Shortlist 2007 G2

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Twitch Solves ISL

Episode 2

Problem

Denote by M midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point X on a smaller arc $\widehat{MA}$ of circumcircle of triangle $\triangle ABM$. Denote by T point inside of angle BMA such that $\angle TMX = 90^\circ$ and $TX = BX$.

Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

Video

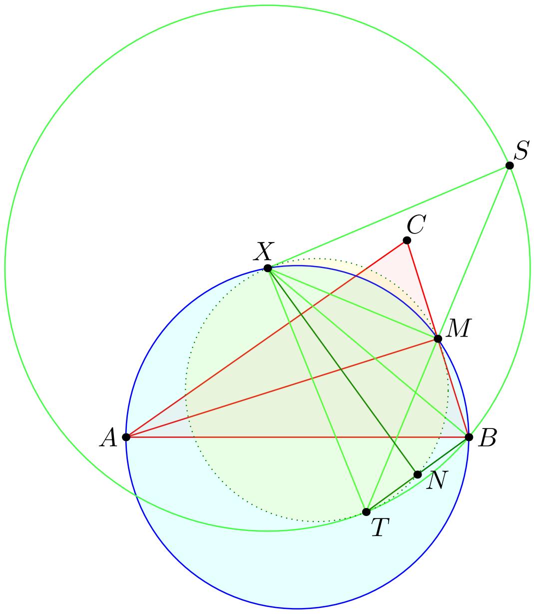
https://youtu.be/vA1iwW41Jmw

External Link

https://aops.com/community/p1143932

Solution

Construct parallelogram $CTBS$ whose diagonals meet at M. Also, let N be the midpoint of $\overline{BT}$.



We first eliminate C from the diagram by noting that

$$\angle CTM = \angle MSB = \angle TSB = \frac{1}{2}\angle TXB = \angle NXB.$$

Also, noting that $XMTN$ are cyclic (as $\angle XMT = \angle XNT = 90^\circ$), we have

$$\angle MTB = \angle MTN = \angle NXM.$$

Thus $\angle MTB - \angle CTM = \angle NXM - \angle NXB = \angle MXB$ which is fixed.