

# Shortlist 2006 G2

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TWITCH SOLVES ISL

Episode 5

## Problem

Let  $ABCD$  be a trapezoid with parallel sides  $AB > CD$ . Points  $K$  and  $L$  lie on the line segments  $AB$  and  $CD$ , respectively, so that  $AK/KB = DL/LC$ . Suppose points  $P$  and  $Q$  on segment  $KL$  obey

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

Prove that  $P, Q, B, C$  are concyclic.

## Video

<https://youtu.be/--mpFb3GbBA>

## Solution

We begin with two simple observations:

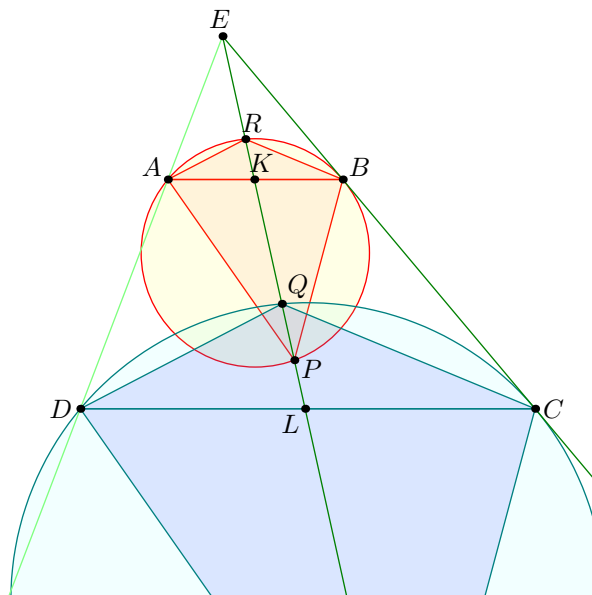
**Claim.** Lines  $BC$ ,  $AD$ ,  $KL$  are concurrent at a point  $E$ .

*Proof.* Follows from  $\overline{AB} \parallel \overline{CD}$  and  $AK/KB = DL/LC$ . □

**Claim.** Line  $BC$  is tangent to  $(ABP)$  and  $(CQD)$ .

*Proof.* Follows from  $\angle APB = \angle BCD$  and  $\angle CQD = \angle ABC$  respectively. □

Now there is a homothety at  $E$  which maps  $\triangle CQD$  to  $\triangle ARB$ , say.



Since  $ER \cdot EP = EB^2$  it follows now that

$$EQ \cdot EP = EB \cdot EC$$

and we're done.