Shortlist 2006 A2

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Problem

The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for } n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

Solution

In fact, it suffices to take the two consecutive equations

$$0 = \frac{a_n}{1} + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + \cdots + \frac{a_0}{n+1}$$
$$0 = \frac{a_{n+1}}{1} + \frac{a_n}{2} + \frac{a_{n-1}}{3} + \frac{a_{n-2}}{4} + \cdots + \frac{a_0}{n+2}$$

and subtract $\frac{n+1}{n+2}$ times the first equation from the second (eliminating the a_0) to obtain a new recursion

$$\begin{aligned} a_{n+1} = &\left(\frac{n+1}{n+2} \cdot \frac{1}{1} - \frac{1}{2}\right) a_n + \left(\frac{n+1}{n+2} \cdot \frac{1}{2} - \frac{1}{3}\right) a_{n-1} \\ &+ \left(\frac{n+1}{n+12} \cdot \frac{1}{3} - \frac{1}{4}\right) a_{n-2} + \cdots + \left(\frac{n+1}{n+2} \cdot \frac{1}{n} - \frac{1}{n+1}\right) a_1. \end{aligned}$$

This recursion has the property that all the coefficients in parentheses are positive. Since $a_1 = 1/2 > 0$, this implies the result by induction.