Evan's PhD Notebook

https://github.com/vEnhance/evans-phd-notebook/

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This has some notes I took on material mostly before my qualifying exams. It's not polished or cleaned up at all, so it's unlikely to be useful as a reference, but maybe can give you an idea of what I was reading about back in 2020.

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I

Classical theory

1 Prerequisites

This chapter covers some theory that is considered "prerequisite" for all the fancy number theory to follow, but isn't already covered in Napkin.

§1.1 Fourier transforms

As usual, $\mathbf{e}(t)$ is shorthand for $\exp(2\pi i t)$.

§1.1.1 Fourier transform of a periodic function

We'll repeatedly need the following.

Theorem 1.1.1 (Fourier coefficients of a periodic function)
Suppose that $g\colon \mathbb{R} \to \mathbb{C}$ is smooth and satisfies $g(x+1) = g(x) + 1$. In that case, it can be expressed in terms of the basis $t \mapsto \mathbf{e}(nt)$ by

$$g(x) = \sum_n a_n \mathbf{e}(nx) \qquad \text{where} \qquad a_m = \int_0^1 g(x)\, \mathbf{e}(-nx)\, dx.$$

Thus if $f\colon \mathcal{H} \to \mathbb{C}$ is holomorphic that satisfies $f(z+1) = f(z)$, then we will also have the relation

$$f(z) = \sum_n a_n \mathbf{e}(nz) = \sum_n a_n q^n \qquad \text{where} \qquad a_n = \int_w^{w+1} f(z)\, \mathbf{e}(-nz)\, dz$$

for the same reason. Here w may be any complex number in $\mathcal{H}$; by the contour theorem, the choice doesn't matter.

If we write $z = x + yi$ it's often more economical to divorce x and y:

$$f(z) = \sum_n a_n(y) \mathbf{e}(nx) \qquad \text{where} \qquad a_n(y) = \int_0^1 f(x+yi) \mathbf{e}(-nx)\, dx$$

§1.1.2 Fourier transform of a real function

Now let us suppose $f\colon \mathbb{R} \to \mathbb{C}$. We say it is

- of **moderate decrease** if $|f(x)| = O((1+x^2)^{-1})$, and
- a **Schwartz function** (or of rapid decrease) if all derivatives decay faster than any polynomial.

In either case, one can define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) \mathbf{e}(-\xi \cdot x)\, dx$$

which converges for any x. The advantage of the Schwartz functions is that $\widehat{\bullet}$ is actually a bijection on this space; whereas $\widehat{f}$ may not be of moderate decrease even if f is.

We'll repeatedly use the following.

Theorem 1.1.2 (Poisson summation formula)
If both f and $\widehat{f}$ are both of moderate decrease then

$$\sum_n f(n) = \sum_n \widehat{f}(n).$$

Proof. More generally, the following is true:

$$\sum_n f(x+n) = \sum_n \widehat{f}(n)\mathbf{e}(nx).$$

You can prove this by applying the previous result to $g(x) = \sum_n f(x+n)$. Indeed, the mth Fourier coefficients of this g is

$$\begin{aligned} a_m &= \int_0^1 \sum_n f(x+n)\mathbf{e}(-mx)\, dx \\ &= \sum_n \int_0^1 f(x+n)\mathbf{e}(-mx)\, dx \\ &= \int_{\mathbb{R}} f(x)\mathbf{e}(-m(x-\lfloor x \rfloor))\, dx \\ &= \int_{\mathbb{R}} f(x)\mathbf{e}(-mx)\, dx = \widehat{f}(m). \end{aligned}$$

□

We'll also occasionally use:

Theorem 1.1.3 (Fourier inversion)
If f is a Schwartz function then

$$\widehat{\widehat{f}}(x) = f(-x).$$

§1.1.3 Applications of Fourier stuff

We do now a few calculations which we will use later.

Proposition 1.1.4
Fix $t > 0$. The Fourier transform of $h_t = \exp(-\pi t x^2)$ is $\widehat{h_t} = \frac{1}{\sqrt{t}} h_{1/t}$.

Proof. We calculate

$$\begin{aligned} \widehat{h}_t(\xi) &= \int_{\mathbb{R}} \exp(-\pi t x^2)\exp(-2\pi i \xi x)\, dx \\ &= \int_{\mathbb{R}} \exp\left(-\frac{\pi}{t}\left[tx + i\xi\right]^2\right) \exp\left(-\frac{\pi \xi^2}{t}\right)\, dx \\ &= \exp\left(-\frac{\pi\xi^2}{t}\right) \int_{\Im z = c} \exp\left(-\pi(\sqrt{t}z)^2\right)\, dz \\ &= \frac{1}{\sqrt{t}} \exp\left(-\frac{\pi\xi^2}{t}\right) \int_{\Im z = c} \exp\left(-\pi z^2\right)\, dz \end{aligned}$$

$$= \frac{1}{\sqrt{t}} \exp\left(-\frac{\pi \xi^2}{t}\right) = \frac{1}{\sqrt{t}} h_{1/t}(\xi).$$

Here we used the Cauchy residue theorem to assert that $\int_{\Im z = c} \exp\left(-\pi z^2\right) \, dz$ is independent of the choice of c, and thus we may replace it with 0 at which point we get the famous Gaussian integral. □

Proposition 1.1.5

Fix $t > 0$. The Fourier transform of $h_t(x) = x \exp(-\pi t x^2)$ is $\widehat{h_t} = \frac{i}{t^{3/2}} h_{1/t}$.

Proof. Integrate by parts and repeat previous proposition. □

§1.2 Mellin transform

§1.2.1 Generalized Mellin transform

We'll follow Zagier's appendix.

Initially, suppose $\phi \colon (0, \infty) \to \mathbb{C}$ is a smooth function which decays rapidly at infinity. We will not assume ϕ decays rapidly at zero since this restriction is too much; we will instead just assume it is has an asymptotic expansion

$$\phi(t) = \sum_n a_n t^{\alpha_n} \qquad t \to 0$$

where $\alpha_1, \alpha_2, \dots$are complex numbers say with $\Re \alpha_1 \le \Re \alpha_2 \le \cdots$. We allow this sequence to be finite, or also infinite if $\lim_{n \to \infty} \Re \alpha_n = \infty$. In the most common case this will be a sort of "Taylor series" at 0. It's not required that this series actually converges at any point.

Then we define its **Mellin transform** $\mathcal{M}\phi \colon \mathbb{C} \to \mathbb{C}$ by

$$\mathcal{M}(\phi, s) = \int_0^\infty t^s \phi(t) \, \frac{dt}{t}$$

which is initially defined as long as $\Re s > -\Re \alpha_1$. (For example, ϕ is smooth at 0, the above equation is okay for $\operatorname{Re} s > 0$.) The following result will be indispensable as a source of meromorphic continuations:

Theorem 1.2.1 (Generalized Mellin transform for rapidly decaying functions)

This Mellin transform $\mathcal{M}(\phi, s)$ has a meromorphic continuation to $\mathbb{C}$ with (at most) simple poles of residue a_n at $s = -\alpha_n$ (and no other poles).

Proof. Let $T > 0$ be arbitrary. For each $N > 0$ write

$$\begin{aligned} \mathcal{M}(\phi, s) &= \int_0^\infty t^s \phi(t) \, \frac{dt}{t} \\ &= \int_0^T t^s \phi(t) \, \frac{dt}{t} + \int_T^\infty t^s \phi(t) \, \frac{dt}{t} \\ &= \int_0^T t^s \left(\phi(t) - \sum_{n=1}^N a_n t^{\alpha_n} \right) \frac{dt}{t} + \sum_{n \le N} \frac{a_n}{s + \alpha_n} T^{s + \alpha_n} + \int_T^\infty t^s \phi(t) \, \frac{dt}{t} \end{aligned}$$

which gives a desired meromorphic continuation to $\Re s > -\Re \alpha_N$ with poles in the specified places, of residue a_n. (Moreover, this choice is independent of T; a meromorphic continuation is going to be unique, anyways.) □

§1.2.2 Applications of Mellin

- Let $\phi(t) = e^{-t}$ which has a Taylor expansion $1 - t + \frac{t^2}{2} - \cdots$ near zero. Then by definition
$$\mathcal{M}(\phi, s) = \int_0^\infty t^s e^{-t} \, \frac{dt}{t} = \Gamma(s)$$
is the famous Gamma function. We now immediately know that Γ has meromorphic continuation with poles of residue $\frac{(-1)^n}{n!}$ at $s = -n$, for $n = 0, 1, \dots$. Note that integration by parts gives the functional equation $\Gamma(s+1) = s\Gamma(s)$, and since $\Gamma(1) = 1$, we could have deduced the result a similar way.

- More generally, if $\phi(t) = e^{-\lambda t}$ for some $\lambda > 0$ then
$$\mathcal{M}(\phi, s) = \lambda^{-s} \cdot \Gamma(s)$$
by a change of variables. (Actually, for $\lambda = 0$ we also find that the Mellin transform of a constant function is zero.)

