Shortlist 2008 N2

Evan Chen

Twitch Solves ISL

Episode 1

Problem

Let $a_1, a_2, \dots, a_n$ be distinct positive integers for $n \ge 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of $3a_1, 3a_2, \dots, 3a_n$.

External Link

https://aops.com/community/p1555929

Solution

Assume for contradiction this is not true. By sorting, we assume $a_1 > a_2 > \cdots > a_n$. We will suppose that $\gcd(a_1, \ldots, a_n) = 1$, else divide through.

Claim. We have $a_i \equiv -a_1 \pmod 3$ for all $i > 1$.

Proof. If not then $a_1 + a_i$ divides some $3a_k$. But $a_1 + a_i \perp 3$, so $a_1 + a_i$ divides some a_k. This is impossible for size reasons. $\square$

Since we assumed the sequence was relatively prime, we have now that

$$a_2 \equiv a_3 \equiv \cdots \equiv a_n \equiv -a_1 \not\equiv 0 \pmod 3.$$

We now work with only a_1, a_2, a_3. The proof proceeds in three steps.

- Again, $a_2 + a_3$ divides $3a_k$ for some k, but since $a_2 + a_3 \not\equiv 0 \pmod 3$, we find that $a_2 + a_3 \mid a_k$ for some k. Hence in fact $\boxed{a_2 + a_3 \mid a_1}$.
- Meanwhile, $a_1 + a_2$ divides either $3a_1$ or $3a_2$ for size reasons. But actually, these two statements are equivalent, since $a_1 + a_2$ divides their sum $3a_1 + 3a_2$. Now from $2a_2 < a_1 + a_2 \mid 3a_2$, we must have $\boxed{a_1 = 2a_2}$.
- Now $a_2 < a_2 + a_3 \mid a_1 = 2a_2$, so we must have $a_2 + a_3 = 2a_2$ but then $a_2 = a_3$, contradiction.

This concludes the proof.

Remark. Note that we can find pretty early on that the $n = 3$ case of the problem is sufficient: the three largest numbers satisfy the condition anyways, so there is no need any longer to look at a_4, a_5, $\ldots$.