Shortlist 2006 G2

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Twitch Solves ISL

Episode 5

Problem

Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose points P and Q on segment KL obey

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

Prove that P, Q, B, C are concyclic.

Video

https://youtu.be/--mpFb3GbBA

External Link

https://aops.com/community/p875014

Solution

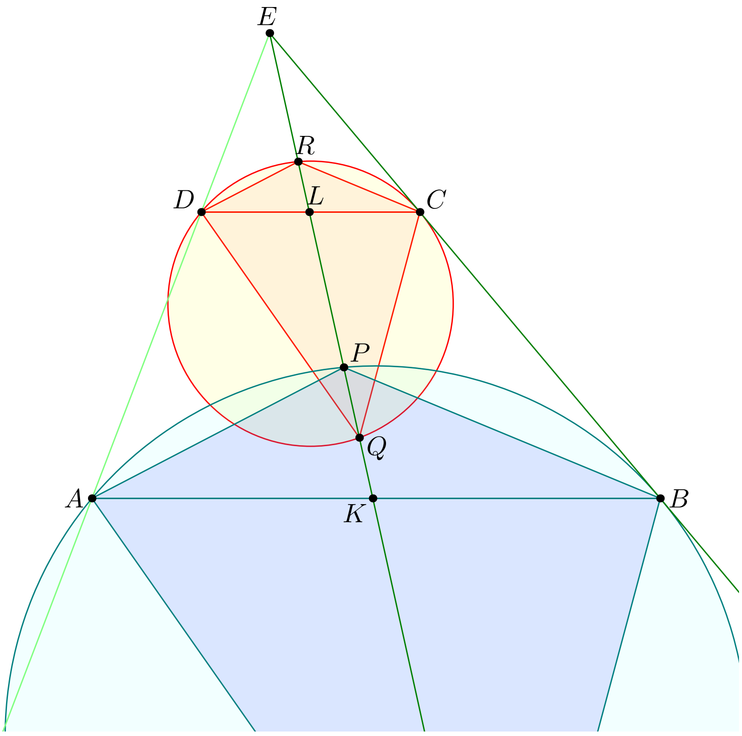
We begin with two simple observations:

Claim. Lines BC, AD, KL are concurrent at a point E.

Proof. Follows from $\overline{AB} \parallel \overline{CD}$ and $AK/KB = DL/LC$. □

Claim. Line BC is tangent to (ABP) and (CQD).

Proof. Follows from $\angle APB = \angle BCD = 180^\circ - \angle EBA$ and $\angle CQD = \angle ABC = \angle ECD$ respectively. □



Let R denote the second intersection of $\overline{EKQ}$ with (CQD). Thus there is a homothety at E which maps $\triangle DRC$ to $\triangle APB$, say.

Since $ER \cdot EP = EB^2$ it follows now that

$$EQ \cdot EP = EB \cdot EC$$

and we're done.