Strong use: we can take ϕ a sum of such exponentials. Consider
$$\begin{aligned} \phi(t) &= \frac{1}{e^t - 1} \\ &= \frac{1}{t + \frac{t^2}{2!} + \dots} = \sum_{n \ge -1} \frac{B_{n+1}}{(n+1)!} t^n \\ &= e^{-t} + e^{-2t} + e^{-3t} + \cdots \end{aligned}$$
where B_{n+1} are the Beronulli numbers. Taking the Mellin transform of the last expression now gives
$$\mathcal{M}(\phi, s) = \sum_{n \ge 1} n^{-s} \Gamma(s) = \Gamma(s)\zeta(s)$$
where ζ is the Riemann zeta function. This gives an identity
$$\zeta(s) = \frac{\mathcal{M}(\phi, s)}{\Gamma(s)}$$
which is now the meromorphic continuation of ζ!

Let's see what we can extract about its poles. For $n \ge -1$, note that $\mathcal{M}(\phi, s)$ has simple poles at $s = -n$ of residue $\frac{B_{n+1}}{(n+1)!}$. For $n \ge 0$, the function Γ has simple poles at $s = -n$ of residue $\frac{(-1)^n}{n!}$. That means ζ has only a simple pole of residue 1 at $s = 1$ (since $B_0/0! = 1$). And the values of the zeta function for $n \ge 0$ are now given by
$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$
In particular, $n \ge 0$ is odd then $B_{n+1} = 0$.

§1.2.3 Mellin transforms for functions not decaying at infinity

This is still not general enough for us. For example, we will later want to take the Mellin transform of a certain θ function corresponding to the Riemann zeta function. However, this function behaves like $\frac{1}{2\sqrt{t}}$ for $t \to 0$ and $1/2$ for $t \to \infty$. This makes the previous definition fail.

We will now consider a case where t may not decay rapidly at either 0 or ∞, but having asymptotic expansions at both

$$\phi(t) = \sum_n a_n t^{\alpha_n} \qquad \text{as } t \to 0$$
$$\phi(t) = \sum_n b_n t^{\beta_n} \qquad \text{as } t \to \infty$$

where $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ are complex numbers with $\Re\alpha_1 \le \Re\alpha_2 \le \cdots$ and $\Re\beta_1 \ge \Re\beta_2 \ge \cdots$; again we assume $\lim_{n\to\infty} \Re\alpha_n = \infty$ and $\lim_{n\to\infty} \Re\beta_n = -\infty$ if either sequence is infinite. In that case, the original integral $\int_0^\infty t^s \phi(t) \frac{dt}{t}$ is defined if $\Re s > -\Re\alpha_1$ and $\Re s < -\Re\beta_1$. The problem is that this might not hold for *any* values of s at all!

This means even defining the function which we want to take extend requires some straightforward but annoying work. Here is the specification. Again pick a real number $T > 0$. The idea is that $\int_0^\infty t^s \phi(t) \frac{dt}{t}$ maybe split into $\int_0^T t^s \phi(t) \frac{dt}{t} + \int_T^\infty t^s \phi(t) \frac{dt}{t}$ and each of the two halves will be defined somewhere and can be extended analytically as before. The explicit definition is to consider

$$\begin{aligned} \mathcal{M}(\phi, s) &= \int_0^T \left(\phi(t) - \sum_{n=1}^N a_n t^{\alpha_n} \right) \frac{dt}{t} + \sum_{n=1}^N \frac{a_n}{s+\alpha_n} T^{s+\alpha_n} \\ &+ \int_T^\infty \left(\phi(t) - \sum_{n=1}^N b_n t^{\beta_n} \right) \frac{dt}{t} - \sum_{n=1}^N \frac{b_n}{s+\beta_n} T^{s+\beta_n} \end{aligned}$$

as $N \to \infty$. The first line is defined as long as $\Re s > -\Re\alpha_N$ while the second line is defined as long as $\Re s < -\Re\beta_N$. Of course, the overall sum is also independent of T. Thus:

Theorem 1.2.2 (Generalized Mellin transform)

Let $\phi\colon (0,\infty) \to \mathbb{C}$ be a function with asymptotic expansions at 0 and ∞ as above. Then $\mathcal{M}(\phi, s)$ as given above is a meromorphic function with at most simple poles only at $s = -\alpha_n$ and $s = -\beta_n$, of residue a_n and $-b_n$ (respectively, and additively).

This means that $\mathcal{M}(-, s)$ is gives an $\mathbb{C}$-linear map from the set of functions $(0,\infty) \to \mathbb{C}$ with asymptotic expansions at 0 and ∞, to the space of meromorphic functions.

An important example we can now note:

Example 1.2.3 (Kernel of the generalized Mellin transform)

Suppose $\phi(t) = 1$. Then $\mathcal{M}(1, s) \equiv 0$, because we have $a_1 = b_1 = 1$ and $\alpha_1 = \beta_1 = 0$, and $a_1 - b_1 = 0$ means the pole there gets cancelled out. More generally, if $\phi(t)$ is any polynomial in t, or any finite sum of t^α terms, then its generalized Mellin transform vanishes.

This means we can shift away any constants when discussing the Mellin transform. One just has to keep in mind the definition.

In general, the following result can be proven by change of variables.

Proposition 1.2.4 (What happens if you u-sub a Mellin transform)
Let $\phi\colon (0,\infty) \to \mathbb{C}$ be a function with asymptotic expansions at 0 and ∞ as above. Then for any $c > 0$, $d \in \mathbb{R}$ and $\alpha \in \mathbb{C}$,

$$\mathcal{M}(t^\alpha \phi(ct^d), s) = \frac{\mathcal{M}(\phi, \frac{s+\alpha}{d})}{(c^{1/d}|d|)^{s+\alpha}}.$$

Proof. This is actually a compact way of abbreviating three changes of variables; the idea is

$$\begin{aligned}\mathcal{M}(t^\alpha \phi(ct^d), s) &= \mathcal{M}(\phi(ct^d), s+\alpha) = |d|^{-(s+\alpha)} \mathcal{M}\left(\phi(ct), \frac{s+\alpha}{d}\right) \\ &= c^{\frac{-(s+\alpha)}{d}} |d|^{-(s+\alpha)} \mathcal{M}\left(\phi(t), \frac{s+\alpha}{d}\right).\end{aligned}$$

□

§1.3 Dirichlet characters

§1.3.1 Sums involving Dirichlet characters

Let $\chi\colon (\mathbb{Z}/N)^\times \to \mathbb{C}$ be a primitive Dirichlet character with conductor N.

We need two things:

- Recall that the **Gauss sum** is defined by

$$\tau(\chi) = \sum_{n \bmod N} \chi(n) \mathbf{e}(n/N)$$

  which satisfies the famous identity $|\tau(\chi)| = \sqrt{N}$.

- We need the interpolation formula for primitive Dirichlet characters:

$$\chi(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{r \bmod N} \overline{\chi}(r) \mathbf{e}(nr/N).$$

  The point of this formula is to re-express $\chi(n)$ as a sum of exponentials in n with certain coefficients (given by $\overline{\chi}$).

  In particular, this extends $\chi\colon \mathbb{R} \to \mathbb{C}$. For example, this means we could, if we wanted to, speak of the Fourier transform of χ (viewed as a function of period N).

Here is one application of the interpolation formula.

Corollary 1.3.1 (Twisted Poisson summation)
If f is of moderate decay then

$$\sum_n \chi(n) f(n) = \frac{\tau(\chi)}{N} \sum_n \overline{\chi}(n) \widehat{f}\left(\frac{n}{N}\right).$$

Proof. Let $F(n) = \chi(n) f(n)$. Then by the interpolation formula, we can extend to

$$F(x) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{r \bmod N} \overline{\chi(r)} f(x) \mathbf{e}(xr/N)$$

Now the Fourier transform of $f(x)\mathbf{e}(xr/N)$ is given by

$$\int_{\mathbb{R}} f(x)\mathbf{e}(xr/N)\mathbf{e}(-\xi x)\, dx = \widehat{f}\left(\xi - \frac{r}{N}\right)$$

Thus, by Poisson summation formula,

$$\begin{aligned}\sum_n \chi(n)f(n) &= \frac{\chi(-1)\tau(\chi)}{N} \sum_{r \bmod N} \sum_m \overline{\chi}(r)\widehat{f}\left(n - \frac{r}{N}\right) \\ &= \frac{\tau(\chi)}{N} \sum_m \sum_{r \bmod N} \overline{\chi}(-r)\widehat{f}\left(m - \frac{r}{N}\right) \\ &= \frac{\tau(\chi)}{N} \sum_n \overline{\chi}(n)\widehat{f}\left(\frac{n}{N}\right). \end{aligned}$$

□

§1.3.2 The L-function of a Dirichlet character

We then attach the L-function defined by

$$L(s,\chi) = \sum_n \chi(n)n^{-s} = \prod_p \left(1 - \frac{\chi(p)}{p^{-s}}\right)$$

for $\Re s > 1$. For example, if χ is the trivial character (so $N = 1$), then $L(s,\chi) = \zeta$ is the Riemann zeta function.

Any time we have an L-function our goal will to be get an *analytic continuation* and a *functional equation*. For Dirichlet characters, both goals will be achieved by using the following so-called *theta function*.

