positive.

By Cauchy-Schwarz though, we have that

$$\left(\sum_{\text{cyc}} \frac{\sqrt{x^2 + xy + xz - yz}}{x}\right)^2 \leq 3 \sum_{\text{cyc}} \frac{x^2 + xy + xz - yz}{x^2}$$

and expanding fully, we see it suffices to show

$$\sum_{\text{sym}} xy^2z^3 \leq (xy)^3 + (yz)^3 + (zx)^3 + 3(xyz)^2$$

which is Schur's inequality applied to  $xy, yz, zx$ .