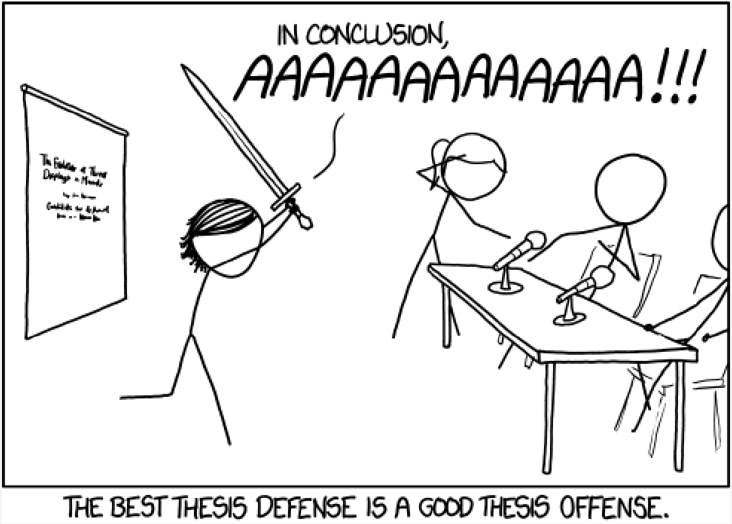
Semi-Lie Arithmetic Fundamental Lemma for full spherical Hecke Algebra



Semi-Lie Arithmetic Fundamental Lemma for full spherical Hecke Algebra

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Notation

- E/F an unramified quadratic extension of p-adic fields, $p > 2$
- q residue characteristic of F.
- $\mathbb{V}_n^+$ split E/F-Hermitian space of dimension n
- $\mathbb{V}_n^-$ non-split E/F-Hermitian space of dimension n
- Symmetric space:

$$S_n(F) := \{\gamma \in \mathrm{GL}_n(E) \mid \gamma\bar{\gamma} = 1_n\}.$$

Hecke algebras:

- $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) := \mathbb{Q}[K \backslash \mathrm{U}(\mathbb{V}_n^+)/K]$ is the Hecke algebra of compactly supported K-bi-invariant functions, where $K := \mathrm{U}(\mathbb{V}_n^+) \cap \mathrm{GL}_n(O_E)$ is hyperspecial maximal compact subgroup.
- $\mathcal{H}(\mathrm{GL}_n(E)) := \mathbb{Q}[K' \backslash \mathrm{GL}_n(E)/K']$, $K' := \mathrm{GL}_n(O_E)$.
- $\mathcal{H}(S_n(F)) := \mathcal{C}_c^\infty(S_n(F))^{K'}$, is an $\mathcal{H}(\mathrm{GL}_n(E))$-module.

Overview

Conjecture	Lemma	Generalization
GGP	Fundamental lemma (Jacquet-Rallis 2011)	Leslie (2023)
Arith GGP	AFL for $\mathrm{U}(n) \times \mathrm{U}(n+1)$ (Zhang 2012)	Li-Rapoport-Zhang (2024)
	$\iff$ AFL for $\mathrm{U}(n) \times \mathrm{U}(n)$ (Liu 2021)	🤔

Table of contents for talk

- Discuss first, second, third column, each from top to bottom.
- Things in first column and first row will be cursory historical overview, not defined or made precise at all.

GGP/AGGP extremely rough statements

Global GGP conjecture, extremely roughly

Let $\mathrm{H} \subset \mathrm{G}$ be "spherical" pair of reductive groups, π a tempered cuspidal automorphic representation of G. Then the following are equivalent:

1. $\mathscr{P}(\phi) := \int_{\mathrm{H}(k)\backslash\mathrm{H}(\mathbb{A}_k)} \phi(h)\,\mathrm{d}h$ is not identically zero for $\phi \in \pi$.
2. $\mathrm{Hom}_{\mathrm{H}(\mathbb{A}_k)}(\pi, \mathbb{C}) \neq 0$ and $L(\frac{1}{2}, \pi, R) \neq 0$ for certain R.