Definition 1.3.2. Let $\varepsilon = \frac{1-\chi(-1)}{2}$ and define

$$\theta_\chi(t) = \frac{1}{2}\sum_n n^\varepsilon \chi(n) \exp(-\pi n^2 t).$$

Note that $\chi(0) = 0$ unless $N = 1$, which would cause $\chi(0) = 1$ (and actually $\chi \equiv 1$). It is clear that $\theta_\chi(t) - \frac{1}{2}\chi(0)$ decays rapidly as $t \to \infty$. We'll now show the following functional equation:

Proposition 1.3.3 (Functional equation of the theta function)

We have

$$\theta_\chi(t) = \frac{(-i)^\varepsilon \tau(\chi)}{N^{1+\varepsilon} t^{\varepsilon+1/2}} \theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right).$$

Proof. By cases on whether $\varepsilon = 0$ or $\varepsilon = 1$; just apply twisted Poisson summation on $\exp(-\pi x^2 t)$ and $x\exp(-\pi x^2 t)$ respectively. □

When $N > 1$, this means $\theta_\chi(t)$ decays rapidly to 0 as well. Let's assume momentarily that $N > 1$, and see what falls out when we compute the Mellin transform of θ_χ. It equals

$$\mathcal{M}(\theta_\chi, s) = \sum_{n>0} n^\varepsilon \chi(n) \mathcal{M}(\exp(-\pi n^2 t), s)$$

$$\begin{aligned} &= \sum_{n>0} n^\varepsilon \chi(n)(\pi n^2)^{-s}\Gamma(s) \\ &= \Gamma(s)\pi^{-s}\sum_{n>0} n^\varepsilon \chi(n) n^{-2s} \\ &= \Gamma(s)\pi^{-s} L(2s-\varepsilon, \chi) \end{aligned}$$

Replacing s with $\frac{1}{2}(s+\varepsilon)$ and rearranging:

$$\mathcal{M}\left(\theta_\chi, \frac{s+\varepsilon}{2}\right) = \pi^{-\frac{1}{2}(s+\varepsilon)}\Gamma\left(\frac{s+\varepsilon}{2}\right) L(s,\chi).$$

Since Γ never vanishes, this gives the analytic continuation of $L(s,\chi)$.

When $N = 1$ and $\chi = \mathbf{1}$ is the trivial character we get $\theta_\chi(t) = \frac{1}{\sqrt{t}}\theta_\chi(1/t)$, and so $\theta_\chi(t)$ behaves like $\frac{1}{2\sqrt{t}}$ at zero, and the Mellin transform is defined. The same calculation gives

$$\begin{aligned} \mathcal{M}(\theta_{\mathbf{1}}, s) &= \mathcal{M}(1/2, 0) + \sum_{n>0} \mathcal{M}(\exp(-\pi n^2 t), s) \\ &= 0 + \Gamma(s)\pi^{-s} L(2s-\varepsilon, \mathbf{1}) = \Gamma(s)\pi^{-s}\zeta(2s) \end{aligned}$$

so

$$\mathcal{M}(\theta_{\mathbf{1}}, s/2) = \Gamma(s)\pi^{-s}L(2s-\varepsilon, \mathbf{1}) = \Gamma(s/2)\pi^{-s/2}\zeta(s)$$

In this case, $\mathcal{M}(\theta_{\mathbf{1}}, s/2)$ is meromorphic except for simple poles at $s = 0$ and $s = 1$ of residue 1.

The Mellin transforms we mentioned are usually called the completed L-functions as follows:

Theorem 1.3.4 (The Mellin transform of the theta function)

Define

$$\Lambda(s,\chi) = \mathcal{M}\left(\theta_\chi, \frac{s+\varepsilon}{2}\right) = \pi^{-\frac{1}{2}(s+\varepsilon)}\Gamma\left(\frac{s+\varepsilon}{2}\right) L(s,\chi).$$

Then $\Lambda(s,\chi)$ is analytic if $\chi \neq \mathbf{1}$; otherwise it is meromorphic, with simple poles at $s = 0$ and $s = 1$ of residue 1. Moreover, we have the functional equation

$$\Lambda(s,\chi) = \Lambda(1-s,\chi).$$

Proof. The first half of the theorem follows from Mellin transform properties. To show the functional equation, use Proposition 1.2.4 to clean everything up once the theta function is used.

$$\begin{aligned} \Lambda(s,\chi) = \mathcal{M}\left(\theta_\chi(t), \frac{s+\varepsilon}{2}\right) &= \mathcal{M}\left(\frac{(-i)^\varepsilon \tau(\chi)}{N^{1+\varepsilon} t^{\varepsilon+1/2}}\theta_{\overline{\chi}}\left(\frac{1}{N^2}t^{-1}\right), \frac{s+\varepsilon}{2}\right) \\ &= \frac{(-i)^\varepsilon\tau(\chi)}{N^{1+\varepsilon}} \cdot \mathcal{M}\left(t^{-(\varepsilon+1/2)}\theta_{\overline{\chi}}\left(\frac{1}{N^2}t^{-1}\right), \frac{s+\varepsilon}{2}\right) \\ &= \frac{(-i)^\varepsilon\tau(\chi)}{N^{1+\varepsilon}} \cdot \frac{\mathcal{M}\left(\theta_{\overline{\chi}}, \frac{\frac{s+\varepsilon}{2}-(\varepsilon+1/2)}{-1}\right)}{(N^2)^{\frac{s+\varepsilon}{2}-(\varepsilon+1/2)}} = \frac{(-i)^\varepsilon\tau(\chi)}{N^s} \cdot \mathcal{M}\left(\theta_{\overline{\chi}}, \frac{(1-s)+\varepsilon}{2}\right) \\ &= \frac{(-i)^\varepsilon\tau(\chi)}{N^s}\Lambda(\theta_{\overline{\chi}}, 1-s). \end{aligned}$$

□

§1.4 Linear algebraic groups

§1.4.1 Reductive groups

Definition 1.4.1. A linear algebraic group G over a field k is a closed subscheme of $\mathrm{GL}_n(k)$, i.e. a smooth affine group scheme over k.

Recall that a generic group H is

- **solvable** if the map $X \mapsto [X, X]$ starting from H eventually stabilizes; equivalently, there needs to be a normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$$

  where each G_i/G_{i-1} is abelian.

- **unipotent** if every element of the group is unipotent; this implies H is a closed subgroup of U_n.

Definition 1.4.2. Given a LAG G, we define

- its **radical** is the largest connected solvable normal subgroup;
- its **unipotent radical** is the largest connected unipotent normal subgroup.

We say G is

- **semisimple** if the radical is trivial;
- **reductive** if the unipotent radical is trivial.

Semisimple groups are considered pretty rigid, but reductive groups are not too much worse; from https://mathoverflow.net/a/223895/70654:

> From the modern perspective the class of (connected) reductive groups is more natural than that of (connected) semisimple groups for the purposes of setting up a robust general theory, due to the fact that Levi factors of parabolics in reductive groups are always reductive but generally are not semisimple when the ambient group is semisimple. However, after some development of the basic theory one learns that reductive groups are just a fattening of semisimple groups via a central torus (e.g., GL_n versus SL_n), so Harish-Chandra had no trouble to get by in the semisimple case by just dragging along some central tori here and there in the middle of proofs.

Following non-obvious theorem is another motivation for why reductive groups are considered really nice:

Theorem 1.4.3

A smooth connected affine group over a field of characteristic 0 is reductive if and only if all of its algebraic representations are completely reducible.

§1.4.2 Parabolic and Borel subgroups

Definition 1.4.4. A **Borel subgroup** of G is

- a connected solvable subgroup variety B for which G/B is complete;
- or equivalently, a maximal Zariski-closed solvable subgroup (but not necessarily normal).

The Borel subgroups are all conjugate to each other.

Definition 1.4.5. A **parabolic subgroup** P of G is

- any subgroup containing a Borel subgroup;
- over an algebraically closed field, equivalently, such that G/P is a complete variety.

Example copied from Wikipedia: for $G = \mathrm{GL}_4(\mathbb{C})$, a Borel subgroup is

$$\left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : \det(A) \neq 0 \right\}$$

and the maximal proper parabolic subgroups of G containing B are:

$$\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \right\}$$

Also, a maximal torus in B is

$$\left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} \neq 0 \right\} \cong \mathbb{G}_{\mathrm{m}}^4$$

§1.4.3 Table

G	Radical	Unipotent radical	Borel subgroup
$\mathrm{GL}(n)$	$\mathbb{C}$ (diagonal matrices)	1	Upper tri
$\mathrm{SL}(n)$	1	1	
$\mathrm{O}(n)$		1	
$\mathrm{SO}(n)$		1	
$\mathrm{Sp}(n)$		1	
$\mathrm{U}(n)$		1	
$\mathbb{G}_{\mathrm{a}}$	$\mathbb{G}_{\mathrm{a}}$	$\mathbb{G}_{\mathrm{a}}$	

§1.4.4 Tori

split vs anisotropic

2 Modular forms

§2.1 The half-plane and the modular group

§2.1.1 The action of $\mathrm{SL}_2(\mathbb{R})$ on the half-plane

As usual, let $\mathcal{H} = \{z \mid \Im z > 0\}$ be the half-plane. Then there is a famous action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

This action is not quite faithful, so commonly one will work in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ instead. The action on $\mathcal{H}$ is transitive though;

$$\begin{bmatrix} y^{1/2} & xy^{-1l2} \\ 0 & y^{-1/2} \end{bmatrix} : i \mapsto x + yi.$$

Let's record a few properties of the action. First, an identity that will come into play surprisingly often.

