Shortlist 2012 G2

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Twitch Solves ISL

Episode 1

Problem

Let $ABCD$ be a cyclic quadrilateral and let $E = \overline{AC} \cap \overline{BD}$. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that the points D, H, F, G are concyclic.

External Link

https://aops.com/community/p3160578

Solution

We present two approaches.

Complex numbers. Let A, B, C, D be the unit circle in the usual manner. In what follows, the notation $z_1 \asymp z_2$ means that z_1/z_2 is real.

Note that $e = \frac{ac(b+d)-bd(a+c)}{ac-bd}$. Also, $g = c + d - e$ and $h = a + d - ad\overline{e}$. We calculate

$$\begin{aligned}
\frac{g-h}{f-h} \asymp \frac{g-h}{\frac{(f-d)^2}{f-e}} &= \frac{(c+d-e)-h}{\frac{(a-d)^2}{f-e}} = (f-e) \cdot \frac{c-a-e+ad\overline{e}}{(a-d)^2} \\
&= (f-e) \cdot \frac{c-a-\frac{ac(b+d)-bd(a+c)}{ac-bd} + ad \cdot \frac{a+c-b-d}{ac-bd}}{(a-d)^2} \\
&= \frac{f-e}{(a-d)^2} \cdot \frac{(c-a)(ac-bd) - [ac(b+d)-bd(a+c)] + ad(a+c-b-d)}{ac-bd} \\
&= \frac{f-e}{(a-d)^2} \cdot \frac{ac^2 - a^2c - abc + abd + a^2d - ad^2}{ac-bd} \\
&= \frac{f-e}{(a-d)^2} \cdot \frac{a(c-d)(c+d-a-b)}{ac-bd}.
\end{aligned}$$

To handle E and F, we cop out using Brokard's Theorem: taking the final intersection $\frac{ab(c+d)-cd(a+b)}{ab-cd}$, we have

$$f - e \sim i \cdot \frac{ab(c+d)-cd(a+b)}{ab-cd}.$$

In summary,

$$\frac{g-h}{f-h} \asymp \frac{ab(c+d)-cd(a+b)}{(a-d)^2(ab-cd)} \cdot \frac{a(c-d)(c+d-a-b)}{ac-bd}$$

On the other hand,

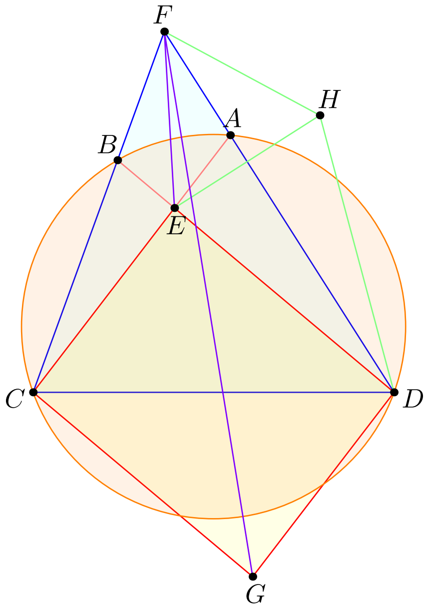
$$\frac{g-d}{f-d} \asymp \frac{c-a}{a-d}.$$

Thus, we have

$$\frac{g-h}{f-h} \div \frac{g-d}{f-d} \asymp \frac{ab(c+d)-cd(a+b)}{(a-d)(c-a)(ab-cd)} \cdot \frac{a(c-d)(c+d-a-b)}{ac-bd}$$

This expression is easily seen to be self-conjugate, completing the proof.

Synthetic approach. By the so-called isogonality lemma (see Geometry Revisited 1.9§3) on $\triangle FCD$, we find that lines FE and FG are isogonal with respect to $\angle CFD$.



In particular, since

$$\measuredangle FBE = \measuredangle CBE = \measuredangle CAD = \measuredangle ADG = \measuredangle FDG$$

the isogonality implies that

$$\triangle FBE \overset{-}{\sim} \triangle FDG.$$

Thus

$$\measuredangle FGD = \measuredangle BEF = \measuredangle DEF = \measuredangle FHD$$

which finishes the problem.