Arithmetic GGP conjecture, extremely roughly

Let π be a tempered cuspidal automorphic representation of $\mathrm{G}(\mathbb{A}_k)$ appearing in the cohomology $H^\bullet(\mathrm{Sh}(\mathrm{G}))$. The following are equivalent:

1. A certain height pairing $\mathscr{P}_{\mathrm{Sh}(\mathrm{H})} \colon \mathrm{Ch}^{n-1}(\mathrm{Sh}(\mathrm{G}))_0 \to \mathbb{C}$ does not vanish on π_f-isotypic component.
2. $\mathrm{Hom}_{\mathrm{H}(\mathbb{A}_k)}(\pi_f, \mathbb{C}) \neq 0$ and $L'(\frac{1}{2}, \pi, R) \neq 0$ for certain R.

Jacquet-Rallis fundamental lemma

- $(\mathbb{V}_n^+)^\flat$ codimension one subspace in $\mathbb{V}_n^+$.
- $G'^\flat := \mathrm{GL}_{n-1}(E)$, $G' := \mathrm{GL}_n(E)$, $G^\flat := \mathrm{U}((\mathbb{V}_n^+)^\flat)(F)$, $G := \mathrm{U}(\mathbb{V}_n^+)(F)$.
- $K'^\flat$, K', $K^\flat$, K maximal hyperspecial compact subgroups.

Fundamental lemma, roughly

For certain “matching” $\gamma \in G'^\flat \times G'$ and $g \in G^\flat \times G$ we get

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'^\flat \times K'}) = \pm \mathrm{Orb}(g, \mathbf{1}_{K^\flat \times K})$$

Arithmetic fundamental lemma

Analogous statement to Jacquet-Rallis fundamental lemma. For simplicity, stating the inhomogeneous version (the full version is more general).

Definition

For $\gamma \in S_n(F)$, $\phi \in \mathcal{H}(S_n(F))$, and $s \in \mathbb{C}$:

$$\mathrm{Orb}(\gamma, \phi, s) := \int_{h \in \mathrm{GL}_{n-1}(F)} \phi(h^{-1}\gamma h)(-1)^{v(\det h)} \left|\det(h)\right|_F^{-s} \, \mathrm{d}h.$$

Theorem (Inhomogeneous AFL)

For matching regular semisimple elements $g \in \mathrm{U}(\mathbb{V}_n^-)_{rs} \longleftrightarrow \gamma \in S_n(F)_{rs}$*:*

$$\mathrm{Int}\left((1, g), \mathbf{1}_{K'^\flat} \otimes \mathbf{1}_{K'}\right) \log q = \pm \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Orb}(\gamma, \mathbf{1}_K, s).$$

Matching definition

Definition

$\begin{pmatrix} A & \mathbf{u} \\ \mathbf{v}^\top & d \end{pmatrix} \in \mathrm{Mat}_n(E)$ is **regular semisimple** if

$$\Delta := \det\left(\left(\mathbf{v}^\top A^{i+j-2}\mathbf{u}\right)_{i,j=1}^{n-1}\right) \neq 0$$

We say $\gamma \in S_n(F)_{\text{rs}}$ **matches** the element $g \in \mathrm{U}(\mathbb{V}_n^\pm)_{\text{rs}}$ if g is conjugate to γ by an element of $\mathrm{GL}_{n-1}(E)$; write $g \in \mathrm{U}(\mathbb{V}_n^\pm)_{\text{rs}} \longleftrightarrow \gamma \in S_n(F)_{\text{rs}}$.

