Shortlist 2007 N4

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Episode 3

Problem

Let $n \geq 2$ be an integer. Prove that $\nu_2(N) = 3n$ where

$$N = \binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}.$$

Video

https://youtu.be/L-gy9W0sY-8

Solution

First, for the "factor out all the unnecessary stuff" part,

$$\begin{aligned}
\frac{\binom{2^{k+1}}{2^k}}{\binom{2^k}{2^{k-1}}} - 1 &= \frac{(2^k+1)(2^k+2)\dots(2^k+2^k)}{\left[(2^{k-1}+1)(2^{k-1}+2)\dots(2^{k-1}+2^{k-1})\right]^2} - 1 \\
&= \frac{2^{2^k}\cdot\left[(2^k+1)(2^k+3)(2^k+5)\dots(2^k+(2^k-1))\right]}{(2^{k-1}+1)(2^{k-1}+2)\dots(2^{k-1}+2^{k-1})} - 1 \\
&= \frac{2^{2^k}\cdot\left[(2^k+1)(2^k+3)(2^k+5)\dots(2^k+(2^k-1))\right]}{2^{2^k}\cdot(2^k-1)!!} - 1 \\
&= \frac{\left[(2^k+1)(2^k+3)(2^k+5)\dots(2^k+(2^k-1))\right]-(2^k-1)!!}{(2^k-1)!!}.
\end{aligned}$$

A calculation shows $\nu_2\left[\binom{2^k}{2^{k-1}}\right]=1$. Therefore it is sufficient to show that:

Claim. We have

$$(2^k+1)(2^k+3)(2^k+5)\dots(2^k+(2^k-1)) \equiv 2^{3k-1}+(2^k-1)!! \pmod{2^{3k}}.$$

Proof. Let $Y=2^k$, so we are working modulo Y^3. Expanding $(Y+1)(Y+3)(Y+5)\dots(Y+(2^k-1))$ modulo Y^3 gives

$$\begin{aligned}
&(2^k-1)!! + (2^k-1)!!\cdot Y\cdot\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2^k-1}\right) \\
&\qquad + (2^k-1)!!\cdot Y^2\cdot\left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^k-3)(2^k-1)}\right) \\
&= (2^k-1)!!\cdot\Bigg[1+Y\cdot\left(\frac{2^k}{1\cdot(2^k-1)}+\frac{2^k}{3\cdot(2^k-3)}+\frac{2^k}{5\cdot(2^k-5)}+\dots+\frac{2^k}{(2^{k-1}-1)(2^{k-1}+1)}\right) \\
&\qquad + Y^2\left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^k-3)(2^k-1)}\right)\Bigg] \\
&= (2^k-1)!!\cdot\Bigg[1+Y^2\cdot\Bigg(\frac{1}{1\cdot(2^k-1)}+\frac{1}{3\cdot(2^k-3)}+\frac{1}{5\cdot(2^k-5)}+\dots+\frac{1}{(2^{k-1}-1)(2^{k-1}+1)} \\
&\qquad + \frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^k-3)(2^k-1)}\Bigg)\Bigg]
\end{aligned}$$

Let S denote the portion inside the round parentheses above. It can be taken modulo

2^k without penalty, so we compute

$$\begin{aligned} S &= -\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2^{k-1}-1)^2}\right) \\ &\quad + \left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^k-3)(2^k-1)}\right) \\ &= \left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^{k-1}-1)^2}\right) \\ &\quad + \left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^k-3)(2^k-1)}\right) \\ &\quad - 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^{k-1}-1)^2}\right) \\ &\equiv \left(\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2^{k-1}-1}\right)^2 \\ &\quad + \left(\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2^{k-1}-1}\right)\left(\frac{1}{2^{k-1}+1}+\dots+\frac{1}{2^k-1}\right) \\ &\quad - \left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^k-1)^2}\right) \pmod{2^k}. \end{aligned}$$

Let $T=\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2^{k-1}-1}$. By a pairing argument again, it follows $2^{k-1}\mid T$, so actually $T^2\equiv 0 \pmod{2^k}$. In conclusion, we have

$$\begin{aligned} S &\equiv -\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^k-1)^2}\right) \\ &\equiv -\left(1^2+3^2+5^2+\dots+(2^k-1)^2\right). \end{aligned}$$

Using the identity $1^2+3^2+\dots+(2m-1)^2=\frac{m(m^2-1)}{6}$, we conclude $S\equiv 2^{k-1} \pmod{2^k}$. Putting S back into the main calculation finally gives

$$(Y+1)(Y+3)(Y+5)\dots(Y+(2^k-1)) = (2^{k-1})!!\left(1+2^{k-1}\right) \pmod{2^{3k}}$$

and the proof ends here. □