Explicit formulas for weighted orbital integrals for the inhomogeneous and semi-Lie arithmetic fundamental lemmas conjectured for the full spherical Hecke algebra

by

Evan Chen

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ABSTRACT

As an analog to the Jacquet-Rallis fundamental lemma that appears in the relative trace formula approach to the Gan-Gross-Prasad conjectures, the arithmetic fundamental lemma was proposed by Wei Zhang and used in an approach to the arithmetic Gan-Gross-Prasad conjectures. The Jacquet-Rallis fundamental lemma was recently generalized by Spencer Leslie to a statement holding for the full spherical Hecke algebra. In the same spirit, there is a recent conjectural generalization of the arithmetic fundamental lemma to the full spherical Hecke algebra. This paper formulates another analogous conjecture for the semi-Lie version of the arithmetic fundamental lemma proposed by Yifeng Liu. Then this paper produces explicit formulas for particular cases of the weighted orbital integrals in the two conjectures mentioned above.

Thesis supervisor: Wei Zhang
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Chapter 1

Introduction

Throughout this whole paper, $p > 2$ is a prime, F is a finite extension of $\mathbb{Q}_p$, and E/F is an unramified quadratic field extension.

1.1 Brief history and motivation for the arithmetic fundamental lemma

The primary motivation for this paper arises from the study of conjectured variants of the arithmetic fundamental lemma for spherical Hecke algebras proposed in [LRZ24]. This section briefly provides an overview of the historical context that led to the formulation of these conjectures. This history is also summarized in Figure 1.1.

Because this subsection is meant for motivation only, in this survey we do not give complete definitions or statements, being content to outline a brief gist. A more detailed account can be found in [Zha24a].

$$\begin{tikzcd}
\text{[Wal85]} \arrow[d] \\
\text{[GP92; GP94]} \arrow[d] \\
\text{GGP [GGP12]} \arrow[d, dotted, leftrightarrow, "\text{analog}"] & \text{FL [JR11]} \arrow[l, Rightarrow, "\text{used to prove}"'] \arrow[r] \arrow[d, dotted, leftrightarrow, "\text{analog}"] & \text{[Les23]} \arrow[d, dotted, leftrightarrow, "\text{analog}"] \\
\text{Arith. GGP [GGP12]} & \text{AFL [Zha12]} \arrow[l, Rightarrow, "\text{used to prove}"'] \arrow[r] & \text{[LRZ24]} \\
\text{[YZZ13]} \arrow[u] \\
\text{[GZ86]} \arrow[u]
\end{tikzcd}$$

Figure 1.1: The history behind the fundamental lemma and its arithmetic counterpart. Unlabeled arrows denote generalizations.

1.1.1 The GGP conjectures, and the fundamental lemma of Jacquet-Rallis

In modern arithmetic geometry, a common theme is that there are deep connections between geometric data with the values of related L-functions.

This story begins with a result of Waldspurger [Wal85] which showed a formula relating the nonvanishing of an automorphic period integral to the central value of the same L-functions. Later, a conjecture that generalizes Waldspurger's formula was proposed by Gross-Prasad in [GP92; GP94]. This was further generalized to a series of conjectures now known as the Gan-Gross-Prasad (GGP) conjectures, which were proposed in 2012 in [GGP12]; they generalize the Gross-Prasad conjecture to different classical groups. Specifically, the GGP conjecture predict the nonvanishing of a period integral based on the values of the L-function of a certain cuspidal automorphic representation.

In 2011, Jacquet-Rallis [JR11] proposed an approach to the Gross-Prasad conjectures for unitary groups via a relative trace formula (RTF). The idea is to compare an RTF for the general linear group to one for a unitary group. This approach relies on a so-called *fundamental lemma*, which links values of certain orbital integrals over two reductive groups over a non-Archimedean local field.

Let's be a bit more precise about what this fundamental lemma says. Let $\mathbb{V}_n^+$ denote the split E/F-Hermitian space of dimension n (unique up to isomorphism), fix a unit vector w_0 in it, and let $(\mathbb{V}_n^+)^\flat$ be the orthogonal complement of the span of w_0. Let $(G')^\flat := \mathrm{GL}_{n-1}(E)$, $G' := \mathrm{GL}_n(E)$, $G^\flat := \mathrm{U}((\mathbb{V}_n^+)^\flat)(F)$ and $G := \mathrm{U}(\mathbb{V}_n^+)(F)$. For certain

$$\gamma \in (G')^\flat \times G', \qquad g \in G^\flat \times G$$

the Jacquet-Rallis fundamental lemma proposes a relation between two orbital integrals. Specifically, it supplies a relation between

- the orbital integral of γ with respect to the indicator function $\mathbf{1}_{(K')^\flat \times K'}$ of the natural hyperspecial compact subgroup

  $$(K')^\flat \times K' \subset (G')^\flat \times G' = \mathrm{GL}_{n-1}(E) \times \mathrm{GL}_n(E);$$

  and

- the orbital integral of g with respect to the indicator function $\mathbf{1}_{K^\flat \times K}$ of the natural hyperspecial compact subgroup

  $$K^\flat \times K \subset G^\flat \times G = \mathrm{U}((\mathbb{V}_n^+)^\flat)(F) \times \mathrm{U}(\mathbb{V}_n^+)(F).$$

In other words, it states that

$$\mathrm{Orb}(\gamma, \mathbf{1}_{(K')^\flat \times K'}) = \omega(\gamma)\, \mathrm{Orb}(g, \mathbf{1}_{K^\flat \times K}) \tag{1.1}$$

where $\omega(\gamma)$ is a suitable *transfer factor*. The fundamental lemma has since been proved completely; a local proof was given by Beuzart-Plessis [Beu21] while a global proof was given for large characteristic by W. Zhang [Zha21b].

1.1.2 The arithmetic GGP conjectures, and the arithmetic fundamental lemma

At around the same time Waldspurger's formula was published, Gross-Zagier [GZ86] proved a formula relating the height of Heegner points on certain modular curves to the derivative at $s = 1$ of certain L-functions. The Gross-Zagier formula was then generalized over several decades, culminating in [YZZ13] where the formula is established for Shimura curves over arbitrary totally real fields.

An arithmetic analogue of the original Gan-Gross-Prasad conjectures, which we henceforth refer to as *arithmetic GGP* [GGP12], can then be formulated, further generalizing Gross-Zagier's formula. Here the modular curves in Gross-Zagier are replaced with higher dimensional Shimura varieties. Rather than the period integrals considered previously, one instead takes intersection numbers of cycles on some Shimura varieties. Specifically, if one considers the Shimura variety associated to a classical group, the arithmetic GGP conjecture predicts a relation between intersection numbers on this Simura variety with the central derivative of automorphic L-functions.

By analogy to the work Jacquet-Rallis [JR11], the arithmetic GGP conjectures should have a corresponding *arithmetic fundamental lemma* (henceforth AFL), which was proposed by W. Zhang [Zha12, Conjecture 2.9]. The arithmetic fundamental lemma then relates the derivative of the weighted orbital integral with respect to the indicator function $\mathbf{1}_{(K')^\flat \times K'} \in \mathcal{H}((G')^\flat \times G, (K')^\flat \times K')$, that is

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Orb}(\gamma, \mathbf{1}_{(K')^\flat \times K'}, s)$$

for $\gamma \in (G')^\flat \times G'$, to arithmetic intersection numbers on a certain Rapoport-Zink formal moduli space. The AFL in [Zha12] has since been proven over p-adic fields for any large prime p in Mihatsch-Zhang [MZ24], W. Zhang [Zha21b], and then for any odd prime p by Z. Zhang [Zha23].

1.1.3 The semi-Lie version of the AFL proposed by Liu

There is another different version of the AFL, proposed by Yifeng Liu in [Liu21, Conjecture 1.12], which is often referred to as the *semi-Lie version* of the AFL. Its statement has been shown to be equivalent to AFL, see [Liu21, Remark 1.13] and is thus now proven. In contrast, the original AFL proposed by Zhang in [Zha12, Conjecture 2.9] is sometimes referred to as the *group version*.

A more detailed account of this equivalence is described in [Liu21, §1.4].