Matching criteria

$$[S_n(F)]_{\text{rs}} \xrightarrow{\sim} [\mathrm{U}(\mathbb{V}_n^+)]_{\text{rs}} \amalg [\mathrm{U}(\mathbb{V}_n^-)]_{\text{rs}}.$$

Matches in $\mathrm{U}(\mathbb{V}_n^+)$ if $v(\Delta)$ is even, and $\mathrm{U}(\mathbb{V}_n^-)$ otherwise.

Rapoport-Zink space

- Let $\breve{F}$ denote the completion of a maximal unramified extension of F, and let $\mathbb{F}$ denote the residue field of $O_{\breve{F}}$.
- Suppose S is a $\operatorname{Spf} O_{\breve{F}}$-scheme.

Definition

Consider triples (X, ι, λ) where:

- X is a formal ϖ-divisible n-dimensional O_F-module over S whose relative height is $2n$.
- $\iota\colon O_E \to \operatorname{End}(X)$ is an action of O_E such that the induced action of O_F on $\operatorname{Lie} X$ is via the structure morphism $O_F \to \mathcal{O}_S$, satisfying the Kottwitz condition of signature $(n-1, 1)$,
- $\lambda\colon X \to X^\vee$ is a principal O_F-relative polarization whose Rosati involution induces $a \mapsto \bar{a}$ on O_F.

Rapoport-Zink space, continued

Choose a *supersingular* triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ called the *framing object*.

Definition

For each $n \geq 1$, we let $\mathcal{N}_n$ denote the functor over $\operatorname{Spf} O_{\breve{F}}$ defined as follows. Let S be an $\operatorname{Spf} O_{\breve{F}}$ scheme, $\overline{S} := S \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \mathbb{F}$. We let $\mathcal{N}_n(S)$ be the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where (X, ι, λ) is one of the triples as we described, and

$$\rho \colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\operatorname{Spec} \mathbb{F}} \overline{S}$$

is a *framing*, meaning it is a height zero O_F-linear quasi-isogeny and satisfies $\rho^*((\lambda_{\mathbb{X}_n})_{\overline{S}}) = \lambda_{\overline{S}}$.

Then $\mathcal{N}_n$ is formally smooth over $O_{\breve{F}}$ of relative dimension $n-1$ and is acted on by $\mathrm{U}(\mathbb{V}_n^-)$.

Intersection number

- $\mathcal{N}_{m,n} \coloneqq \mathcal{N}_m \times \mathcal{N}_n$.
- Let $\Delta\colon \mathcal{N}_{n-1} \to \mathcal{N}_{n-1,n}$ be the graph morphism of $\delta\colon \mathcal{N}_{n-1} \to \mathcal{N}_n$, with image $\Delta_{\mathcal{N}n-1}$.
- Realize $\mathbb{V}_n^- = \mathrm{Hom}^\circ_{O_E}(\mathbb{E}, \mathbb{X}_n)$.

Definition

$$\begin{aligned}\mathrm{Int}((1,g), \mathbf{1}_{K'^\flat} \otimes \mathbf{1}_{K'}) &\coloneqq \left(\Delta_{\mathcal{N}_{n-1}}, (1,g)\cdot \Delta_{\mathcal{N}_{n-1}}\right)_{\mathcal{N}_{n-1,n}} \\ &\coloneqq \chi_{\mathcal{N}_{n-1,n}}\left(\mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n-1,n}}}{\otimes}} \mathcal{O}_{(1,g)\cdot\Delta_{\mathcal{N}_{n-1}}}\right).\end{aligned}$$

($\otimes^{\mathbf{L}}$ is derived tensor product, χ is Euler-Poincare characteristic)

Semi-Lie version of AFL by Yifeng Liu (2021)