Proposition 2.1.1 (The stupid identity)
For any $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z = x + yi \in \mathcal{H}$,

$$\Im \gamma(z) = \frac{y}{|cz+d|^2}.$$

Proof. Trivial. Compute it directly. □

Definition 2.1.2. An element $\gamma \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm I$ is called

- **elliptic** if $|\operatorname{Tr}\gamma| < 2$;
- **parabolic** if $|\operatorname{Tr}\gamma| = 2$;
- **hyperbolic** if $|\operatorname{Tr}\gamma| > 2$.

We extend the action of γ to include the "boundary" of $\mathcal{H}$, which is $\mathbb{RP}^1 = \mathbb{R} \cup \{i\infty\}$. It still has the same formula $z \mapsto \frac{az+b}{cz+d}$ on $\mathbb{RP}^1$. Warning: in pictures, ∞ is typically drawn at $i\infty$.

Proposition 2.1.3 (Number of fixed points)
For $\gamma \neq \pm I$,

- An elliptic element has exactly one fixed point in $\mathcal{H}$.
- A parabolic element has two fixed points, both in $\mathbb{RP}^1$.
- A hyperbolic element has exactly one fixed point in $\mathbb{RP}^1$.

Proof. Set $z = \frac{az+b}{cz+d}$ and look at the corresponding quadratic $cz^2 + (d-a)z - b = 0$. It is a quadratic in z with real coefficients, so either it has a pair of complex conjugate roots, or it has 1-2 real roots. Its discriminant is $(d-a)^2 + 4bc = (d+a)^2 - 4$, and the cases correspond to whether the determinant is positive, zero, or negative. □

Proposition 2.1.4

The stabilizer of an elliptic element is a cyclic group.

Proof. Let γ be an elliptic element fixing z. By conjugating (since the action on $\mathcal{H}$ is transitive), we may as well assume $z = i$. The stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$ is

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : a^2 + b^2 = 1 \right\}$$

and any discrete subgroup of $\mathrm{SO}(2)$ is cyclic. □

§2.1.2 Fuchsian groups

Generally, we're going to mod out $\mathcal{H}$ by the action of subgroups of $\mathrm{SL}_2(\mathbb{R})$ which are discrete. We name them now:

Definition 2.1.5. A **Fuchsian group** is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 2.1.6

A subgroup of $\mathrm{SL}_2(\mathbb{R})$ is Fuchsian if and only if it acts *discontinuously* on $\mathcal{H}$, meaning for any two compact subsets K_1 and K_2, the set $\{\gamma \mid \gamma(K_1) \text{ meets } K_2\}$ is finite.

The most important Fuchsian groups are the so-called congruence subgroups.

Definition 2.1.7. For every positive integer N, we will let

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$

In other words, these are matrices congruent to the identity modulo N. (In still other words, $\Gamma(N)$ is the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N)$.) Note in particular $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. A subgroup of $\Gamma(1)$ is a **congruence subgroup** if it contains $\Gamma(N)$ for some N.

Note that "most congruence subgroups contain no elliptic elements".

§2.1.3 Cusps

Let Γ be a Fuchsian group, and consider its action on $\mathbb{RP}^1$.

Definition 2.1.8. A **cusp** of Γ is an element of $\mathbb{RP}^1$ which is fixed by some parabolic element of Γ. By abuse of language we refer to an equivalence class of cusps under the action of Γ as "a" cusp as well.

Proposition 2.1.9
The cusps of $\Gamma(1)$ are exactly $\mathbb{Q} \cup \{i\infty\}$.

Proof. Note $i\infty$ is a cusp by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \Gamma(1)$. And if $q = m/n$ for $\gcd(m,n) = 1$,

$$\gamma(q) = q \text{ for } \gamma = \begin{bmatrix} 1+mn & -m^2 \\ n^2 & 1-mn \end{bmatrix}$$

Conversely, if $r \in \mathbb{R}$ is the cusp of some $\gamma \in \Gamma(1)$, then r is the double root of $\frac{az+b}{cz+d} = z$, so it must be rational. □

In general, given a Fuchsian group Γ, we denote by $\mathcal{H}^*_\Gamma$ (or just $\mathcal{H}^*$ if Γ is implied) the extended half-plane with the cusps added. Thus Γ acts on $\mathcal{H}^*$ too.

§2.1.4 The classical picture of $\mathrm{SL}_2(\mathbb{Z})$

TO BE WRITTEN

§2.1.5 Compactification

The quotient space $\Gamma \backslash \mathcal{H}^*$ can be made into a Riemann surface.

§2.2 Modular forms

§2.2.1 Definition on $\Gamma(1)$

Let k be an even nonnegative integer.

Definition 2.2.1. A **modular form** for $\Gamma(1)$ of *weight* k is a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

and which is holomorphic at the cusp $i\infty$. If it also vanishes at the cusp, we say it is **cuspidal**.

To explain "holomorphic at the cusp", note that $f(z+1) = f(z) + 1$ and so one ought to be able to write the Fourier expansion

$$f(z) = \sum_n a_n \mathbf{e}(nz) = \sum_n a_n q^n$$

for some coefficients a_n, with $q = \mathbf{e}(z)$ the nome. One can then discuss holomorphic/meromorphic/vanishing at the cusp by taking $z \to i\infty$ (equivalently, $q \to 0$); we say f is holomorphic at the cusp if $a_n = 0$ for all $n < 0$, vanishes at the cusp if $a_n = 0$ for all $n \le 0$ and is meromorphic at the cusp if $a_n = 0$ for all $n < -N$ for some N.

We may as well define the slash operator now for notational convenience.

Definition 2.2.2. If $f \colon \mathcal{H} \to \mathbb{C}$ is holomorphic, and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$, then we let

$$(f|\gamma)(z) = (\det \gamma)^{k/2} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

The symbol is called a **slash operator**. Then the modularity of f is just asserting $f|\gamma = f$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proposition 2.2.3 (Slash operator is a right acttion)
We have $f|(\gamma_1\gamma_2) = (f|\gamma_1)|\gamma_2$ for any $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{R})$.

Proof. Obvious. □

§2.2.2 Definition for a general Fuchsian group

We may as well introduce the notation now for these forms.

Let Γ be a Fuchsian group.

Definition 2.2.4. A modular form for Γ of weight k is a holomorphic $f \colon \mathcal{H} \to \mathbb{C}$ for which $f|\gamma = f$ holds for any $\gamma \in \Gamma$, and which is holomorphic at every cusp in the following sense:

Given a cusp $t \in \mathbb{R} \cup \{i\infty\}$, we choose $\rho = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$ with $\rho(t) = i\infty$, and consider $f|\rho^{-1}$ instead. There will exist h with $f(z+h) = f(z)$, and we can thus take a Fourier expansion $f(z) = \sum_n a_n q^{n/h}$. We can then define holomorphic/vanishing/meromorphic at the cusp.

Then $M_k(\Gamma)$ denotes the set of modular forms for Γ, and $S_k(\Gamma)$ the set of cusp forms. We will not usually need this level of generality:

- When Γ is a congruence group, there will exist $\left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$ for some h, so $i\infty$ will be a cusp anyways ($\rho = \mathrm{id}$).
- If we specialize further to $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$, defined later, then we may take $h = 1$.

We will almost always restrict to the case we mentioned where Γ is a congruence subgroup containing $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$; in which case the modular form f simply obeys $f(z+1) = f(z)$ and no further antics are needed to discuss the Fourier coefficients.

§2.2.3 Growth rate

It will be useful to know that cusp forms decay quickly as $y \to \infty$.

Lemma 2.2.5 (Cusp forms decay rapidly at infinity and are bounded)
For any cusp form f for a congruence group, we have $|f(z)| = O(e^{-cy})$ as $y \to \infty$, uniformly in x,i for some constant c in terms of f.

Moreover, there is a constant C_f such that $|f(z)| < C_f y^{-k/2}$ for all complex numbers z.

Proof. We know $f(z) = \sum_n a_n q^{n/h}$, and since $|q| = e^{-2\pi y}$ we have

$$|f(z)| \le \sum_{n \ge 1} a_n e^{-2\pi n y/h} = O(e^{-2\pi y/h})$$

as claimed, as $y \to \infty$.

To get the bound for all y, the stupid identity implies that the function $z \mapsto |f(z) y^{k/2}|$ is invariant under $\Gamma(1)$, and (since f is a cusp form) approaches zero rapidly as $y \to \infty$. In particular, this function is bounded on some fundamental domain, and thus is bounded above by some constant C_f. □

§2.2.4 Important special case with $\Gamma_0(N)$ and $\Gamma_1(N)$; twisting by nebentypus

In particular, we now define two important congruence subgroups that will become relevant later on.