1.1.4 Generalizations of FL and AFL to the full spherical Hecke algebra

Recently it was shown by Leslie [Les23] that in fact (1.1) holds in greater generality where the indicator function $\mathbf{1}_{K^\flat \times K}$ can be replaced by any element in the spherical Hecke algebra $\varphi \in \mathcal{H}((G')^\flat \times G', (K')^\flat \times K')$. In that case, $\mathbf{1}_{(K')^\flat \times K}$ needs to be replaced by the corresponding element φ' under a certain base change homomorphism

$$\mathcal{H}((G')^\flat \times G', (K')^\flat \times K') \to \mathcal{H}(G^\flat \times G, K^\flat \times K)$$
$$\varphi' \mapsto \varphi$$

In that case, the identity (1.1) still hold as

$$\operatorname{Orb}(\gamma, \varphi') = \omega(\gamma) \operatorname{Orb}(g, \varphi). \tag{1.2}$$

To complete the analogy illustrated in Figure 1.1, there should thus be a generalization of the

AFL in which $\mathbf{1}_{(K')^\flat \times K}$ is replaced by any element of the Hecke algebra $\mathcal{H}((G')^\flat \times G, (K')^\flat \times K)$. This conjectural generalization (for the group version of the AFL) is [LRZ24] and we discuss it momentarily; while the analogous conjectural generalization for the semi-Lie version of the AFL is our Conjecture 1.2.2 also discussed in the next section.

1.2 Formulation of AFL conjectures to the full spherical Hecke algebra

1.2.1 The inhomogeneous version of the arithmetic fundamental lemma for spherical Hecke algebras proposed by Li-Rapoport-Zhang

In contrast to the vague motivational cheerleading in the previous section, starting in this section we will give more precise statements, even though we will necessarily need to reference definitions appearing in later sections.

Retain the notation $G' \coloneqq \mathrm{GL}_n(E)$, and $G \coloneqq \mathrm{U}(\mathbb{V}_n^+)(F)$, with $K' \subset G'$ and $K \subset G$ the natural hyperspecial compact subgroups. Also, let q denote the residue characteristic of F. Moreover, define the symmetric space

$$S_n(F) \coloneqq \{g \in \mathrm{GL}_n(E) \mid g\bar{g} = \mathrm{id}_n\}.$$

Finally, let $\mathbb{V}_n^-$ be the non-split Hermitian space of dimension n (unique up to isomorphism), while $\mathbb{V}_n^+$ continues to denote the split one (again unique up to isomorphism).

For concreteness, we focus on the inhomogeneous version of the arithmetic fundamental lemma, which is [LRZ24, Conjecture 6.2.1]. This allows us to deal with just G' instead of $(G')^\flat \times G'$, etc., so that the Hecke algebra $\mathcal{H}((G')^\flat \times G', (K')^\flat \times K')$ can be replaced by the simpler one $\mathcal{H}(\mathrm{GL}_n(E)) \coloneqq \mathcal{H}(G', K')$. Similarly, $\mathcal{H}(G^\flat \times G, K^\flat \times K)$ can be replaced by the simpler $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) := \mathcal{H}(G, K)$.

The AFL conjecture provides a bridge between a geometric left-hand side (given by an intersection number) and an analytic right-hand side (given by a weighted orbital integral). Stating it requires several pieces of data. We only mention these pieces by name here, with definitions given later:

- On the geometric side, we have an intersection number. It uses the following ingredients.
  - We choose a regular semisimple element $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$. (The notation $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ denotes the regular semisimple elements of $\mathrm{U}(\mathbb{V}_n^-)$, etc. The notion of regular semisimple is defined in Definition 3.1.4.)
  - We choose a function $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ from the spherical Hecke algebra, defined in Chapter 2.
  - We define a certain *intersection number* $\mathrm{Int}(g, f)$ in Definition 13.3.2. These intersection numbers take place in a Rapoport-Zink space described in Chapter 13.
- On the analytic side, we have a weighted orbital integral. It uses the following ingredients.
  - We choose a regular semisimple element $\gamma \in S_n(F)_{\mathrm{rs}}$.
  - We choose a test function ϕ which comes from a certain $\mathcal{H}(\mathrm{GL}_n(E))$-module that we will denote $\mathcal{H}(S_n(F))$. This module $\mathcal{H}(S_n(F))$ is defined in Chapter 2.
  - The weighted orbital integral $\mathrm{Orb}(\gamma, \phi, s)$ is itself defined in Definition 5.1.1. (It is connected to an unweighted orbital integral on the unitary group according to Theorem 5.1.3.) We abbreviate

$$\partial \operatorname{Orb}(\gamma, \phi) := \left.\frac{\partial}{\partial s}\right|_{s=0} \operatorname{Orb}(\gamma, \phi, s).$$

  - There is also an extra transfer factor $\omega \in \{\pm 1\}$ which we define in Chapter 12.

- We need a way to connect the inputs between the two parts of our conjecture. Specifically, f and ϕ need to be linked, and g and γ need to be linked. This is done as follows.
  - Once the regular semisimple element $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ is chosen, we require $\gamma \in S_n(F)_{\mathrm{rs}}$ to be a *matching* element. This matching is defined in Definition 3.2.1. (Alternatively, one could imagine picking $\gamma \in S_n(F)_{\mathrm{rs}}$ first and finding corresponding g. It turns out γ will match an element of $\mathrm{U}(\mathbb{V}_n^{\pm})_{\mathrm{rs}}$ in general, and the conjecture is only formulated in the case where $g \in \mathrm{U}(\mathbb{V}_n^-)$.)
  - Once $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ is chosen, we select

$$\phi = (\mathrm{BC}_{S_n}^{\eta^{n-1}})^{-1}(f)$$

    to be the image of f under a *base change*. This base change is defined and then calculated explicitly for $n = 3$ in Chapter 4.

With all our protagonists now having names and references, we can now state the conjecture proposed in [LRZ24].

Conjecture 1.2.1 (Inhomogeneous version of the AFL for the full spherical Hecke algebra, [LRZ24, Conjecture 6.2.1])**.** *Let $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ be any element of the Hecke algebra, and let*

$$\phi = (\mathrm{BC}_{S_n}^{\eta^{n-1}})^{-1}(f) \in \mathcal{H}(S_n(F))$$

be its image under base change as defined in Chapter 4. Then for matching (as defined in Definition 3.2.1) regular semisimple elements

$$g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}} \longleftrightarrow \gamma \in S_n(F)_{\mathrm{rs}}$$

we have

$$\mathrm{Int}\left((1, g), \mathbf{1}_{K^\flat} \otimes f\right) = -\frac{\omega(\gamma)}{\log q} \partial \, \mathrm{Orb}(\gamma, \phi) \tag{1.3}$$

where the weighted orbital integral $\mathrm{Orb}(\dots)$ *is defined in Definition 5.1.1, the transfer factor* ω *is defined in Chapter 12, and the intersection number* $\mathrm{Int}(\dots)$ *is defined in Chapter 13.*

At present, the (inhomogeneous) AFL is the case where $f = \mathbf{1}_K$, and is thus proven. Note that in the case of interest where $\gamma \in S_n(F)_{\mathrm{rs}}$ matches an element of $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ (rather than $\mathrm{U}(\mathbb{V}_n^+)_{\mathrm{rs}}$), the actual value of $\mathrm{Orb}(\gamma, \phi, s)$ at $s = 0$ is always zero by Theorem 5.1.3; so the conjecture instead looks at the first derivative at $s = 0$.

The generalized conjecture is also proved in full for $n = 2$ in [LRZ24, Theorem 1.0.1] (in that reference, our n denotes the $n+1$ in *loc. cit.*). The part of the calculation involving the weighted orbital integral has two parts:

- The calculation makes $\mathrm{BC}_{S_n}^{\eta^{n-1}}$ completely explicit in a natural basis for $n = 2$. The result is [LRZ24, Lemma 7.1.1].
- The calculation makes explicit the value of the weighted orbital integral

  $$\mathrm{Orb}(\gamma, \phi, s)$$

  for any $\gamma \in S_n(F)_{\mathrm{rs}}$ and $\phi \in \mathcal{H}(S_n(F))$, in terms of invariants of γ and a decomposition of ϕ in a natural basis. The result is [LRZ24, Proposition 7.3.2].

Combining these two (hence obtaining the right-hand side of (1.3)) with a calculation of intersection numbers in [LRZ24, Corollary 7.4.3] (which is the left-hand side of (1.3)) shows that Conjecture 1.2.1 holds for $n = 2$, cf. [LRZ24, Theorem 7.5.1].

1.2.2 A proposed arithmetic fundamental lemma for spherical Hecke algebras in the semi-Lie case

The primary focus of this paper is an analogous conjecture to Conjecture 1.2.1 for the semi-Lie version (also called the Fourier-Jacobi case). It serves to complete the analogy given in Table 1.1.

