# Shortlist 2012 C2

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### Twitch Solves ISL

Episode 1

## Problem

Let $n \geq 1$ be an integer. What is the maximum number of possible disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the pairs are different integers not exceeding $n$?

## External Link

## Solution

The answer is $N = N(n) \le \left\lfloor \frac{2n-1}{5} \right\rfloor$. The proof is nearly identical to that of IMO SL 2009 C2.

To prove the bound, suppose the pairs are $(a_1, b_1)$, ..., $(a_N, b_N)$. Then on the one hand

$$\sum_1^N (a_i + b_i) \le \underbrace{n + \cdots + (n - N + 1)}_{N \text{ largest sums } <n} = \frac{1}{2} N (2n - N + 1).$$

On the other hand,

$$\sum_1^N (a_i + b_i) \ge \underbrace{1 + 2 + \cdots + 2N}_{2N \text{ smallest possible entries}} = N \cdot (2N + 1)$$

Putting these two bounds together and solving works.

For the construction, it suffices to exhibit the construction when $n \equiv 1 \pmod 5$ and $n \equiv 3 \pmod 5$, since for all other $n$ we have $N(n) = N(n-1)$. We just give examples which generalize readily.

- When $n = 18$, we use the following:

$$
\begin{array}{cccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
+ & + & + & + & + & + & + \\
14 & 12 & 10 & 8 & 13 & 11 & 9
\end{array}
$$

  The general construction for $n = 5k + 3$ is analogous, using $(k + 1) + k = 2k + 1$ pairs.

- When $n = 21$, we use the following:

$$
\begin{array}{ccc|ccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
+ & + & + & + & + & + & + & + \\
14 & 12 & 10 & 17 & 15 & 13 & 11 & 9
\end{array}
$$

  The general construction for $n = 5k + 1$ is analogous, using $(k - 1) + (k + 1) = 2k$ pairs.