Definition 2.2.6. We let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod N, \; c \equiv 0 \bmod N \right\}$$

Note that $\Gamma_0(N) \supset \Gamma_1(N) \supset \Gamma(N)$.

It's common to call $\Gamma_0(N)$ the **level group**, and we will use it more than $\Gamma_1(N)$. For $\Gamma_0(N)$ we will also permit twisting by a Dirichlet character:

Definition 2.2.7. Let χ be a Dirichlet character modulo N (not necessarily primitive). A **modular form of level N with weight k and nebentypus χ** is an element of $M_k(\Gamma_1(N))$ which satisfies the additional relation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \chi(d) f(z)$$

for $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$. The set is denoted $M_k(\Gamma_0(N), \chi)$.

The cusp forms of level N with weight k and nebentypus χ are denoted similarly by $S_k(\Gamma_0(N), \chi)$; they are the above modular forms which also vanish at the cusp.

Note that $S_k(\Gamma_0(N), \mathbf{1}) = S_k(\Gamma_1(N))$. In fact,

Theorem 2.2.8

We have

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \bmod N} M_k(\Gamma_0(N))$$

where the direct sum is over all Dirichlet characters modulo N, not necessarily primitive.

Proof. □

We remind the reader that $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma_1(N)$, so any modular forms in this group will satisfy $f(z+1) = f(z)$ and thus have a Fourier expansion $\sum_n a_n q^n$ (with no floating h).

§2.3 Classification of modular forms for $\Gamma(1)$

§2.3.1 First important example

To proceed further, it will be convenient to know there *exists* some modular forms.

When $k = 0$, a modular form of weight zero must be constant according to the maximum modulus principle. We will show later on that there are no modular forms of weight 2 either; but there is a modular form of any even weight $k \geq 4$, and we construct it as follows.

Definition 2.3.1. For $k \geq 4$ an even integer, the **Eisenstein series** is defined by

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} (mz+n)^{-k}.$$

The sum is absolutely convergent for $z \in \mathcal{H}$. Indeed, the contribution of terms with $m = 0$ is $\zeta(k)$, the contribution of (absolute values) of remaining terms is

$$\begin{aligned}
\sum_{m \neq 0} \sum_{n} |mz+n|^{-k} &= \sum_{m \neq 0} \sum_{n} \frac{1}{((n + m\Re z)^2 + (m\Im z)^2)^{k/2}} \\
&< \sum_{m \neq 0} \left[\frac{3}{(m\Im z)^k} + \sum_{n \neq 0} \frac{1}{(n^2 + (m\Im z)^2)^{k/2}} \right] \\
&< \sum_{m \neq 0} \left[\frac{3}{(m\Im z)^k} + \sum_{n \neq 0} \frac{1}{(2mn\Im z)^{k/2}} \right] \\
&= \frac{6\zeta(k)}{(\Im z)^k} + \frac{4}{(2\Im z)^{k/2}} \zeta(k/2)^2 < \infty.
\end{aligned}$$

It also satisfies the functional equation as

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-k} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} ((ma+nc)z + (mb+nd))^{-k}$$

and since $ad - bc = 1$, the map $(m,n) \mapsto (ma+nc, mb+nd)$ is a bijection on $\mathbb{Z}^2 - \{(0,0)\}$ (multiplying by a matrix in $\mathrm{SL}_2(\mathbb{Z})$), so the right-hand side is $(cz+d)^{-k} E_k(z)$.

§2.3.2 Second important example

Consider the Dedekind eta function

$$\eta(z) = \sum_{n=1}^{\infty} \chi(n) q^{n^2/24}$$

where χ is primitive quadratic with conductor 12 (unique). Then we define

$$\Delta(z) = \eta(z)^{24} = q \prod_{1}^{\infty} (1-q^n)^{24}$$

which is a weight 12 modular form with only a simple zero at $i\infty$, vanishing nowhere else. The equality here comes from Jacobi triple product formula.

§2.3.3 Main result

Proposition 2.3.2

The space of cusp forms of weight 12 is one-dimensional.

Proof. Δ is in it. Also if f is in it, then f/Δ is an modular form of weight 0 with no poles, ergo it is constant. □

Theorem 2.3.3

The space of modular forms for $\Gamma(1)$ is generated by E_4 and E_6.

Proof. Analogous to the earlier the argument, dimension counting: multiplication by Δ generally gives an isomorphism

$$M_{k-12}(\Gamma(1)) \to S_k(\Gamma(1)).$$

□

§2.4 The L-function of a modular form for a congruence group

§2.4.1 Definition

Let $f \in M_k(\Gamma_1(N))$. Consider its Fourier expansion $f(z) = \sum_n a_n q^n$. We want to attach to it an L-function defined by

$$L(s, f) = \sum_n a_n n^{-s}.$$

For this to work, we need to first prove that the Fourier coefficients a_n decay quickly enough that $L(s, f)$ converges.

Lemma 2.4.1 (Trivial estimate, due to Hardy and Hecke)

If $f = \sum_n a_n q^n$ is a cusp form, then $a_n = C_f n^{k/2}$ (with the implied constant depending on f).

Proof. Recall that

$$a_n = \int_w^{w+1} f(z)\mathbf{e}(-nz)\, dz.$$

We choose $[w, w+1]$ to be $[yi, 1+yi]$ and calculate

$$\begin{aligned} |a_n| &< \int_{yi}^{yi+1} |f(z)\mathbf{e}(-nz)|\, dz = \int_{yi}^{yi+1} |f(z)|\, e^{2\pi ny}\, dz \\ &= e^{2\pi ny} \int_0^1 |f(x+yi)|\, dx < e^{2\pi ny} \int_0^1 C_f y^{-k/2}\, dx \\ &= e^{2\pi ny} C_f y^{-k/2}. \end{aligned}$$

Taking $y = 1/n$ completes the proof. □

> **Remark 2.4.2.** In fact, the so-called Ramanujan conjecture (proved in 1970) implies $a_n \le C_{f,\varepsilon} n^{(k-1)/2+\varepsilon}$.

If f is not a cusp form, then we can choose a constant c such that $f - cE_k(z)$ is a cusp form. Since the coefficients of $E_k(z)$ are bounded by $n^{k-1} \log n$, we conclude altogether that $L(s, f)$ converges if $\Re s > 1 + (k-1) = k$. This completes our definition of the L-function.

§2.4.2 Completed L-function

Much like for ζ, we may consider the generalized Mellin transform

$$\Lambda(s,f) \overset{\text{def}}{=} \mathcal{M}\left[y \mapsto f(iy)\right].$$

Note that $f(iy)$ approaches a constant c rapidly as $y \to \infty$. From $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \Gamma(1)$, we have

$$f(iy) = (-1)^{k/2} y^{-k} f(i/y)$$

so as $y \to 0$ we have $f(iy) \to c(-1)^{k/2}y^{-k}$ rapidly. Thus, the generalized Mellin transform will have at most simple poles at $s=0$ and $s=k$, and otherwise be analytic.

Now, we go ahead and actually compute

$$\begin{aligned} \Lambda(s,f) = \mathcal{M}\left[y \mapsto f(iy)\right] &= \int_0^\infty y^s f(iy) \frac{dy}{y} \\ &= \int_0^\infty y^{s-1} \sum_n a_n e^{-2\pi n y} dy = \sum_n \int_0^\infty y^{s-1} a_n e^{-2\pi n y} dy \\ &= \sum_n a_n (2\pi n)^{-s} \Gamma(s) = (2\pi)^{-s}\Gamma(s) \sum_n a_n n^{-s} \\ &= (2\pi)^{-s}\Gamma(s) L(s,f). \end{aligned}$$

Also, from a change of variables $y \mapsto 1/y$, we can also get that $\Lambda(s,f) = \Lambda(k-s,f)$.

As before, it is more conventional to define Λ through the L-function and hide the role of the Mellin transform. So we have proved the following:

Theorem 2.4.3 (L-function for $\Gamma(1)$)

Let f be a modular form for $\Gamma(1)$. The function

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f)$$

has analytic continuation; it has simples poles at $s=0$ and $s=k$ if f is not cuspidal, and otherwise is analytic. Moreover, it satisfies a functional equation $\Lambda(s,f) = \Lambda(k-s,f)$.

§2.5 Petersson inner product

We will now make the space of modular/cusp forms into a Hilbert space.

§2.5.1 Poincaré metric

The space $\mathcal{H}$ is usually endowed with the metric

$$d\mu = \frac{dx\,dy}{y^2}.$$

The reason is that:

Proposition 2.5.1

This measure is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

Proof. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$ and let $S \subseteq \mathcal{H}$ be measurable. Define $u(x,y) = \Re\gamma(z)$ and $v(x,y) = \Im\gamma(z)$. We wish to show

$$\int_{\gamma(S)} \frac{du\,dv}{v^2} = \int_S \frac{dx\,dy}{y^2}.$$

The change-of-variables formula implies that the right-hand side can be transformed by multiplying by Jacobian:

$$\int_{\gamma(S)} \frac{du\,dv}{v^2} = \int_S \frac{1}{v(x,y)^2} \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} dx\,dy$$

At this point, we can verify the identity be explicitly calculating $u(x,y) = \frac{(ad+bc)x+bd+ac(x^2+y^2)}{c^2(x^2+y^2)+2cdx+d^2}$ and $v(x,y) = \frac{y}{c^2(x^2+y^2)+2cdx+d^2}$, and then bashing the right-hand side until it simplifies to $1/y^2$.

