Shortlist 2006 A5

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Twitch Solves ISL

Episode 6

Problem

If a, b, c are the sides of a triangle, prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 3.$$

Video

https://youtu.be/DGLLEYuvHVA

Solution

To deal with the long denominators, let $x = \sqrt{b} + \sqrt{c} - \sqrt{a}$, etc. for brevity. Note that $x > 0$, and similarly for the others. Then we have that

$$0 < b + c - a = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x+z}{2}\right)^2 - \left(\frac{y+z}{2}\right)^2 = x^2 + xy + xz - yz$$

Thus the problem asks us to prove that

$$\sum_{\text{cyc}} \frac{\sqrt{x^2+xy+xz-yz}}{x} \le 3\sqrt{2}$$

for positive real numbers x, y, z satisfying the extra condition that all radicals are positive.

By Cauchy-Schwarz though, we have that

$$\left(\sum_{\text{cyc}} \frac{\sqrt{x^2+xy+xz-yz}}{x}\right)^2 \le 3\sum_{\text{cyc}} \frac{x^2+xy+xz-yz}{x^2}$$

and expanding fully, we see it suffices to show

$$\sum_{\text{sym}} xy^2z^3 \le (xy)^3 + (yz)^3 + (zx)^3 + 3(xyz)^2$$

which is Schur's inequality applied to xy, yz, zx.