Version	AFL for $\mathbf{1}$ (now proven)	Full spherical Hecke
Group	[Zha12, Conjecture 2.9]	[LRZ24, Conjecture 6.2.1]
Semi-Lie	[Liu21, Conjecture 1.12]	Conjecture 1.2.2

Table 1.1: Table showing the analogy between the proposed Conjecture 1.2.2 and the existing conjectures.

In this variation, as in [Liu21], rather than matching $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ to $\gamma \in S_n(F)_{\mathrm{rs}}$, we consider an augmented space larger than $\mathrm{U}(\mathbb{V}_n^-)$ and $S_n(F)$. Specifically, one considers a matching between tuples of regular semisimple elements

$$(g, u) \in (\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}} \longleftrightarrow (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$$

where $V_n'(F) = F^n \times (F^n)^\vee$ (defined in Definition 3.3.1) consists of pairs of column vectors and row vectors of length n, and the space $\mathbb{V}_n^-$ is defined in Definition 13.2.2. The notion of *matching* is defined in Definition 3.3.1 as well.

Meanwhile, we still use the same test functions f and ϕ, as we did for [LRZ24, Conjecture 6.2.1]. The derivative of interest will be denoted

$$\partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi) := \left.\frac{\partial}{\partial s}\right|_{s=0} \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s).$$

Finally, we also update the definition of intersection number to accommodate the new term u in Definition 13.4.3.

Conjecture 1.2.2 (Semi-Lie version of the AFL for the full spherical Hecke algebra)**.** *Let $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ be any element of the Hecke algebra, and let*

$$\phi = (\mathrm{BC}_{S_n}^{\eta^{n-1}})^{-1}(f) \in \mathcal{H}(S_n(F))$$

be its image under base change defined in Chapter 4. Then for matching (as defined in

Definition 3.3.1) regular semisimple elements

$$(g, u) \in (\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}} \longleftrightarrow (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$$

we have

$$\operatorname{Int}((g,u), f) = -\frac{\omega(\gamma, \mathbf{u}, \mathbf{v}^\top)}{\log q} \partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}) \tag{1.4}$$

where the orbital integral $\operatorname{Orb}(\dots)$ *is defined in Definition 8.1.1, the transfer factor is defined in Chapter 12, and the intersection number* $\operatorname{Int}(\dots)$ *is defined in Chapter 13.*

Note that in this version the new orbital integral $\operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s)$ is defined similarly. However, as far as we know, no analog of Theorem 5.1.3 (linking it to an unweighted orbital integral on the unitary side) appears in the literature. Thus we record the corresponding statement as Conjecture 8.1.3. Like before, Conjecture 8.1.3 predicts that $\operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, 0) = 0$ in the case of interest, which in this case can be checked independently.

Remark 1.2.3. For $n = 1$, the Hecke algebra $\mathcal{H}(S_n(F))$ is trivial and therefore Conjecture 1.2.2 becomes a special case of the known result [Liu21]. Therefore $n = 2$ is the first case of Conjecture 1.2.2 worth examining.

1.2.3 A proposed large image conjecture

We let $S_n(F)^-_{\mathrm{rs}}$ (resp. $(S_n(F) \times V_n'(F))^-_{\mathrm{rs}}$) denote the subsets of $S_n(F)_{\mathrm{rs}}$ (resp. $(S_n(F) \times V_n'(F))_{\mathrm{rs}}$) consisting of regular semisimple elements that also match with an element of $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ (resp. $(\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}}$), i.e. those for which Conjecture 1.2.1 and Conjecture 1.2.2 apply.

In [LRZ24], it was observed that for $n = 2$ there was a rather large space of $\phi \in \mathcal{H}(S_2(F))$ such that

$$\partial \operatorname{Orb}(\gamma, \phi) = 0$$

held identically across all $\gamma \in S_2(F)^-_{\mathrm{rs}}$. If we consider the map

$$\begin{aligned} \partial \operatorname{Orb}\colon \mathcal{H}(S_2(F)) &\to \mathcal{C}^\infty(S_2(F)^-_{\mathrm{rs}}) \\ \phi &\mapsto (\gamma \mapsto \partial \operatorname{Orb}(\gamma, \phi)) \end{aligned}$$

then [LRZ24, Theorem 8.2.3] in fact shows this map has a kernel of codimension 2 (equivalently, the image of the map was only two-dimensional). They thus stated [LRZ24, Conjecture 1.0.2] which asserts that for $n \geq 2$, a similarly defined map (albeit for more than one Hecke algebra) has a large kernel.

It is therefore natural to ask whether a similar large kernel result could hold for the analogous orbital integral in the semi-Lie case. We instead propose the following *large image* conjecture, which we have proved for $n = 2$.

Conjecture 1.2.4 (Large image conjecture for $(S_n(F) \times V_n'(F))^-_{\mathrm{rs}}$)**.** *Let $n \geq 2$. The map*

$$\begin{aligned} \partial \operatorname{Orb}\colon \mathcal{H}(S_n(F)) &\to \mathcal{C}^\infty\left((S_n(F) \times V_n'(F))^-\right) \\ \phi &\mapsto \left((\gamma, \mathbf{u}, \mathbf{v}^\top) \mapsto \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi\right)\right) \end{aligned}$$

is injective.

In fact for $n = 2$ we have a more precise result showing that the injectivity essentially comes from $v(\mathbf{u}\mathbf{v}^\top)$ alone in this case. See Theorem 1.3.5 and Theorem 1.3.6 momentarily.

1.3 Results

Most of the results here are dedicated toward the semi-Lie version of the AFL, which is the new contribution provided by this paper. But in Section 1.3.4 we mention some other results we proved for the group version of the AFL.

1.3.1 Formulas for the orbital side of the semi-Lie AFL conjecture for $n = 2$

The main case of interest in this thesis is the new conjectured AFL for the spherical Hecke algebra in the semi-Lie situation in the specific case $n = 2$ where one can provide evidence for the conjecture. On the orbital side, the various ingredients can be described concretely in the following way:

- The Hecke algebra $\mathcal{H}(S_2(F))$ has a natural basis of indicator functions $\mathbf{1}_{K',\leq r}$ for each $r \geq 0$; see Chapter 8 for a definition.
- Suppose $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$. Then under the action $\mathrm{GL}_2(F)$ we may assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is of the form

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix} \right) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$$

  (that is, we can find a $\mathrm{GL}_2(F)$-orbit representative of this form). The parameters a, b, c, d need to satisfy certain dependencies for the matching to hold; these requirements are documented in Lemma 8.4.2.

Then we were able to derive the following fully explicit formula in terms of the representative detailed in Lemma 8.4.2. See Section 10.3 for some concrete examples and illustrations of Theorem 1.3.1.

Theorem 1.3.1 (Explicit orbital integral on $S_2(F) \times V_2'(F)$)**.** *Let*

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix} \right) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$$

satisfy the requirements in Lemma 8.4.2. Let $r \geq 0$.

If $v(e) < 0$ or $v(b)+v(c) < -2r$, then

$$\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}} \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s) = 0$$

holds identically for all $s \in \mathbb{C}$.

Otherwise define

$$\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) := \min\left(\left\lfloor \tfrac{k+(v(b)+r)}{2} \right\rfloor, \left\lfloor \tfrac{(2v(e)+v(c)+r)-k}{2} \right\rfloor, N\right)$$

where

$$N := \min\left(v(e), \tfrac{v(b)+v(c)-1}{2} + r, v(d-a)+r\right).$$

Also, if $v(d-a) < v(e) - r$ and $v(b)+v(c) > 2v(d-a)$, then additionally define

$$\begin{aligned}\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = \min\big(&k - (2v(d-a) - v(b) + r),\\ &(2v(e)+v(c)-2v(d-a)-r) - k, v(e) - v(d-a) - r\big).\end{aligned}$$

Otherwise define $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = 0$. Then we have

$$\begin{aligned}\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}}, s) = &\sum_{k=-(v(b)+r)}^{2v(e)+v(c)+r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)}\right)(q^s)^k\\ &+ \sum_{k=2v(d-a)-v(b)+r}^{2v(e)+v(c)-2v(d-a)-r} (-1)^k \mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) q^{v(d-a)+r} (q^s)^k.\end{aligned}$$

Differentiating this yields the following result:

Corollary 1.3.2 (Derivative at $s=0$ for $S_2(F) \times V_2'(F)$)**.** *Retain the setting of Theorem 1.3.1. Also define $\varkappa := v(e) - (v(d-a)+r)$. If both $\varkappa \ge 0$ and $v(b)+v(c) > 2v(d-a)$, then we have the formula*

$$\frac{(-1)^{v(c)+r}}{\log q} \partial\, \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}})$$

$$= \sum_{j=0}^{N} \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j$$
$$- q^{v(d-a)+r} \cdot \begin{cases} \frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r \right) - \frac{\varkappa}{2} & \text{if } \varkappa \equiv 1 \pmod 2. \end{cases}$$

Otherwise we instead have the formula

$$\frac{(-1)^{v(c)+r}}{\log q} \partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq r}}) = \sum_{j=0}^{N} \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j.$$

The formula simplifies even further if one considers instead $\mathbf{1}_{K'_{S,\leq r}} + \mathbf{1}_{K'_{S,\leq (r-1)}}$; and indeed we will see that this particular combination comes up naturally in Chapter 15 with a special role as the base change mentioned in [LRZ24, Lemma 7.1.1].

