Shortlist 2005 G3

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Twitch Solves ISL

Episode 3

Problem

Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that the measure of angle $\angle KCL$ is independent of the line g.

External Link

https://aops.com/community/p512720

Solution

Here is a synthetic solution. Let $\alpha = \angle BAK = \angle KAX$ and $\beta = \angle XAL = \angle LAD$. Since $\angle ABK = \angle ADL = 180^\circ - (\alpha + \beta)$, it follows that

$$\triangle ABK \sim \triangle LDA \implies \frac{AB}{LD} = \frac{BK}{DA} \implies \frac{CD}{LD} = \frac{BK}{BC} \implies \triangle LDC \sim \triangle CBK.$$

Then, $\angle BCK + \angle LCD = 180^\circ - (\alpha + \beta)$, which is fixed. On the other hand, $\angle BCD = 2(\alpha + \beta)$ is fixed too. Therefore $\angle KCL$ is fixed as desired.

Moving points should also work; see `https://aops.com/community/c6h87927p8743952` for example.