Definition

For $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in S_n(F) \times F^n \times (F^n)^\vee$, $\phi \in \mathcal{H}(S_n(F))$, and $s \in \mathbb{C}$,

$$\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{\mathcal{O}_F^n \times (\mathcal{O}_F^n)^\vee}, s)$$

$$:= \int_{h \in \mathrm{GL}_n(F)} \phi(h^{-1}\gamma h) \mathbf{1}_{\mathcal{O}_F^n \times (\mathcal{O}_F^n)^\vee}(h\mathbf{u}, \mathbf{v}^\top h^{-1}) (-1)^{v(\det h)} \left|\det(h)\right|_F^{-s} \, \mathrm{d}h.$$

Theorem (AFL, semi-Lie version)

If $(g, u) \in (\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{rs} \longleftrightarrow (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times F^n \times (F^n)^\vee)_{rs}$*:*

$$\mathrm{Int}\left((g, u), \mathbf{1}_{K'}\right) \log q = \pm \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_K \otimes \mathbf{1}_{\mathcal{O}_F^n \times (\mathcal{O}_F^n)^\vee}, s).$$

Equivalent to Zhang's AFL; induction uses both!

Matching (analogous)

Matching is defined in the same way via $(g, u) \mapsto \begin{pmatrix} g & u \\ u^* & 0 \end{pmatrix} \in \mathrm{GL}_{n+1}(E)$

and $(\gamma, \mathbf{u}, \mathbf{v}^\top) \mapsto \begin{bmatrix} \gamma & \mathbf{u} \\ \mathbf{v}^\top & 0 \end{bmatrix} \in \mathrm{Mat}_{n+1}(E)$.

Matching criteria in semi-Lie case

$$[S_n(F) \times F^n \times (F^n)^\vee]_{\mathsf{rs}} \xrightarrow{\sim} [\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+]_{\mathsf{rs}} \amalg [\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-]_{\mathsf{rs}}.$$

To see which one, define

$$\Delta := \det \left(\left(\mathbf{v}^\top \gamma^{i+j-2} \mathbf{u} \right)_{i,j=1}^{n} \right) \neq 0.$$

We get $\mathbb{V}_n^+$ if $v(\Delta)$ is even and $\mathbb{V}_n^-$ if $v(\Delta)$ is odd.

Intersection number (briefly)

Definition

Let $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ be the unique triple over $\mathbb{F}$ which has signature $(1, 0)$. Then the formal O_F-module has a unique lifting called its *canonical lifting*, denoted $(\mathcal{E}, \iota_{\mathcal{E}}, \lambda_{\mathcal{E}})$. The Kudla-Rapoport divisor $\mathcal{Z}(u)$ is the locus where the quasi-homomorphism $\mathbb{E} \to \mathbb{X}_n$ lifts to a homomorphism from $\mathcal{E}$ to the universal object over $\mathcal{N}_n$.

Let $\Delta_{\mathcal{Z}(u)}$ be the image of $\mathcal{Z}(u) \to \mathcal{N}_n \to \mathcal{N}_{n,n}$; let $\Gamma_g \subseteq \mathcal{N}_{n,n}$ be the graph of the automorphism of $\mathcal{N}_n$ induced by g.

Definition

$$\mathrm{Int}((g, u), \mathbf{1}_{K'}) := \left(\Delta_{\mathcal{Z}(u)}, \Gamma_g\right)_{\mathcal{N}_{n,n}} := \chi_{\mathcal{N}_{n,n}} \left(\mathcal{O}_{\Delta_{\mathcal{Z}(u)}} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n,n}}}{\otimes}} \mathcal{O}_{\Gamma_g} \right).$$

Base change

Denote by $\operatorname{proj}\colon \mathrm{GL}_n(E) \twoheadrightarrow S_n(F)$ the map $\operatorname{proj}(g) := g\bar{g}^{-1}$.

$$\operatorname{proj}_*^{\eta^{n-1}} : \mathcal{H}(\mathrm{GL}_n(E)) \to \mathcal{H}(S_n(F))$$

$$\operatorname{proj}_*^{\eta^{n-1}}(f')\left(g\bar{g}^{-1}\right) = \int_{\mathrm{GL}_n(F)} f'(gh)\eta^{n-1}(gh)\,\mathrm{d}h$$

