Shortlist 2004 N1

Evan Chen

Twitch Solves ISL

Episode 5

Problem

Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does *not* have a positive integer solution n.

Video

https://youtu.be/yiBPjy49z80

External Link

https://aops.com/community/p212395

Solution

The construction is quite specific: we claim that

$$a = p^{p-1}$$

works for sufficiently large primes $p > 100$.

Indeed, assume for contradiction $\tau(an) = n$ for some integer n. Then:

- If $p \nmid n$, then $\nu_p(an) = p - 1$ so $p \mid \tau(an)$, a contradiction. Thus we must have $p \mid n$.
- Now let $n = p^e t$ for some $e > 0$. Then

$$\underbrace{(e+p) \cdot \tau(t)}_{=\tau(an)} = \underbrace{p^e \cdot t}_{=n}.$$

  Now we split into two further cases.

  - If $e > 1$ then $p^e > p + e$ and $t \ge \tau(t)$, give a size contradiction together.
  - If $e = 1$ we get $\tau(t) = \frac{p}{p+1} t \ge \frac{101}{102} t$.

    However, it's known (and easy to show) that $\tau(t) < 2\sqrt{t}$ for all integers t. Hence we get $\frac{101}{102} t < 2\sqrt{t}$, so $t < 5$.

    But the equation $\tau(t) = \frac{p}{p+1} t$ gives no solution for $t = 1, 2$, $p = 2$ for $t = 3$, and $p = 3$ for $t = 4$. So when $p > 100$ the situation $t < 5$ cannot occur either.

Having checked everything, the proof is complete.