Shortlist 2009 C1

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Episode 2

Problem

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

External Link

Solution

Interpret gold as 1 and black as 0. The answer to (a) is obviously yes, since viewing the number as a binary string of length 2009 we find that it decreases at each step.

As for (b), we claim that

Claim. The second player must win *regardless* of what moves occur.

Proof. Consider the 50th, 100th, 150th card, and so on, up to the 2000th card *from the right* (which is the 10th card from the left). Denote the set of these 40 cards by C.

The number of 1's in C must change by exactly 1 every turn, and it is initially 40. The game could only end when all cards in C are zero. So this can only occur on the first player's turn. $\square$

Remark. It's important that the cards are counted from the right rather than from the left in (b). If one counts from the left, then the 2000th card from the left being gold does not imply a player has a legal move.