Corollary 1.3.3 (The special case $\partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq r}} + \mathbf{1}_{K'_{S,\leq (r-1)}})$). *Retain the setting of Theorem 1.3.1. Also define $\varkappa := v(e) - (v(d-a) + r)$. For $r \geq 1$ define*

$$C := \begin{cases} \frac{\varkappa - 1}{2} & \text{if } \varkappa > 0 \text{ is odd and } v(b) + v(c) > 2v(d-a) \\ \frac{\varkappa + v(b) + v(c) - 2v(d-a) - 1}{2} & \text{if } \varkappa \geq 0 \text{ is even and } v(b) + v(c) > 2v(d-a) \\ v(e) - N & \text{if } v(e) \geq \frac{v(b)+v(c)-1}{2} + r \text{ and } 2v(d-a) > v(b) + v(c) \\ 0 & \text{otherwise} \end{cases}$$
$$C' := \begin{cases} C + 1 & \text{if } \varkappa \geq 0 \text{ and } v(b) + v(c) > 2v(d-a) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\frac{(-1)^{v(c)+r}}{\log q} \partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq r}} + \mathbf{1}_{K'_{S,\leq (r-1)}})$$
$$= (q^N + q^{N-1} + \cdots + 1) + Cq^N + C'q^{N-1}$$

Example 1.3.4 (Examples of Corollary 1.3.3)**.** We show some examples of Corollary 1.3.3:

- When $r = 5$, $v(b) = -20$, $v(c) = 37$, $v(e) = 35$ and $v(d-a) > \frac{v(b)+v(c)}{2} = 8.5$ the derivative in Corollary 1.3.3 equals

$$\log q \cdot (23q^{13} + q^{12} + q^{11} + q^{10} + q^9 + \cdots + q + 1).$$

- When $r = 6$, $v(b) = 10$, $v(c) = 5$, $v(e) = 7$, $v(d-a) > v(e) - r = 1$, the derivative in Corollary 1.3.3 equals

$$-\log q \cdot (q^7 + q^6 + q^5 + \cdots + q + 1).$$

- When $r = 8$, $v(b) = -101$, $v(c) = 1000$, $v(e) = 29$, $v(d-a) = 11$, the derivative in Corollary 1.3.3 equals

$$\log q \cdot (444q^{19} + 445q^{18} + q^{17} + q^{16} + q^{15} + \cdots + q + 1).$$

1.3.2 Kernel and image results for the semi-Lie orbital integral when $n = 2$

As we mentioned our earlier conjecture Conjecture 1.2.4 is true for $n = 2$. More precisely, we have the following two theorems.

Theorem 1.3.5 ($\partial\,\mathrm{Orb}$ is injective even for fixed $\gamma \in S_2(F)$)**.** *Fix any $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$. Then there doesn't exist any nonzero function $\phi \in \mathcal{H}(S_2(F))$ such that*

$$\partial\,\mathrm{Orb}\big((\gamma, \mathbf{u}, \varpi^i \mathbf{v}^\top), \phi\big) = 0$$

holds for every integer i. Thus Conjecture 1.2.4 holds for $n = 2$.

In particular, for $n = 2$ the map

$$\begin{aligned} \partial \operatorname{Orb}\colon \mathcal{H}(S_2(F)) &\to \mathcal{C}^\infty\left((S_2(F) \times V_2'(F))^-\right) \\ \phi &\mapsto \left((\gamma, \mathbf{u}, \mathbf{v}^\top) \mapsto \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi\right)\right) \end{aligned}$$

is indeed injective.

Theorem 1.3.6 (The kernel of $\partial \operatorname{Orb}$ is large for fixed $(\mathbf{u}, \mathbf{v}^\top) \in V_2'(F)$)**.** *Let $N \geq 0$ be an integer. Consider all $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$ for which $v(\mathbf{u}\mathbf{v}^\top) \leq N$. Then the space of $\phi \in \mathcal{H}(S_2(F))$ for which*

$$\partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi\right) = 0$$

holds for all such $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is a $\mathbb{Q}$-vector subspace of $\mathcal{H}(S_2(F))$ whose codimension is at most $N + 2$.

Moreover, this subspace of $\mathcal{H}(S_2(F))$ is not contained in any maximal ideal of $\mathcal{H}(S_2(F))$ when $\mathcal{H}(S_2(F))$ is viewed as a ring under the isomorphism of Chapter 4.

Remark 1.3.7 (On formalizing large kernels)**.** In each case the Hecke algebra is isomorphic as a $\mathbb{Q}$-algebra to $\mathbb{Q}[T]$ for a single variable $T = Y + Y^{-1}$. So actually it's mildly surprising that we get a result on finite codimension. In general, a finite codimension vector subspace of $\mathbb{Q}[T]$ could be contained in a maximal ideal, such as the codimension one subspace $T\mathbb{Q}[T] \subset \mathbb{Q}[T]$. Conversely, a *finite* dimension vector subspace might still not be contained in any maximal ideal, such as the one-dimensional space $\mathbb{Q} \subseteq \mathbb{Q}[T]$.

Thus neither finite codimension nor generating all of $\mathbb{Q}[T]$ as an ideal imply each other. It might be interesting to consider other different ways of formalizing the notion of "large kernel".

1.3.3 The geometric side of the semi-Lie AFL conjecture for $n = 2$

On the geometric side, we were also able to determine the intersection numbers subject to the provisionsal Conjecture 14.4.4.

Theorem 1.3.8 (Semi-Lie AFL for the full Hecke algebra for $n = 2$)**.** *Assume Conjecture 14.4.4. Then our generalized AFL conjecture, Conjecture 1.2.2, holds for $n = 2$.*

The proof of Theorem 1.3.8 is built up gradually throughout the paper, culminating in Chapter 15.

We comment briefly on the strategy of the proof. The proof is made possible because the intersection numbers for $n = 2$ are easier to work with for a few reasons.

- First, one can identify $\mathbb{V}_2^-$ with an E/F-quaternion division algebra, equipped with a compatible Hermitian form defined via quaternion multiplication. This makes it possible to describe $\mathrm{U}(\mathbb{V}_2^-)$ concretely as transformations obtained via left multiplication by an element of E and right multiplication by a quaternion.

- Secondly, it becomes possible to replace the so-called Rapoport-Zink spaces $\mathcal{N}_2$ used in the definition of the intersection number with a Lubin-Tate space $\mathcal{M}_2$. Thus the problem of computing the intersection number $\mathrm{Int}\left((g, u), f\right) \log q$ can be reduced to calculating the intersection of certain special Kudla-Rapoport divisors on the space $\mathcal{M}_2$.

  However, on the Lubin-Tate space $\mathcal{M}_2$, there is a result known as the Gross-Keating formula [GK93] which allows one to make this intersection number fully explicit. One can then match the resulting equation to the formulas described in Corollary 1.3.2 and verify that, under the base change Chapter 4 and the matching condition described in Chapter 3, the two obtained formulas are identical.

The hypothesis Conjecture 14.4.4 is a stipulation that the pullback of two of the divisors behaves in the way one would expect.

1.3.4 Results for $n = 3$ for the group AFL

For the group AFL, we were able to fully compute the orbital integral as well. The result is too involved to state in the introduction, but we give the following summary.