And let $\eta(g) = (-1)^{v(\det g)}$ be the nontrivial quadratic character for E/F.

$$\begin{array}{ccccc} & & \mathcal{H}(\mathrm{GL}_n(E)) & \xrightarrow[\sim]{\text{Satake}} & \mathbb{Q}[X_1^\pm, \dots, X_n^\pm]^{\mathrm{Sym}(n)} \\ & \swarrow^{\operatorname{proj}_*^{\eta^{n-1}}} & \downarrow{\scriptstyle \mathrm{BC}} & & \downarrow{\scriptstyle \mathrm{BC}} \\ \mathcal{H}(S_n(F)) & \xrightarrow[\mathrm{BC}_{S_n}^{\eta^{n-1}}]{\sim} & \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) & \xrightarrow[\sim]{\text{Satake}} & \mathbb{Q}[Y_1^\pm, \dots, Y_m^\pm]^{W_m} \end{array}$$

Theorem (Leslie 2023)

BC *factors through* $\operatorname{proj}_*^{\eta^{n-1}}$ *and gives an isomorphism* $\mathrm{BC}_{S_n}^{\eta^{n-1}}$.

Leslie's generalization of Jacquet-Rallis fundamental lemma to the full spherical Hecke algebra

As before $G'^\flat := \mathrm{GL}_{n-1}(E)$, $G' := \mathrm{GL}_n(E)$, $G^\flat := \mathrm{U}((\mathbb{V}_n^+)^\flat)(F)$, $G := \mathrm{U}(\mathbb{V}_n^+)(F)$.

Theorem (Leslie 2023)

Suppose φ' and φ are related by base change; then still for certain "matching" $\gamma \in G'^\flat \times G'$ and $g \in G^\flat \times G$:

$$\mathrm{Orb}(\gamma, \varphi') = \pm \mathrm{Orb}(g, \varphi).$$

The original Jacquet-Rallis fundamental lemma is the special case

$$\varphi' = \mathbf{1}_{K'^\flat \times K'} \in \mathcal{H}(\mathrm{GL}_{n-1}(E)) \otimes \mathcal{H}(\mathrm{GL}_n(E))$$
$$\varphi = \mathbf{1}_{K^\flat \times K} \in \mathcal{H}(\mathrm{U}((\mathbb{V}_n^+)^\flat)) \otimes \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)).$$

AFL conjectured for the full spherical Hecke algebra

Conjecture (Li-Rapoport-Zhang 2024)

Let $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ and $\phi \in \mathcal{H}(S_n(F))$ be related by base change. Then for matching $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}} \longleftrightarrow \gamma \in S_n(F)_{\mathrm{rs}}$:

$$\mathrm{Int}\left((1,g), \mathbf{1}_{K^\flat} \otimes f\right) \log q = \pm \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Orb}(\gamma, \phi, s).$$

AFL is special case $f = \mathbf{1}_K$ and $\phi = \mathbf{1}_{K'}$. Verified for $n = 2$.
(For $n = 3$, I brute-force computed RHS but not LHS.)

Updated intersection number (with new Hecke operator $\mathbb{T}_{\mathbf{1}_{K^\flat} \otimes f}$)

$$\mathrm{Int}((1,g), \mathbf{1}_{K^\flat} \otimes f) := \chi_{\mathcal{N}_{n-1,n}} \left(\mathcal{O}_{\mathbb{T}_{\mathbf{1}_{K^\flat} \otimes f}(\Delta_{\mathcal{N}_{n-1}})} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n-1,n}}}{\otimes}} \mathcal{O}_{(1,g)\cdot\Delta_{\mathcal{N}_{n-1}}} \right).$$

Semi-Lie AFL conjectured for full spherical Hecke algebra

Conjecture (C., October 2024)