This is almost comedically painful, so we mention that a less obnoxious approach is possible if one knows that $\mathrm{SL}_2(\mathbb{R})$ is generated by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$, so it suffices to just check these two special cases which is much easier. □

This makes it possible to integrate over $\Gamma \backslash \mathcal{H}$ whenever Γ is a congruence subgroup.

We may as well do the following classical calculation now.

Proposition 2.5.2

We have $\int_{\Gamma(1)\backslash\mathcal{H}} \frac{dx\,dy}{y^2} = \frac{1}{3}\pi^2$.

Proof. Take the classical fundamental domain... □

We won't need the value, just the finiteness.

§2.5.2 The Petersson inner product

Definition 2.5.3. If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $f, g \in S_k(\Gamma)$ we define the Petersson inner product by

$$\langle f, g\rangle = \frac{1}{[\Gamma(1) : \Gamma(N)]} \int_{\Gamma(N)\backslash\mathcal{H}} f(z)\overline{g(z)} \frac{dx\, dy}{y^2}$$

where N is large enough that f and g are modular forms for $\Gamma(N)$.

Here, the stupid identity implies $f(z)\overline{g(z)}y^k$ is invariant under the action of $\Gamma(N)$, so this is well-defined. In addition, because $f(z)$ and $g(z)$ decay rapidly as $z \to \infty$ (they are cusp forms), the integral is bounded on the typical (and hence every) fundamental domain of $\Gamma(1)$, and thus converges.

§2.5.3 Genus

§2.6 Hecke operators for $\Gamma(1)$

§2.6.1 Double cosets of $\Gamma(1)$ in $\mathrm{GL}_2^+(\mathbb{Q})$

In order to define the Hecke operators, we'll need to discuss *double cosets* of $\mathrm{GL}_2^+(\mathbb{Q})$. It turns out we can actually just describe all the double cosets of $\Gamma(1)\backslash \mathrm{GL}_2^+(\mathbb{Q})/\Gamma(1)$.

Theorem 2.6.1 (Complete description of double cosets of $\Gamma(1)$ in $\mathrm{GL}_2^+(\mathbb{Q})$)

The double coset $\Gamma(1)\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\Gamma(1)$ consists of all matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{Q})$ with determinant d_1d_2 and $\gcd(a, b, c, d) = d_2$ (with gcd defined over the rational numbers).

Consequently, there is a disjoint union

$$\mathrm{GL}_2^+(\mathbb{Q}) = \bigsqcup_{d_1 \mid d_2} \Gamma(1)\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\Gamma(1).$$

Proof. □

Theorem 2.6.2 (Double coset partition into right cosets)

If d_1 and d_2 are positive integers with $d_2 \mid d_1$ then

$$\Gamma(1)\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\Gamma(1) = \bigsqcup_{\substack{a,d>0 \\ ad=d_1d_2 \\ b \bmod d \\ \gcd(a,b,d)=d_2}} \Gamma(1)\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

If d_1 and d_2 are not integers, then one can factor out the common denominator and then appeal to the theorem.

Proof. □

Okay, that is all for double cosets for now.

§2.6.2 The Hecke operator

$$(T(n)f)(z) = \sum_{ad=n} \sum_{b \bmod d} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{d}\right).$$

Also defined via double cosets.

Proposition 2.6.3

The Hecke operators commute, and are self-adjoint.

This means you can find a basis of eigen-stuff.

If f is a Hecke eigenform, that means it has an eigenvalue $\lambda(n)$. We can then expand and normalize the coefficients: $A(1) \neq 0$, and scaling so that $A(1) = 1$, we have $A(\bullet)$ is multiplicative and

$$L(s,f) = \prod_p \left(1 - A(p)p^{-s} + p^{k-1-2s}\right)^{-1}.$$

II

Towards adeles

3 Adeles

§3.1 The adele ring

Let K be a global field.

Definition 3.1.1. The **adele ring** of K is defined as the restricted product

$$\mathbb{A}_K = \prod_v^{\text{restrict}} (K_v, \mathcal{O}_v)$$

across all the places v of K, both Archimedean and non-Archimedean. Elements of $\mathbb{A}_K$ are called **adeles**.

This means that it consists of tuples $(a_v)_v$ for which $a_v \in \mathcal{O}_v$ for all but finitely many v.

Definition 3.1.2. The **idele group** of K is defined as the restricted product

$$\mathbb{A}_K^\times = \prod_v^{\text{restrict}} (K_v^\times, \mathcal{O}_v^\times)$$

across all the places v of K, both Archimedean and non-Archimedean. Elements of $\mathbb{A}_K^\times$ are called ideles.

(The topology of the idele group is not the subspace topology of $\mathbb{A}_K$, so the inclusion $\mathbb{A}_K^\times \subseteq \mathbb{A}_K$ occurs only at the level of sets.)

There are obviously diagonal inclusions $K \hookrightarrow \mathbb{A}_K$ and $K^\times \hookrightarrow \mathbb{A}_K^\times$. By abuse of notation we're going to just treat K and $K^\times$ as subsets of $\mathbb{A}_K$ and $\mathbb{A}_K^\times$, hence e.g. $\mathbb{A}_K/K$ refers to the coimage of the former map.

The adelic absolute value is the map

$$\mathbb{A}_K^\times \to \mathbb{R}_{>0}$$

given by $(a_v) \mapsto \prod_v |a_v|_v$.

Proposition 3.1.3 (Product formula)

We have $|a| = 1$ for $a \in K^\times$.

§3.2 Example

By "strong approximation",

$$\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}_{>0} \times \mathbb{R} \times \prod_p \mathbb{Z}_p^\times.$$

But if K is a number field with class number greater than 1 then such a clean decomposition is usually impossible.

§3.3 First properties of the adele ring

Upshots of using the restricted pruduct include the following results.

Theorem 3.3.1 ($\mathbb{A}_K$ is LCA Hausdorff and self-dual)

$\mathbb{A}_K$ is locally compact and Hausdorff, and in fact equal to its own Pontryagin dual.

Theorem 3.3.2 (K is the Pontryagin dual of $\mathbb{A}_K/K$)

K is a discrete co-compact subgroup of $\mathbb{A}_K$. In fact, K and $\mathbb{A}_K/K$ are Pontryagin duals.

Remark. In fact, it's also true that $K^\times$ is discrete and co-compact in

$$\mathbb{A}_K^1 \overset{\text{def}}{=} \ker\left(|\bullet| : \mathbb{A}_K^\times \to \mathbb{R}^\times\right)$$

the set of ideles of adelic absolute value 1.

In general, it may be shown there is a short exact sequence

$$1 \to \frac{K_\infty^1 \times \prod_{v<\infty} \mathcal{O}_v^\times}{\mathcal{O}_K^\times} \to \frac{\mathbb{A}_K^1}{K^\times} \to \operatorname{ClassGrp}(K) \to 1$$

which does not necessarily split, though when K has trivial class group it does explicitly characterize $\mathbb{A}^1$ (e.g. $K = \mathbb{Q}$, as we saw in the earlier example).

This short exact sequence is conceptually nice because it implies both Dirichlet's unit theorem, and the finiteness of the class group, solely from the compactness of the center term.

citation needed on Dirichlet unit theorem

§3.4 Calculus on adeles

The following results are included here for completeness, to signal that they do not depend on any of the automorphic-flavored stuff to follow. But you could skip this section for now and come back to it later when it's quoted, if you prefer.

§3.4.1 Fourier setup for local fields

In this section, let F be a local field, and define the following *standard character* on it:

$$\psi_F(x) = \begin{cases} e^{-2\pi i x} & F = \mathbb{R} \\ e^{-2\pi i |x|_p} & F = \mathbb{Q}_p \\ e^{2\pi i a_{-1}/p} & F = \mathbb{F}_p((t)) \\ \psi_0\left(\operatorname{Tr}_{F/F_0}(x)\right) & F/F_0 \text{ separable, } F_0 \text{ and } \psi_0 \text{ as above} \end{cases}$$

These choices are contrived so that later, if K is a global field, then $\prod_v \psi_{K_v} : \mathbb{A} \to \mathbb{C}$ will vanish on K.

Moreover, we can choose a Haar measure such that

- dx is the standard Lebesgue measure for $F = \mathbb{R}$
- dx is twice the standard Lebesgue measure on $F = \mathbb{C}$

- dx is selected such that $\mathrm{Vol}(\mathcal{O}) = (\#\mathcal{O}/\mathcal{D})^{-1/2}$, where $\mathcal{D}$ is the different, when F is a non-Archimedean local field.