Theorem 1.3.9 (Summary of the weighted orbital integral for $S_3(F)$)**.** *Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$. Then the weighted orbital integral $\mathrm{Orb}(\gamma, \phi, s)$ takes the form*

$$\sum_k P_k(q)(-q^s)^k$$

for some polynomials $P_k(q) \in \mathbb{Z}[q]$, where

- *the summation variable k is some contiguous range of integers,*
- *the polynomials $P_k \in \mathbb{Z}[q]$ are nonzero and satisfy the property that every coefficient of P_k besides possibly the leading coefficient is 1;*
- *both $\deg P_k$ and leading coefficient of P_k are the integer parts of piecewise linear functions in k with slopes in $\{0, \pm\frac{1}{2}, \pm 1\}$.*

The range of the summation, and the aforementioned piecewise linear function(s), can be written explicitly in terms of a particular representative in the orbit of γ.

For a full statement, see Theorems 5.5.2, 5.5.7 and 5.5.10. The calculation corresponds directly to the earlier results [LRZ24, Lemma 7.1.1 and Proposition 7.3.2] which were the case $n = 2$ of the inhomogeneous group version of the AFL (note what [LRZ24] calls n is $n + 1$ in our notation). The methods, which are local in nature, are rather similar to those employed in [Zha12], which can be thought of as the case $r = 0$.

Remark 1.3.10 (Theorem 1.3.9 applies to Theorem 1.3.1)**.** Interestingly, the formula Theorem 1.3.1 for $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))_{\mathrm{rs}}$ actually fits the same template as Theorem 1.3.9, although the semi-Lie formula is more pleasant. We do not have a good explanation why the shapes of the orbital integrals end up being so similar.

We were also able to determine the relevant base changes in Chapter 4. However, we did not complete the comparison on the geometric side in this situation. Thus we do not claim a proof of $n = 3$ of Conjecture 1.2.1, although we imagine such a proof could be completed once a method for determining the intersection numbers explicitly is devised. On the other hand, the orbital data is enough to prove the following result, which serves as an analog to Conjecture 1.2.4.

Theorem 1.3.11 ($\partial\,\mathrm{Orb}\colon \mathcal{H}(S_3(F)) \to \mathcal{C}^\infty(S_3(F)^-_{\mathrm{rs}})$ has large image)**.** *There is no nontrivial $\phi \in \mathcal{H}(S_3(F))$ such that*

$$\partial\,\mathrm{Orb}\,(\gamma, \phi) = 0$$

holds for every $\gamma \in S_3(F)^-_{\mathrm{rs}}$. In other words, the map

$$\begin{aligned} \partial\,\mathrm{Orb}\colon \mathcal{H}(S_3(F)) &\to \mathcal{C}^\infty(S_3(F)^-_{\mathrm{rs}}) \\ \phi &\mapsto (\gamma \mapsto \partial\,\mathrm{Orb}\,(\gamma, \phi)) \end{aligned}$$

is injective, i.e., has image as large as possible, for $n = 3$.

Remark 1.3.12 (Comparison to [LRZ24, Conjecture 1.0.2])**.** Note that Theorem 1.3.11 does not falsify [LRZ24, Conjecture 1.0.2], because the conjecture in *loc. cit.* is formulated for a larger Hecke algebra $\mathcal{H}(S_{n-1}(F)) \otimes \mathcal{H}(S_n(F))$. Meanwhile, here we have only considered an inhomogeneous version of the AFL where the first component is $\mathbf{1}_{K'}$.

1.4 Roadmap

The rest of the paper is organized as follows.

1.4.1 Background information

The paper begins with some general background information.

- In Chapter 2 we provide some preliminary background on the spaces appearing in the overall paper and the Hecke algebras $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ and $\mathcal{H}(S_n(F))$.

- Further background is stated in Chapter 3, where we describe the matching of regular semisimple elements so that we may speak of $S_n(F)^-_{\mathrm{rs}}$ and $(S_n(F) \times V'_n(F))^-_{\mathrm{rs}}$.

- In Chapter 4 we provide reminders on the Satake transform. We also derive concrete formulas for base change when $n = 3$ (in comparison, the analogous results for $n = 2$ are [LRZ24, Lemma 7.1.1]); but these formulas are only used towards the end of Chapter 11 later.

1.4.2 Introduction and calculation of the orbital integrals

We then proceed to introduce the orbital integrals.

- In Chapter 5 we introduce the weighted orbital integral for the group version of the AFL for full spherical Hecke algebra, and state the parameters to be used for concrete calculation in Lemma 5.3.3. In Chapters 6 and 7 we prove Theorem 1.3.9 by providing the full formulas Theorems 5.5.2, 5.5.7 and 5.5.10.

- In Chapter 8 we introduce the weighted orbital integral for the semi-Lie version of the AFL for full spherical Hecke algebra, and state the parameters to be used in Lemma 8.4.2. The analogous calculation is in Chapters 9 and 10, which is used to prove Theorem 1.3.1 and its corollaries.

1.4.3 Large kernel and image

Having completely computed the orbital integrals in these cases, we take a side trip in Chapter 11 to prove the large image results asserted. We establish Conjecture 1.2.4 for $n = 2$ by proving the asserted Theorem 1.3.5 and Theorem 1.3.6 for the orbital integral on $(S_2(F) \times V'_2(F))^-_{\mathrm{rs}}$. We also prove Theorem 1.3.11 for the orbital integral on $S_3(F)^-_{\mathrm{rs}}$.

1.4.4 Geometric side

We then turn our attention to the other parts of the two versions of the AFL. In addition to stating the relevant definitions, the subsequent chapters aim to prove Theorem 1.3.8.

- In Chapter 12 we briefly define the transfer factors $\omega \in \{\pm 1\}$.
- In Chapter 13, we describe the Rapoport-Zink spaces $\mathcal{N}_n$ that the geometric side is based on, and define the intersection numbers $\operatorname{Int}((1, g), \mathbf{1}_{K^\flat} \otimes f)$ and $\operatorname{Int}((g, u), f)$ that appear in the two version of the generalized AFL. The main ingredients are the Hecke operator introduced in [LRZ24] and the KR-divisor $\mathcal{Z}(u)$ introduced in [KR11].
- In Chapter 14, we specialize to the situation $n = 2$ for the intersection numbers in the semi-Lie AFL only. The Rapoport-Zink space $\mathcal{N}_2$ become replaced with Lubin-Tate space $\mathcal{M}_2$, and we introduce the Gross-Keating formula that will be our primary tool for the calculation.

Finally, in Chapter 15 we tie everything together and establish Theorem 1.3.8.

An approximate dependency chart between the chapters is given in Figure 1.2.

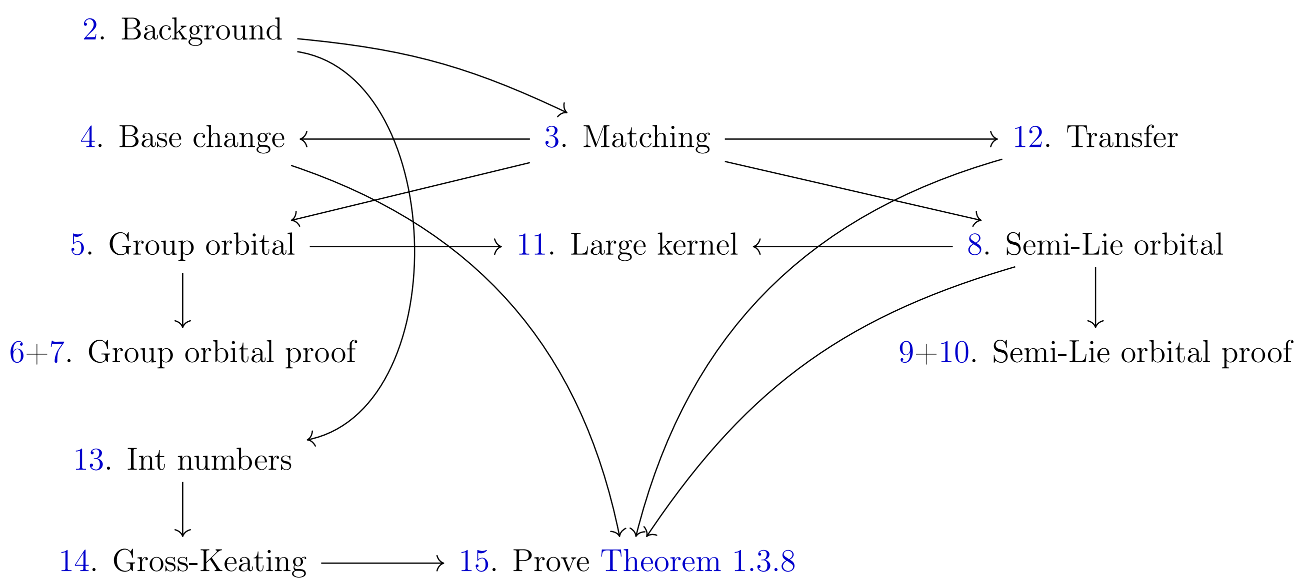

Figure 1.2: Dependency chart of the chapters in this paper, arranged to loosely resemble the Batman logo.