Let $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ and $\phi \in \mathcal{H}(S_n(F))$ be related by base change, and

$$(g, u) \in (\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}} \longleftrightarrow (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times \mathcal{O}_F^n \times (\mathcal{O}_F^n)^\vee)_{\mathrm{rs}}.$$

Then

$$\mathrm{Int}\left((g,u), f\right) \log q = \pm \left. \frac{\partial}{\partial s} \right|_{s=0} \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{\mathcal{O}_F^n \times (\mathcal{O}_F^n)^\vee}, s).$$

Updated definition of intersection number

$$\mathrm{Int}((g,u), f) := \chi_{\mathcal{N}_{n,n}} \left(\mathcal{O}_{\mathbb{T}_{\mathbf{1}_K \otimes f}(\Delta_{\mathcal{Z}(u)})} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n,n}}}{\otimes}} \mathcal{O}_{\Gamma_g} \right).$$

Main theorem

Semi-Lie AFL conjecture holds when $n = 2$.

Strategy of proof for $n = 2$

- For $n = 2$, on the orbital side use indicator functions of

$$K'_{S,\leq r} := S_2(F) \cap \varpi^{-r}\,\mathrm{GL}_2(O_E)$$

  as a basis of $\mathcal{H}(S_2(F))$, for $r \geq 0$.
- Straightforward to get base change to $\mathcal{H}(\mathrm{U}(\mathbb{V}_2^+))$ which has a similar basis

$$\mathbf{1}_{K,r} := \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_2(O_E) \cap \mathrm{U}(\mathbb{V}_2^+)} \in \mathcal{H}(\mathrm{U}(\mathbb{V}_2^+)).$$

- On the geometric side, work instead with a Lubin-Tate space $\mathcal{M}_2$ by pulling back with Serre tensor construction.

Example $\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}} \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee})$ when $n = 2$

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix} \right)$$

For $r = 2$, $v(d - a) = 1$, $v(b) = 3$, $v(c) = 0$, $v(e) = 5$:

$$\begin{aligned} \mathrm{Orb}(\dots) = -q^{-5s} & + (3q^3 + q^2 + q + 1)q^{4s} \\ + q^{-4s} & - (2q^3 + q^2 + q + 1)q^{5s} \\ - (q+1)q^{-3s} & + (q^3 + q^2 + q + 1)q^{6s} \\ + (q+1)q^{-2s} & - (q^2 + q + 1)q^{7s} \\ - (q^2 + q + 1)q^{-s} & + (q^2 + q + 1)q^{8s} \\ + (q^2 + q + 1) & - (q+1)q^{9s} \\ - (q^3 + q^2 + q + 1)q^{s} & + (q+1)q^{10s} \\ + (2q^3 + q^2 + q + 1)q^{2s} & - q^{11s} \\ - (3q^3 + q^2 + q + 1)q^{3s} & + q^{12s}. \end{aligned}$$

Definition (Lubin-Tate space $\mathcal{M}_2$)

We let $\mathcal{M}_2$ denote the functor over $\operatorname{Spf} O_{\breve{F}}$ similarly by taking the set of isomorphism classes of quadruples $(Y, \iota, \lambda, \rho)$ where (Y, ι, λ) is as before and $\rho \colon Y \times_S \overline{S} \to \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}$ is a framing.

Definition (The divisor $\mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(u)$ on $\mathcal{M}_2 \times \mathcal{M}_2$)

Define divisor $\mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(u)$ to be the pairs $(Y, Y') \in \mathcal{M}_2 \times \mathcal{M}_2$ such that there exists $\varphi \colon Y \to Y'$ for which the given $u \colon \mathbb{E} \to \mathbb{E}$ lifts to

$$\begin{array}{ccc} Y \times_S \overline{S} & \xrightarrow{\varphi \times_S \overline{S}} & Y' \times_S \overline{S} \\ \downarrow \rho & & \downarrow \rho' \\ \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S} & \xrightarrow{u \times_{\operatorname{Spec} \mathbb{F}} \overline{S}} & \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}. \end{array}$$