This lets us define a Fourier transform

$$\widehat{f} = \int_F f(x)\psi_F(xy)\,dx.$$

§3.4.2 Poisson summation

write this

4 Tate's thesis

§4.1 Goals moving forward (road map of the future)

In brief, take a unitary character $\omega\colon \mathbb{A}^\times/K^\times \to \mathbb{C}$, known as a **Hecke character** (analogous to the nebentypus Dirichlet character from earlier). An adelic automorphic form with central quasicharacter ω will be a function

$$\phi\colon \operatorname{GL}_n(\mathbb{A}_K) \to \mathbb{C}$$

obeying the conditions

- For all $g \in \operatorname{GL}_n(\mathbb{A}_K)$ and $z \in \mathbb{A}_K^\times$, we have

$$\phi\left(\begin{bmatrix} z & & \\ & \ddots & \\ & & z \end{bmatrix} g\right) = \omega(z)\phi(g).$$

  The choice of the letter z comes from the fact that the center of the group $\operatorname{GL}(n, \mathbb{A}_K)$ is exactly the diagonal matrices appearing above.

- For all $g \in \operatorname{GL}_n(\mathbb{A}_K)$ and $\gamma \in \operatorname{GL}_n(K^\times)$ we have

$$\phi(\gamma g) = \phi(g).$$

  The name "automorphic" comes from here, and means we may equally regard ϕ as a function on $\operatorname{GL}_n(K^\times) \backslash \operatorname{GL}_n(\mathbb{A}_K)$.

- Four other technical niceness conditions defined later.

We are going to have two goals:

First goal — tying forms to representations Connect automorphic forms to certain "representations" on some space of functions $\operatorname{GL}_n(K^\times) \backslash \operatorname{GL}_n(\mathbb{A}_K) \to \mathbb{C}$. This is analogous to how we saw that modular forms and Maass forms turned out to correspond exactly with representations of $L^2(\Gamma \backslash \operatorname{SL}(2, \mathbb{R}))$.

Second goal — L-functions For each such representation, construct an L-function, give it an analytic continuation, and exhibit a functional equation. (One common trend in analytic number theory is that L-functions are worth their weight in gold.)

There is one place where the analogy is slightly weaker. Rather than attaching L-functions directly to the automorphic *forms*, it turns out to be more convenient to attach them to the automorphic *representations* directly.

make a table?

§4.2 Motivation

Even for $n = 2$, this task we described will be rather technical (let alone replacing GL_n with a general algebraic group).

So we will first examine the case where $n = 1$; a result widely known now as Tate's thesis. In this case much of the theory simplifies immensely:

Exercise 4.2.1. When $n = 1$, in the notation of the earlier section, prove that $\phi = c\omega$ for some constant c.

In other words, there's only a one-dimensional space of forms anyways (once ω is fixed) and they are literally just multiples of ω. Hence Tate's thesis only involves constructing L-functions for each given Hecke character ω.

It is traditional that for Tate's thesis, the Hecke character is denoted by χ rather than ω.

§4.3 Local functional equation

Let F be a local field, and $\eta \colon F^\times \to S^1$ a unitary character on it. Recall that η is unramified if it is trivial on $\mathcal{O}^\times$, which is equivalent to $\eta(x) = |x|^{i\lambda}$ for $\lambda \in \mathbb{R}$

§4.3.1 Non-Archimedean definition

If F is non-Archimedean, the local L-factors are defined in the following way:

$$L(s, \eta) = \begin{cases} 1 & \text{ramified} \\ (1 - \eta(\varpi)q^{-s})^{-1} & \text{unramified.} \end{cases}$$

More generally, we may define for any Schwartz function f the local zeta integral

$$Z(s, \eta, f) = \int_{F^\times} |x|^s \eta(x) f(x) \; dx^\times.$$

We justify "more generally" right away with the following calculation.

Proposition 4.3.1 (Zeta integral generalizes L-factor, at least in unramified case)
Suppose F is non-Archimedean and η is a unramified unitary character on F. Then we have

$$L(s, \eta) = Z(s, \eta, 1_{\mathcal{O}}).$$

In other words, L is a special case of Z with $f = 1_{\mathcal{O}}$.

Proof. By condition, $\eta(x) = |x|^{i\lambda}$ for some $\lambda \in \mathbb{R}$ with η trivial on $\mathcal{O}$. Show both equal to $Z(s + i\lambda, 1, 1_{\mathcal{O}})$. Poonen left this as homework so I didn't do it. □

The ramified case is more annoying.

Proposition 4.3.2 (Ramified case)
Suppose F is non-Archimedean and η is a unitary character on F with conductor $\mathfrak{p}^n$. Then

$$Z(s, \eta, 1_{1+\mathfrak{p}^n}) = q^{-n}.$$

Of course since $L(s, \eta) = 1$ in this situation we could also write $Z(s, \eta, 1_{1+\mathfrak{p}^n}) = q^{-n} L(s, \eta)$, but that would be silly.

§4.3.2 Archimedean definition

If $F = \mathbb{R}$, we instead have the complicated formula

$$L(s, \underbrace{|x|^\lambda \operatorname{sign}(x)^\varepsilon}_{=\eta}) = \pi^{-\frac{1}{2}(s+\lambda+\varepsilon)/2}\Gamma\left(\frac{s+\lambda+\varepsilon}{2}\right)$$

and when $F = \mathbb{C}$ the formula

$$L(s, \underbrace{|x|^{2\lambda}(x/|x|)^n}_{=\eta}) = 2(2\pi)^{-(s+\lambda+\frac{1}{2}|n|)}\Gamma\left(s+\lambda+\frac{|n|}{2}\right).$$

Again, these can be viewed as special cases of the zeta integral.

Proposition 4.3.3 (Archimedean relation)
We have

$$L(s,\eta) = \begin{cases} Z(s, e^{-\pi x^2}, \eta) & \eta = |x|^\lambda, F = \mathbb{R} \\ iZ(s, e^{-\pi x^2}, \eta) & \eta = |x|^\lambda \operatorname{sign}(x), F = \mathbb{R} \\ (-i)^a \dots Z(s, \dots, \dots) & F = \mathbb{C} \end{cases}$$

Proof. Bash. □

§4.3.3 The local functional equation for a test function

We can coalesce the earlier three results in the following lemma.

Proposition 4.3.4 (The local functional equation holds for our test function)
Let F be a local field, and η a unitary character on it. Then there exists *single, nice explicit choice*, of function f, such that

$$\frac{Z(1-s, \widehat{f}, \overline{\eta})}{L(1-s, \overline{\eta})} = \varepsilon(s,\eta) \cdot \frac{Z(s,f,\eta)}{L(s,\eta)}.$$

holds for $0 < \sigma < 1$, with ε of exponential type in s. Moreover, the left-hand side is nonvanishing holomorphic when $\sigma > 0$, while the right-hand side is nonvanishing holomorphic when $\sigma < 1$.

The choice of f is called a *test function*, and it's just the function we chose in the earlier proofs. We will see this lemma verifies the functional equation for a single value of f.

Proof. Earlier, we saw that each $L(s,\eta)$ is basically equal (up to some constant) to $Z(f,s,\eta)$ for some choice of f:

- $f = \mathcal{O}$ if η is unramified on non-Archimedean F
- $f = 1_{1+\mathfrak{p}^n}$ if η is ramified on non-Archimedean F
- $f = e^{-\pi x^2}$ if $F = \mathbb{R}$
- …if $F = \mathbb{C}$.

So naturally, we use this as our function f. This means the right-hand side is basically known at this point. The difficulty is to calculate the left-hand side.

For the $\mathbb{R}$ and $\mathbb{C}$ case, this is not much different. It's more difficult in the non-Archimedean case and requires a Gauss sum. □

§4.3.4 The local functional equation

The main result is the earlier local functional equation holds for any function f and the epsilon factor does

Theorem 4.3.5 (Local functional equation)

For any Schwartz function f and unitary character η,

$$\frac{Z(1-s,\widehat{f},\overline{\eta})}{L(1-s,\overline{\eta})} = \varepsilon(s,\eta)\cdot\frac{Z(s,f,\eta)}{L(s,\eta)}.$$

Moreover, the left-hand side is nonvanishing holomorphic when $\sigma > 0$, while the right-hand side is nonvanishing holomorphic when $\sigma < 1$.

Proof. It suffices to prove that for arbitrary Schwartz functions f and g we have

$$Z(s,\eta,f)Z(1-s,\overline{\eta},\widehat{g}) = Z(s,\eta,g)Z(1-s,\overline{\eta},\widehat{f}).$$

Bash with Fubini theorem. □

§4.4 Global functional equation

A Schwartz function on $\mathbb{A}_K$ is a function $\prod_{v\in K} f_v$ such that $f_v = 1_{\mathcal{O}_v}$ almost everywhere. For such a function we may define the global zeta integral

$$Z(s,f,\eta) = \int_{\mathbb{A}^\times} |x|^s \eta(x) f(x)\, d^\times x$$

in nearly the same way as before. In fact,

$$Z(s,f,\eta) = \prod_v Z(s,f_v,\eta_v).$$

Theorem 4.4.1

This integral converges when $\sigma > 1$ and satisfies

$$Z(s,f,\eta) = Z(1-s,\widehat{f},\overline{\eta}).$$

Proof. Poisson summation. □

Finally, we may define

$$L(s,\eta) = \prod_v L(s,\eta_v)$$

$$\varepsilon(s,\eta) = \prod_v \varepsilon(s,\eta_v)$$

and multiply everything together to get that

$$L(s,\eta) = \varepsilon(s,\eta)L(1-s,\overline{\eta}).$$

5 Adelization

write

(how to make a modular form into an automorphic form)

§5.1 Derivation of modular forms via representation theory

TODO: I have no memory of writing this section or what it's about

Let f be a classical cusp form of weight k. Then we can associate it to a function ϕ on $\Gamma\backslash \operatorname{SL}(2,\mathbb{R})$ by

$$\phi_f(g) = f(g(i))(c \cdot g(i) + d)^{-k}.$$

This is a bijection between modular forms and functions satisfying certain conditions.