Chapter 2

General background

2.1 Notation

We provide a glossary of notation that will be used in this paper. As mentioned in the introduction, $p > 2$ is a prime, F is a finite extension of $\mathbb{Q}_p$, and E/F is an unramified quadratic field extension.

- For any $a \in E$, we let $\bar{a}$ denote the image of a under the nontrivial automorphism of $\mathrm{Gal}(E/F)$. (Hence $a = \bar{a}$ exactly when $a \in F$.)
- Fix $\varepsilon \in O_F^\times$ such that $E = F[\sqrt{\varepsilon}]$.
- Denote by ϖ a uniformizer of O_F, such that $\bar{\varpi} = \varpi$.
- Let $q \coloneqq |O_F/\varpi|$ be the residue characteristic. (Hence $|O_E/\varpi| = q^2$.)
- Let v be the associated valuation for ϖ.
- Let η be the quadratic character attached to E/F by class field theory, so that $\eta(x) = -1^{v(x)}$.
- $\mathbb{V}_n^+$ denotes a split E/F-Hermitian space of dimension n (unique up to isomorphism).

- Let β denote the $n \times n$ antidiagonal matrix

$$\beta := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

and pick the basis of $\mathbb{V}_n^+$ such that the Hermitian form on $\mathbb{V}_n^+$ is given by

$$\mathbb{V}_n^+ \times \mathbb{V}_n^+ \to E \qquad (x, y) \mapsto x^* \beta y.$$

- Set

$$\mathrm{U}(\mathbb{V}_n^+) = \{g \in \mathrm{GL}_n(O_E) \mid g^* \beta g = \beta\}$$

the unitary group over $\mathbb{V}_n^+$. Note that β is *antidiagonal*, in contrast to the convention $\beta = \mathrm{id}_n$ that is often used for unitary matrices with entries in $\mathbb{C}$. The natural hyperspecial maximal compact subgroup is

$$\mathrm{U}(\mathbb{V}_n^+) \cap \mathrm{GL}_n(O_E).$$

In some parts of the paper we abbreviate $G = \mathrm{U}(\mathbb{V}_n^+)$ and $K = \mathrm{U}(\mathbb{V}_n^+) \cap \mathrm{GL}_n(O_E)$ following the convention in [LRZ24].

- Let $K' := \mathrm{GL}_n(O_E)$ denote the hyperspecial maximal compact subgroup of $G' := \mathrm{GL}_n(E)$.

- Let $\mathbb{V}_n^-$ denote the non-split E/F-Hermitian space of dimension n (unique up to isomorphism), and $\mathrm{U}(\mathbb{V}_n^-)$ the corresponding unitary group. This space will be realized in Chapter 13.

2.2 Intersection of disks in an ultrametric space

The following two lemmas will be useful for both versions of the orbital integral. It is a slight rephrasing of [Zha12, Lemma 4.4].

Lemma 2.2.1 (One-disk volume lemma). *Let $\xi \in O_E^\times$, $\rho \in \mathbb{Z}$, and $n \geq \max(\rho, 1)$ an integer. Then*

$$\begin{aligned}&\operatorname{Vol}\left(\{x \in E \mid v(1 - x\bar{x}) = n,\ v(x - \xi) \geq \rho\}\right) \\ &= \begin{cases} 0 & \text{if } v(1 - \xi\bar{\xi}) < \rho \\ q^{-n}(1 - q^{-2}) & \text{if } \rho \leq 0 \\ q^{-(n+\rho)}(1 - q^{-1}) & \text{if } v(1 - \xi\bar{\xi}) \geq \rho \geq 1. \end{cases}\end{aligned}$$

Proof. When $\rho \leq 0$, the condition $v(x - \xi) \geq \rho$ is vacuously true, so we just are computing

$$\operatorname{Vol}\left(\{x \in E \mid v(1 - x\bar{x}) = n\}\right).$$

However, in general for any Schwartz function ψ on E we have an identity

$$\int_E \psi(x)\, \mathrm{d}x = \frac{1}{1 - q^{-1}} \int_{y \in F} \int_{t \in O_E^\times} \psi\left(x_t \cdot \frac{y}{\bar{y}}\right) \mathrm{d}y\, \mathrm{d}t \tag{2.1}$$

where x_t is any choice of element $x_t \in E$ such that $t = x_t \bar{x}_t$ (see the proof of [Zha12, Lemma 4.4]). Note the measures here are additive despite $t \in O_E^\times$. So if one selects

$$\psi(x) = \mathbf{1}_{v(1 - \mathrm{N}(x)) = n}$$

then we get

$$\operatorname{Vol}\left(\{x \in E \mid v(1 - x\bar{x}) = n\}\right) = \frac{1}{1 - q^{-1}} \int_{t \in F} \int_{y \in O_E^\times} \mathbf{1}_{v\left(1 - \mathrm{N}\left(x_t \cdot \frac{y}{\bar{y}}\right)\right) = n}\, \mathrm{d}y\, \mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{1-q^{-1}} \int_{t\in F} \int_{y \in O_E^\times} \mathbf{1}_{v(1-t)=n} \, \mathrm{d}y \, \mathrm{d}t \\
&= \frac{\mathrm{Vol}_E(O_E^\times)}{1-q^{-1}} \cdot \mathrm{Vol}_F(1 + \varpi^{-n} O_F^\times) \\
&= \frac{1-\frac{1}{q^2}}{1-q^{-1}} \cdot \left(q^{-n} \cdot \left(1 - \frac{1}{q} \right) \right) \\
&= q^{-n}(1-q^{-2}).
\end{aligned}
$$

The case $\rho > 0$ is proved in [Zha12, Lemma 4.4] using the same method of using (2.1). $\square$

We also comment on the well-known fact that in an ultrametric space, any two disks are either disjoint or one is contained in the other. See Figure 2.1.

Lemma 2.2.2 (No ultrametric MasterCard logo)**.** *Choose $\xi_1, \xi_2 \in E$ and $\rho_1 \geq \rho_2$. Consider the two disks:*

$$
\begin{aligned}
D_1 &= \{x \in E \mid v(x - \xi_1) \geq \rho_1\} \\
D_2 &= \{x \in E \mid v(x - \xi_2) \geq \rho_2\} .
\end{aligned}
$$

Then, if $v(\xi_1 - \xi_2) \geq \rho_2$, we have $D_1 \subseteq D_2$. If not, instead $D_1 \cap D_2 = \varnothing$.

Proof. Because E is an ultrametric space and $\mathrm{Vol}(D_1) \leq \mathrm{Vol}(D_2)$, we either have $D_1 \subseteq D_2$ or $D_1 \cap D_2 = \varnothing$. The latter condition checks which case we are in by testing if $\xi_1 \in D_2$, since $\xi_1 \in D_1$. $\square$

We package both of these results together in this lemma that will be used repeatedly.

Lemma 2.2.3 (Two-disk volume lemma)**.** *Let $\xi_1, \xi_2 \in O_E^\times$ and let $\rho_1 \geq \rho_2$ be integers. Also let $n \geq \max(\rho_1, 1)$ be an integer. Then the set of points $x \in E$ satisfying all of the equations*

$$
\begin{aligned}
v(x - \xi_1) &\geq \rho_1 \\
v(x - \xi_2) &\geq \rho_2
\end{aligned}
$$

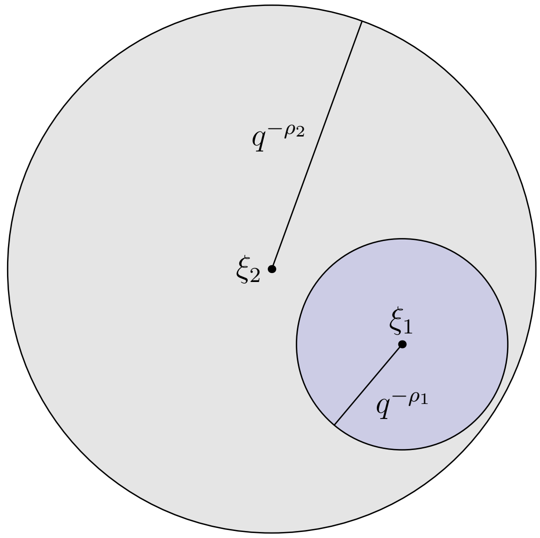

Figure 2.1: Figure corresponding to Lemma 2.2.2.

$$v(1 - x\overline{x}) = n$$

has positive volume if and only if

$$v(1 - \xi_1\bar{\xi_1}) \geq \rho_1, \qquad \rho_2 \leq v(\xi_1 - \xi_2).$$

In that case, the volume is equal to

$$\begin{cases} q^{-(n+\rho_1)}(1 - q^{-1}) & \text{if } \rho_1 \geq 1 \\ q^{-n}(1 - q^{-2}) & \text{if } \rho_1 \leq 0. \end{cases}$$

In the situation where $\xi_i \notin O_E^\times$, the quantity $v(x - \xi_i) = \min(0, v(\xi_i))$ becomes independent of the value of x, and so Lemma 2.2.3 becomes unnecessary (Lemma 2.2.1 will suffice). We will deal with this situation when it arises; it turns out this will only occur when $v(b) \neq 0$.

