Shortlist 2007 G8

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Twitch Solves ISL

Episode 4

Problem

Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD. Suppose that ω is tangent to the incircles of $\triangle ADP$ and $\triangle BPC$ at points K and L, respectively. Let $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{AK} \cap \overline{BL}$. Prove that line EF passes through the center of ω.

Video

https://youtu.be/syyRZfWhoyI

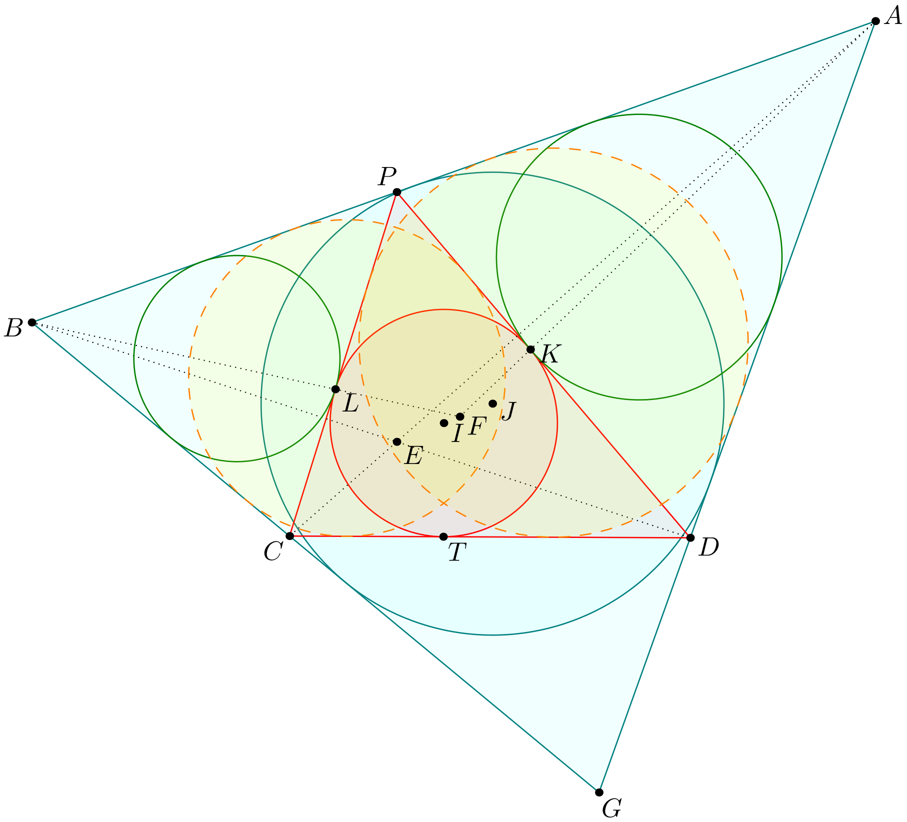
Solution

We begin with the following claim.

Claim. The quadrilaterals $APCD$ and $PBCD$ both have an incircle.

Proof. Follows by Pitot theorem since $AP - AD = KP - KD = CP - CD$ (see SL 2017 G7 for another example of this lemma). $\square$

Let I be the center of ω (the incenter of $\triangle CPD$). Moreover, if we let $G = \overline{AD} \cap \overline{BC}$, we will denote by J the incenter of $\triangle GBA$, and by (J) the circumcircle.



Claim. The point F is the insimilicenter of (I) and (J).

Proof. By Monge d'Alembert theorem on (I), (J), and the incircle of $\triangle APB$, we get that $\overline{BL}$ passes through said insimilicenter. By symmetry, so does $\overline{CK}$, as needed. $\square$

Claim. The point E is the exsimilicenter of (I) and (J).

Proof. By Monge d'Alembert theorem on (I), (J), and the incircle of $BPDC$, we get that $\overline{BD}$ passes through said exsimilicenter. By symmetry, so does $\overline{CA}$, as needed. $\square$

The previous two claims imply all four points E, F, I, J are collinear.