Similarly, given a Maass form ϕ we can define $\phi_f(i) = f(g(i))$. Again so-and-so bijection.

On the other hand the center of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ is given by

$$\mathcal{Z} = \mathbb{C}[\mathrm{id}, \Delta]$$

where Δ is the Laplacian (Casimir element) mentioned earlier. Since Δ is in the center, it acts as a scalar in every irreducible representation.

III

Automorphic forms and representations

6 Automorphic representations

Towards the start of the text, we saw how the classical modular forms and their counterpart Maass forms naturally arose from representations of $L^2(\Gamma\backslash \mathrm{GL}_2(\mathbb{R}))$.

§6.1 Automorphic forms

The definition of an adelic automorphic form was stated earlier, except for the technical niceness condition.

- Smooth
- K-finite
- $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the center of the universal enveloping algebra of $U(\mathfrak{gl}(n, K_v))$,
- moderate growth.

The **cusp forms** are those which obey the additional hypothesis

$$\int \phi\left(\begin{bmatrix} I_r & X \\ & I_s \end{bmatrix}\right)\, dX = 0.$$

§6.2 Automorphic representations, and admissible representations

Definition 6.2.1. An **admissible representation** (π, V) of $\mathrm{GL}_2(\mathbb{A})$ is a representation of $\mathrm{GL}_2(\mathbb{A}_{\text{fin}})$ with a commuting $(\mathfrak{g}_\infty, K_\infty)$-module structure with the additional constraint that every vector of V is K-finite and every isotypic part is finite dimensional.

Definition 6.2.2. An **automorphic representation** is a irreducible representation of $\mathrm{GL}_2(\mathbb{A}_{\text{fin}})$ with a commuting $(\mathfrak{g}_\infty, K_\infty)$-module structure which can be realized as the quotient of a submodule of the space of automorphic forms of some central quasicharacter.

Proposition 6.2.3

Automorphic representations are admissible.

Definition 6.2.4. If F is a non-Archimedean local field, an admissible representation of $\mathrm{GL}_2(F)$ is an actual representation such that every vector has open stabilizer and every $K = \mathrm{GL}_2(\mathcal{O}_F)$-isotypic part is finite dimensional. We say a spherical representation is one with a nonzero K-fixed vector, which we also say is spherical.

If F is an Archimedean local field, we instead want a $(\mathfrak{g}_\infty, K_v)$-module with finite-dimensional isotypic parts.

Theorem 6.2.5 (Tensor product theorem)

Let (V, π) be an irreducible admissible representation. Then we can choose (V_v, π_v) for every place v such that there exists a nonzero spherical vector $\xi_v^0 \in V_v$ for almost all v, such that

$$V = \widehat{\bigotimes}(V_v, \pi_v)$$

where the restricted tensor product uses ξ_0^v.

The space of cusp forms contains each representation at most once; actually following stronger result holds.

Theorem 6.2.6 (Strong multiplicity one theorem)

Let (π, V) and (π, V') be irreducible admissible subrepresentations of the space $\operatorname{CuspForms}(\operatorname{GL}(n, K) \backslash \operatorname{GL}(n, \mathbb{A}_k), \omega)$. If $\pi_v = \pi'_v$ for almost all v then $V = V'$.

7 Whittaker models

§7.1 Local uniqueness

Let F be a non-Archimedean local field, $G = \mathrm{GL}_2(F)$, $N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$, and ψ_F a fixed additive character. Since $N \cong F$, we can think of ψ_F as a one-dimensional representation of N.

Recall that we have the **induced representation**

$$\mathrm{Ind}_N^G \psi_F = \left\{ W \colon G \to \mathbb{C} \mid W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot g \right) = \psi_F(x) W(g) \ \forall x \in F, g \in \mathrm{GL}_2(F) \right\}.$$

Definition 7.1.1. A **Whittaker model** $\mathcal{W}$ of (V, π) is a subrepresentation of $\mathrm{Ind}_N^G \psi_F$ isomorphic to $\mathcal{W}$.

Theorem 7.1.2 (Local uniqueness)

An irreducible admissible representation of $\mathrm{GL}_2(F)$ has at most one Whittaker model.

Proof. By Frobenius reciprocity reduce to fact about functional. □

§7.2 Global uniqueness

If K is a global field, a **Whittaker model** is a subspace of

$$\left\{ W \colon G \to \mathbb{C} \mid W\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot g \right) = \psi_{\mathbb{A}/K}(x) W(g) \ \forall x \in K, g \in \mathrm{GL}_2(K) \right\}$$

closed under translation by G. We also assume the functions in $\mathcal{W}$ are smooth, K-finite, and of moderate growth.

Theorem 7.2.1 (Global uniqueness)

An irreducible admissible representation π of $G = \mathrm{GL}_2(\mathbb{A})$ has a Whittaker model if and only if each π_v has a Whittaker model. If so it is unique and equals the sums of $g \mapsto \prod_v W_v(g_v)$, where $W_v \in \mathcal{W}_v$ and for almost all v we have W_v equal to the spherical element of $\mathcal{W}_v$, normalized to equal 1 on $K_v = \mathrm{GL}_2(\mathcal{O}_v)$.

§7.3 Automorphic cuspidal representations have Whittaker models

Theorem 7.3.1 (Whittaker model for an automorphic cuspidal representation)
Let V be an automorphic *cuspidal* representation. For each $\phi \in V$ define

$$W_\phi(g) = \int_{\mathbb{A}/K} \phi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) \psi_K(-x)\, dx.$$

The space $\mathcal{W} = \{W_\phi \mid \phi \in V\}$ is a Whittaker model. It satisfies Fourier expansion

$$\phi(g) = \sum_{\alpha \in K^\times} W_\phi\left(\begin{bmatrix} \alpha & \\ & 1 \end{bmatrix} g\right).$$

The multiplicity one theorem follows from this.

§7.4 The zeta integral

Given an automorphic cuspidal ϕ and a central quasicharacter ξ we define

$$\begin{aligned} Z(s,\phi,\xi) &\overset{\text{def}}{=} \int_{\mathbb{A}^\times/K^\times} \phi\left(\begin{bmatrix} y & \\ & 1 \end{bmatrix}\right) |y|^{s-\frac12} \xi(y)\, d^\times y \\ &= \sum_{\alpha \in K^\times} \int_{\mathbb{A}^\times/K^\times} W_\phi\left(\begin{bmatrix} y & \\ & 1 \end{bmatrix}\right) |y|^{s-\frac12} \xi(y)\, d^\times y \\ &= \int_{\mathbb{A}^\times} W_\phi\left(\begin{bmatrix} y & \\ & 1 \end{bmatrix}\right) |y|^{s-\frac12} \xi(y)\, d^\times y. \end{aligned}$$

If $\phi = \bigotimes_v \phi_v$ and $W_\phi = \bigotimes_v W_v$ then $Z(s,\phi,\xi) = \prod_v Z_v(s, W_v, \xi_v)$.

In general, if F is a non-Archimedean local field then most irreps of F are isomorphic to $\pi(\chi_1,\chi_2)$, the so-called principle series. Then $\alpha_1 = \chi_1(\varpi)$ and $\alpha_2 = \chi_2(\varpi)$ are called the Satake parameters.

Theorem 7.4.1
If v is unramified then

$$W_v\left(\begin{bmatrix} y & \\ & 1 \end{bmatrix}\right) = \begin{cases} q^{-m/2} \cdot \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} & m \ge 0 \\ 0 & m < 0 \end{cases}$$

where $m = \operatorname{ord}_v(y)$. Consequently, a direct calculation shows

$$Z_v(s, W_v, \xi_v) = L_v(s, \pi_v, \xi_v) = \left(1 - \alpha_1 \xi(\varpi) q^{-s}\right)^{-1} \left(1 - \alpha_2 \xi(\varpi) q^{-s}\right)^{-1}.$$

Since the Satake parameters of the contragredient $\widehat{\pi_v}$ are exactly α_1^{-1}, α_2^{-1}, it follows that

$$L_v(s, \widehat{\pi}_v, \xi_v^{-1}) = L_v(s, \pi_v, \omega_v^{-1}\xi_v^{-1})$$

Using this, we get a functional equation

$$L(s,\pi,\xi) = L(1-s, \widehat{\pi}, \xi^{-1}) \cdot \text{gamma crap}.$$

Dubious: discuss