2.3 The spaces $S_n(F)$ and $S_n(F) \times V_n'(F)$

For the analytic side of the two AFL conjectures we investigate, the following two spaces will be used as inputs to our weighted orbital integrals.

Definition 2.3.1 ($S_n(F)$; [Zha24b, (4.10)])**.** We define the symmetric space

$$S_n(F) \coloneqq \{g \in \mathrm{GL}_n(E) \mid g\bar{g} = \mathrm{id}_n\}.$$

It has a natural left action of $\mathrm{GL}_n(E)$ by

$$\begin{aligned} \mathrm{GL}_n(E) \times S_n(F) &\to S_n(F) \\ g \cdot \gamma &\coloneqq g\gamma\bar{g}^{-1}. \end{aligned}$$

Definition 2.3.2 ($V_n'(F)$; [Zha24b, (4.11)])**.** We set

$$V_n'(F) \coloneqq F^n \times (F^n)^\vee$$

where $-^\vee$ denotes the F-dual space, i.e., $(F^n)^\vee = \mathrm{Hom}_F(F^n, F)$. Then we may also consider the augmented space

$$S_n(F) \times V_n'(F).$$

If we identify F^n with column vectors of length $n-1$ and $(F^n)^\vee$ with row vectors of length n then we have a left action of $\mathrm{GL}_n(F)$ by

$$\begin{aligned} \mathrm{GL}_n(F) \times (S_n(F) \times V_n')(F) &\to S_n(F) \times V_n'(F) \\ h \cdot (\gamma, \mathbf{u}, \mathbf{v}^\top) &\coloneqq (h\gamma h^{-1}, h\mathbf{u}, \mathbf{v}^\top h^{-1}). \end{aligned}$$

Note that according to the embedding

$$\begin{aligned} S_n(F) \times V_n'(F) &\hookrightarrow \mathrm{Mat}_{n+1}(E) \\ (\gamma, \mathbf{u}, \mathbf{v}^\top) &\mapsto \begin{pmatrix} \gamma & \mathbf{u} \\ \mathbf{v}^\top & 0 \end{pmatrix} \end{aligned}$$

we can consider elements of $S_n(F) \times V_n'(F)$ as elements of $\mathrm{Mat}_{n+1}(E)$ too. In that case the action of $h \in \mathrm{GL}_{n+1}(F)$ coincides with $h \cdot g \mapsto hg\bar{h}^{-1}$ as well.

Definition 2.3.3 (K_S')**.** For brevity, let

$$K_S' := S_n(F) \cap \mathrm{GL}_n(O_F).$$

2.4 Definition of Hecke algebra

We reminder the reader the definition of a Hecke algebra. For this subsection, G will denote *any* unimodular locally compact topological group, and K any closed subgroup of G.

Definition 2.4.1 ($\mathcal{H}(G, K)$)**.** The *Hecke algebra*

$$\mathcal{H}(G, K) := \mathbb{Q}[K \backslash G / K]$$

is defined as the space of compactly supported K-bi-invariant locally constant functions on G. (The adjective *spherical* Hecke algebra refers to the special case where K is a maximal compact subgroup of G, which is the main case of interest for us.)

Given two such functions f_1 and f_2 in $\mathcal{H}(G, K)$, one can consider define the convolution

$$(f_1 * f_2)(g) := \int_G f_1(g^{-1}x) f_2(x) \; \mathrm{d}x$$

which makes $\mathcal{H}(G, K)$ into a $\mathbb{Q}$-algebra, whose identity element is $\mathbf{1}_K$.

In other sources, this may be denoted $\mathcal{H}_K(G)$ or even just $\mathcal{H}_K$. (So what is written in $\mathcal{H}_{K'^{\flat}\otimes K'}$ in other places will be written as $\mathcal{H}(G'^{\flat} \otimes G', K'^{\flat} \otimes K')$ here).

In the case where G is a reductive Lie group and K is the maximal compact subgroup (or more generally whenever (G, K) is a Gelfand pair), this Hecke algebra is actually commutative.

2.5 The specific Hecke algebras $\mathcal{H}(\mathrm{GL}_n(E))$ and $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ and the module $\mathcal{H}(S_n(F))$

For our purposes, we define shorthands for two specific Hecke algebras that will come up consistently:

$$\begin{aligned}\mathcal{H}(\mathrm{GL}_n(E)) &\coloneqq \mathcal{H}(\mathrm{GL}_n(E), \mathrm{GL}_n(O_E)) \\ \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) &\coloneqq \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+), \mathrm{U}(\mathbb{V}_n^+) \cap \mathrm{GL}_n(O_E)).\end{aligned}$$

Note that $\mathrm{GL}_n(O_E)$ and $\mathrm{U}(\mathbb{V}_n^+) \cap \mathrm{GL}_n(O_E)$ are the natural hyperspecial maximal compact subgroups of $\mathrm{GL}_n(E)$ and $\mathrm{U}(\mathbb{V}_n^+)$, respectively.

Now the symmetric space $S_n(F)$ is not a group, so it does not make sense to define the same thing here. Nevertheless, we introduce

$$\mathcal{H}(S_n(F)) \coloneqq \mathcal{C}_c^\infty(S_n(F))^{K'}$$

as the set of smooth compactly supported functions on $S_n(F)$ which are invariant under the action of $K' \subseteq G'$; this is an $\mathcal{H}(\mathrm{GL}_n(E))$-module, where the action of $f \in \mathcal{H}(\mathrm{GL}_n(E))$ on $\phi \in \mathcal{H}(S_n(F))$ is given by

$$(f \cdot \phi)(\gamma) \coloneqq \int_G f(g)\phi(g \cdot \gamma) \, \mathrm{d}g$$

for $\gamma \in S_n(F)$. This does **not** have a multiplication structure at the moment, *a priori*. However, we will later (in Chapter 4) give an isomorphism from $\mathcal{H}(S_n(F))$ to $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ as

$\mathbb{Q}$-vector spaces; since the latter is a $\mathbb{Q}$-algebra, this induces a multiplication structure on $\mathcal{H}(S_n(F))$ and consequently we may speak of $\mathcal{H}(S_n(F))$ as a ring under this isomorphism.

Throughout this paper, to be consistent with the notation, we denote

- elements of $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ using f or f_i or similar (i.e. lowercase Roman letters);
- elements of $\mathcal{H}(\mathrm{GL}_n(E))$ by f' or f_i' or similar (i.e. lowercase Roman letters with apostrophes);
- elements of $\mathcal{H}(S_n(F))$ by ϕ or ϕ_i (i.e. lowercase Greek letters).

Chapter 3

Regular semi-simplicity and matching

3.1 Regular semi-simple elements

We first recall the notion of regularity that first appeared in [RS07, §6].

Definition 3.1.1 (Regular semisimple in $\mathrm{Mat}_n(E)$)**.** Consider a $n \times n$ matrix

$$\begin{pmatrix} A & \mathbf{u} \\ \mathbf{v}^\top & d \end{pmatrix} \in \mathrm{Mat}_n(E)$$

where A is an $(n-1) \times (n-1)$ matrix. Then we say this matrix is *regular semi-simple* if

$$\left\langle \mathbf{u}, A\mathbf{u}, \dots, A^{n-2}\mathbf{u} \right\rangle$$

and

$$\left\langle \mathbf{v}^\top, \mathbf{v}^\top A, \dots, \mathbf{v}^\top A^{n-2} \right\rangle$$

are each a basis of E^{n-1}. Equivalently, the matrix

$$\left[\mathbf{v}^\top A^{i+j-2} \mathbf{u} \right]_{i,j=1}^{n-1}$$

should be nonsingular.

