# Shortlist 2004 N1 

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## Twitch Solves ISL

Episode 5

## Problem

Let $\tau(n)$ denote the number of positive divisors of the positive integer $n$. Prove that there exist infinitely many positive integers $a$ such that the equation $\tau(a n)=n$ does not have a positive integer solution $n$.

## Video

https://youtu.be/yiBPjy49z80

## External Link

https://aops.com/community/p212395

## Solution

The construction is quite specific: we claim that

$$
a=p^{p-1}
$$

works for sufficiently large primes $p>100$.
Indeed, assume for contradiction $\tau(a n)=n$ for some integer $n$. Then:

- If $p \nmid n$, then $\nu_{p}(a n)=p-1$ so $p \mid \tau(a n)$, a contradiction. Thus we must have $p \mid n$.
- Now let $n=p^{e} t$ for some $e>0$. Then

$$
\underbrace{(e+p) \cdot \tau(t)}_{=\tau(a n)}=\underbrace{p^{e} \cdot t}_{=n} .
$$

Now we split into two further cases.

- If $e>1$ then $p^{e}>p+e$ and $t \geq \tau(t)$, give a size contradiction together.
- If $e=1$ we get $\tau(t)=\frac{p}{p+1} t \geq \frac{101}{102} t$.

However, it's known (and easy to show) that $\tau(t)<2 \sqrt{t}$ for all integers $t$. Hence we get $\frac{101}{102} t<2 \sqrt{t}$, so $t<5$.
But the equation $\tau(t)=\frac{p}{p+1} t$ gives no solution for $t=1,2, p=2$ for $t=3$, and $p=3$ for $t=4$. So when $p>100$ the situation $t<5$ cannot occur either.

Having checked everything, the proof is complete.

