Shortlist 2012 C2

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Twitch Solves ISL

Episode 1

Problem

Let $n \geq 1$ be an integer. What is the maximum number of possible disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the pairs are different integers not exceeding n?

Solution

The answer is $N = N(n) \le \left\lfloor \frac{2n-1}{5} \right\rfloor$. The proof is nearly identical to that of IMO SL 2009 C2.

To prove the bound, suppose the pairs are $(a_1, b_1), \dots, (a_N, b_N)$. Then on the one hand

$$\sum_1^N (a_i + b_i) \le \underbrace{n + \cdots + (n - N + 1)}_{N \text{ largest sums } < n} = \frac{1}{2} N \left(2n - N + 1\right).$$

On the other hand,

$$\sum_1^N (a_i + b_i) \ge \underbrace{1 + 2 + \cdots + 2N}_{2N \text{ smallest possible entries}} = N \cdot (2N + 1)$$

Putting these two bounds together and solving works.

For the construction, it suffices to exhibit the construction when $n \equiv 1 \pmod 5$ and $n \equiv 3 \pmod 5$, since for all other n we have $N(n) = N(n-1)$. We just give examples which generalize readily.

- When $n = 18$, we use the following:

1	2	3	4	5	6	7
+	+	+	+	+	+	+
14	12	10	8	13	11	9

  The general construction for $n = 5k + 3$ is analogous, using $(k + 1) + k = 2k + 1$ pairs.

- When $n = 21$, we use the following:

1	2	3	4	5	6	7	8
+	+	+	+	+	+	+	+
14	12	10	17	15	13	11	9

  The general construction for $n = 5k + 1$ is analogous, using $(k - 1) + (k + 1) = 2k$ pairs.