Gross-Keating

- Since $\mathcal{N}_2 \simeq \mathcal{M}_2$ work on $\mathcal{M}_2$ instead.
- The Hecke correspondence gives $\mathbb{T}_{\mathbf{1}_K \otimes f}(\Delta_{\mathcal{N}_2}) \simeq \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\varpi^r)$.
- $\mathrm{U}(\mathbb{V}_2^-)$ can basically be identified with $\mathrm{SO}(3)$, up to certain tori.
- Use Gross-Keating triple product formula on $\mathcal{M}_2$.

Theorem (Gross-Keating triple product formula)

$n_1 = \min\left(v((x,x)), v((x,y)), v((y,y))\right)$; $n_1 + n_2 = v\left((x,x)(y,y) - (x,y)^2\right)$

$$\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(x),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(y) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2}$$
$$= \begin{cases} \sum_{j=0}^{\frac{n_1-1}{2}} (n_1 + n_2 - 4j) q^j & n_1 \text{ odd} \\ \frac{n_2 - n_1 + 1}{2} q^{n_1/2} + \sum_{j=0}^{n_1/2-1} (n_1 + n_2 - 4j) q^j & n_1 \text{ even.} \end{cases}$$

An example of part of the intersection number

$$\mathcal{Z}(u)^\circ := \mathcal{Z}(u) - \mathcal{Z}\left(\frac{u}{\varpi}\right).$$

Let $r \geq 1$ and $v(\mathrm{Nm}\, u) > 0$ for $u \in \mathbb{V}_2^-$, and let

$$g = \lambda^{-1}\begin{pmatrix} \alpha & \bar{\beta}\varpi \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathrm{U}(\mathbb{V}_2^-)$$

where $v(\lambda) = 0$. With $\mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,r}}$ the Hecke operator on $\mathcal{M}_2$,

$$\left\langle \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,r}} \Delta_{\mathcal{Z}(u)^\circ}, \Gamma_g \right\rangle_{\mathcal{N}_{2,2}} = \begin{cases} (C+1)q^N + (C+2)q^{N-1} \\ 2q^N \\ q^N + q^{N-1} \end{cases}$$

where $N = \min(v(\mathrm{Nm}\, u), v(\beta) + r, v(\alpha - \bar{\alpha}) + r)$ and $C = v(\beta) - v(\alpha - \bar{\alpha})$.

Large kernel

- Li-Rapoport-Zhang noticed the orbital derivative had a large kernel in group AFL.
- For $n = 2$, the set of $\phi \in \mathcal{H}(S_2(F))$ for which

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \operatorname{Orb}(\gamma, \phi, s) = 0$$

  holds identically for all applicable $\gamma \in S_2(F)_{\mathrm{rs}}$ in fact has codimension 2.

We hope for something similar for semi-Lie, but the situation is different!

Semi-Lie large image conjecture

$$\partial\,\mathrm{Orb}\colon \phi \mapsto \left((\gamma, \mathbf{u}, \mathbf{v}^\top) \mapsto \partial\,\mathrm{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi\right)\right)$$

Conjecture (C.)

$\partial\,\mathrm{Orb}\colon \mathcal{H}(S_n(F)) \to C^\infty\left((S_n(F) \times V'_n(F))^-\right)$ is injective.

Theorem (C.)

True for $n = 2$. In fact for $n = 2$:

(a) *Still injective even if one only considers $(\gamma, \mathbf{u}, \varpi^{\mathbb{Z}}\mathbf{v}^\top)$.*

(b) *But if one requires $v(\mathbf{u}\mathbf{v}^\top) \le N$ then the kernel has codimension at most $N + 2$.*

The end

Slides and thesis draft at github.com/vEnhance/evans-phd-notebook