Shortlist 2013 A2

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Twitch Solves ISL

Episode 5

Problem

Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$, with *either* $a \neq c$ or $b \neq d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

External Link

https://aops.com/community/p3543341

Solution

WLOG, we denote the set of 2000 real numbers by

$$S = \{0 = x_1 < x_2 < \cdots < x_{2000} = 1\}.$$

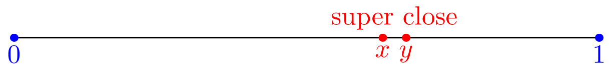
We proceed by contradiction and assume this is not the case.

Claim. There exists $0 \le x < y \le 1$ in S such that

$$x - y < \left(\frac{1}{1 + 10^{-5}}\right)^{\binom{2000}{2} - 1}.$$

Proof. Our contradiction hypothesis assumes that any two differences of elements in S differ by at least a factor of $1 + 10^{-5}$. Since the largest difference is $1 - 0 = 1$ and there are $\binom{2000}{2}$ differences, the conclusion follows. □

We let $0 \le x < y \le 1$ denote the elements with smallest difference. Let $\rho = x - y$. An arithmetic calculation shows that $\rho < 3 \cdot 10^{-9}$.



Note that we have

- If $x > 0.499$ then
$$0 < \frac{y - 0}{x - 0} - 1 < \frac{\rho}{0.499} < 10^{-5}$$
as needed.
- If $y < 0.501$ then similarly
$$0 < \frac{1 - x}{1 - y} - 1 < \frac{\rho}{0.499} < 10^{-5}$$
as needed.