# Shortlist 2007 N4


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Twitch Solves ISL

Episode 3

## Problem

Let $n \geq 2$ be an integer and let

$$N = \binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}.$$

Compute $\nu_2(N)$ in terms of $n$.

## Video

https://youtu.be/L-gy9W0sY-8

## External Link

https://aops.com/community/p1187214

## Solution

We will prove $\nu_2(N) = 3n$.

First, for the "factor out all the unnecessary stuff" part,

$$\begin{aligned}
\frac{N}{\binom{2^n}{2^{n-1}}} = \frac{\binom{2^{n+1}}{2^n}}{\binom{2^n}{2^{n-1}}} - 1 &= \frac{(2^n+1)(2^n+2)\dots(2^n+2^n)}{[(2^{n-1}+1)(2^{n-1}+2)\dots(2^{n-1}+2^{n-1})]^2} - 1 \\
&= \frac{2^{2^n}\cdot[(2^n+1)(2^n+3)(2^n+5)\dots(2^n+(2^n-1))]}{(2^{n-1}+1)(2^{n-1}+2)\dots(2^{n-1}+2^{n-1})} - 1 \\
&= \frac{2^{2^n}\cdot[(2^n+1)(2^n+3)(2^n+5)\dots(2^n+(2^n-1))]}{2^{2^n}\cdot(2^n-1)!!} - 1 \\
&= \frac{[(2^n+1)(2^n+3)(2^n+5)\dots(2^n+(2^n-1))] - (2^n-1)!!}{(2^n-1)!!}.
\end{aligned}$$

Legendre's formula (or just direct calculation) shows $\nu_2\left[\binom{2^n}{2^{n-1}}\right] = 1$. Therefore it is sufficient to show that:

**Claim.** We have

$$(2^n+1)(2^n+3)(2^n+5)\dots(2^n+(2^n-1)) \equiv 2^{3n-1} + (2^n-1)!! \pmod{2^{3n}}.$$

*Proof.* Let $Y = 2^n$, so we are working modulo $Y^3$. Expanding $(Y+1)(Y+3)(Y+5)\dots(Y+(2^n-1))$ modulo $Y^3$ gives

$$\begin{aligned}
&(2^n-1)!! + (2^n-1)!!\cdot Y\cdot\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2^n-1}\right) \\
&\qquad + (2^n-1)!!\cdot Y^2\cdot\left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^n-3)(2^n-1)}\right) \\
&= (2^n-1)!!\cdot\left[1 + Y\cdot\left(\frac{2^n}{1\cdot(2^n-1)}+\frac{2^n}{3\cdot(2^n-3)}+\frac{2^n}{5\cdot(2^n-5)}+\dots+\frac{2^n}{(2^{n-1}-1)(2^{n-1}+1)}\right)\right. \\
&\qquad \left. + Y^2\left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^n-3)(2^n-1)}\right)\right] \\
&= (2^n-1)!!\cdot\left[1 + Y^2\cdot\left(\frac{1}{1\cdot(2^n-1)}+\frac{1}{3\cdot(2^n-3)}+\frac{1}{5\cdot(2^n-5)}+\dots+\frac{1}{(2^{n-1}-1)(2^{n-1}+1)}\right.\right. \\
&\qquad \left.\left. + \frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^n-3)(2^n-1)}\right)\right]
\end{aligned}$$

Let $S$ denote the portion inside the round parentheses above. It can be taken modulo $2^n$ without penalty, so we compute

$$\begin{aligned}
S &= -\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2^{n-1}-1)^2}\right) \\
&\qquad + \left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^n-3)(2^n-1)}\right) \\
&= \left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^{n-1}-1)^2}\right) \\
&\qquad + \left(\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}+\dots+\frac{1}{(2^n-3)(2^n-1)}\right)
\end{aligned}$$

$$\begin{aligned}
&\quad - 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^{n-1}-1)^2}\right) \\
&\equiv \left(\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2^{n-1}-1}\right)^2 \\
&\quad + \left(\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2^{n-1}-1}\right)\left(\frac{1}{2^{n-1}+1}+\dots+\frac{1}{2^n-1}\right) \\
&\quad - \left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^n-1)^2}\right) \pmod{2^n}.
\end{aligned}$$

Let $T = \frac{1}{1} + \frac{1}{3} + \dots + \frac{1}{2^{n-1}-1}$. By a pairing argument again, it follows $2^{n-1} \mid T$, so actually $T^2 \equiv 0 \pmod{2^n}$. In conclusion, we have

$$\begin{aligned}
S &\equiv -\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots+\frac{1}{(2^n-1)^2}\right) \\
&\equiv -\left(1^2+3^2+5^2+\dots+(2^n-1)^2\right).
\end{aligned}$$

Using the identity $1^2+3^2+\dots+(2m-1)^2 = \frac{m(m^2-1)}{6}$, we conclude $S \equiv 2^{n-1} \pmod{2^n}$. Putting $S$ back into the main calculation finally gives

$$(Y+1)(Y+3)(Y+5)\dots(Y+(2^n-1)) = (2^{n-1})!!\left(1+2^{n-1}\right) \pmod{2^{3n}}$$

and the proof ends here. $\square$