Remark 3.1.2 (Equivalent definition of regular semisimple)**.** In [RS07, Theorem 6.1], this definition is shown to be equivalent to requiring that, under the action of conjugation by $\mathrm{GL}_{n-1}(E)$:

- the matrix has trivial stabilizer; and
- the $\mathrm{GL}_{n-1}(\overline{E})$-orbit is a Zariski-closed subset of $\mathrm{GL}_n(\overline{E})$.

Here $\overline{E}$ is as usual an algebraic closure of E.

Remark 3.1.3 (Invariants under $\mathrm{GL}_{n-1}(E)$ conjugation; [RS07, Proposition 6.2])**.** It turns out we can detect whether two regular semisimple elements

$$\begin{pmatrix} A_1 & \mathbf{u}_1 \\ \mathbf{v}_1^\top & d_1 \end{pmatrix}, \begin{pmatrix} A_2 & \mathbf{u}_2 \\ \mathbf{v}_2^\top & d_2 \end{pmatrix} \in \mathrm{Mat}_n(E)$$

are conjugate by an element of $\mathrm{GL}_{n-1}(E)$. This happens if and only if the following conditions all hold:

- The matrices A_1 and A_2 have the same characteristic polynomial;
- We have
$$\mathbf{v}_1^\top A_1^i \mathbf{u}_1 = \mathbf{v}_2^\top A_2^i \mathbf{u}_2$$
for every $i = 0, 1, \dots, n-2$; and
- We have $d_1 = d_2$.

Thus, this gives a set of invariants that completely classify the orbits under the action of $\mathrm{GL}_{n-1}(E)$.

Put another way, the invariants of

$$\begin{pmatrix} A & \mathbf{u} \\ \mathbf{v}^\top & d \end{pmatrix} \in \mathrm{Mat}_n(E)$$

are the (monic) characteristic polynomial of A (which has $n-1$ coefficients besides the leading coefficient), the values of $\mathbf{v}^\top A^i \mathbf{u}$ for $i = 0, \dots, n-2$ and the number d, for a total of $2n-1$ numbers.

We can now speak of regular-simplicity in each of the four particular cases relevant to this paper.

Definition 3.1.4 (Regular semisimple)**.** In the group version of the AFL:

- We say $\gamma \in S_n(F)$ is regular semisimple if its image under the inclusion $S_n(F) \subseteq \mathrm{Mat}_n(E)$ is regular semisimple. We write $\gamma \in S_n(F)_{\mathrm{rs}}$.
- For $g \in \mathrm{U}(\mathbb{V}_n^\pm)$, we say g is regular semisimple if its image under the inclusion $\mathrm{U}(\mathbb{V}_n^\pm) \subseteq \mathrm{Mat}_n(E)$ is regular semisimple. We write $g \in \mathrm{U}(\mathbb{V}_n^\pm)_{\mathrm{rs}}$.

In the semi-Lie version of the AFL:

- We say $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in S_n(F) \times V_n'(F)$ is regular semisimple if its image under the embedding

$$\begin{aligned} S_n(F) \times V_n'(F) &\hookrightarrow \mathrm{Mat}_{n+1}(E) \\ (\gamma, \mathbf{u}, \mathbf{v}^\top) &\mapsto \begin{pmatrix} \gamma & \mathbf{u} \\ \mathbf{v}^\top & 0 \end{pmatrix} \end{aligned} \tag{3.1}$$

  is regular semisimple. In other words, we require that both of the sets $(\mathbf{u}, \gamma\mathbf{u} \dots, \gamma^{n-1}\mathbf{u})$ and $\left(\mathbf{v}^\top, \mathbf{v}^\top\gamma, \dots, \mathbf{v}^\top\gamma^{n-1}\right)$ are bases of E^n. In this case we write $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$.

- For $(g, u) \in \mathrm{U}(\mathbb{V}_n^\pm) \times \mathbb{V}_n^\pm$ we say (g, u) is regular semisimple if its image under the embedding

$$\begin{aligned} \mathrm{U}(\mathbb{V}_n^\pm) \times \mathbb{V}_n^\pm &\hookrightarrow \mathrm{Mat}_{n+1}(E) \\ (g, u) &\mapsto \begin{pmatrix} g & u \\ u^* & 0 \end{pmatrix} \end{aligned} \tag{3.2}$$

is regular semisimple. Here u^* is the conjugate transpose.

This is equivalent to the set $(u, gu, \dots, g^{n-1}u)$ being linearly independent (i.e. form a basis of $\mathbb{V}_n^{\pm}$); in this case the independence of $(u^*, u^*g, \dots, u^*g^{n-1})$ is redundant, so it's enough to verify one condition. We write $(g, u) \in (\mathrm{U}(\mathbb{V}_n^{\pm}) \times \mathbb{V}_n^{\pm})_{\mathrm{rs}}$.

3.2 Matching in the group version of the inhomogeneous AFL

We now describe the matching condition used in the group version of AFL.

Definition 3.2.1 (Matching $S_n(F)_{\mathrm{rs}} \longleftrightarrow \mathrm{U}(\mathbb{V}_n^{\pm})_{\mathrm{rs}}$; [Zha12, p. 202])**.** We say $\gamma \in S_n(F)_{\mathrm{rs}}$ matches the element $g \in \mathrm{U}(\mathbb{V}_n^{\pm})_{\mathrm{rs}}$ if g is conjugate to γ by an element of $\mathrm{GL}_{n-1}(E)$. By Remark 3.1.3, this is an assertion that the invariants for γ and g coincide.

In that case, we have the following result.

Proposition 3.2.2 ([Zha12, Lemma 2.3]; see also [LRZ24, (3.3.2)])**.** *This definition of matching gives a bijection of regular semisimple orbits*

$$[S_n(F)]_{\mathrm{rs}} \xrightarrow{\sim} [\mathrm{U}(\mathbb{V}_n^{+})]_{\mathrm{rs}} \amalg [\mathrm{U}(\mathbb{V}_n^{-})]_{\mathrm{rs}}.$$

Moreover, we can detect whether $\gamma \in S_n(F)_{\mathrm{rs}}$ matches an orbit of $\mathrm{U}(\mathbb{V}_n^{+})_{\mathrm{rs}}$ or $\mathrm{U}(\mathbb{V}_n^{-})_{\mathrm{rs}}$ as follows. Suppose we write γ in the format of Definition 3.1.1 and consider

$$\Delta \coloneqq \det \left[\mathbf{v}^{\top} A^{i+j-2} \mathbf{u}\right]_{i,j=1}^{n-1} \neq 0.$$

Then

- *γ matches an orbit in $\mathrm{U}(\mathbb{V}_n^{+})_{\mathrm{rs}}$ if $v(\Delta)$ is even;*

- γ *matches an orbit in* $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ *if* $v(\Delta)$ *is odd.*

In this paper, Conjecture 1.2.1 requires that γ should match an element of $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ and consequently we will usually only be interested in the latter case. We write the following abbreviation:

Definition 3.2.3 ($S_n(F)^{\pm}_{\mathrm{rs}}$)**.** We let

$$S_n(F)^-_{\mathrm{rs}} \subset S_n(F)_{\mathrm{rs}}$$

denote the subset of elements in $S_n(F)_{\mathrm{rs}}$ that match with an element in $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$. Define $S_n(F)^+_{\mathrm{rs}}$ similarly. Hence $S_n(F)_{\mathrm{rs}} = S_n(F)^-_{\mathrm{rs}} \amalg S_n(F)^+_{\mathrm{rs}}$.

3.3 Matching in the semi-Lie version of the AFL

For the semi-Lie version matching is defined analogously:

Definition 3.3.1 (Matching $(S_n(F) \times V_n'(F))_{\mathrm{rs}} \longleftrightarrow (\mathrm{U}(\mathbb{V}_n^{\pm}) \times \mathbb{V}_n^{\pm})_{\mathrm{rs}}$; [Liu21, §1.3])**.** We say $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$ matches the element $(g, u) \in (\mathrm{U}(\mathbb{V}_n^{\pm}) \times \mathbb{V}_n^{\pm})_{\mathrm{rs}}$ if their images under the embeddings (3.1) and (3.2) are conjugate by an element of $\mathrm{GL}_n(E)$.

Unwrapping this with Remark 3.1.3, an equivalent definition is to require both of the following conditions:

- As elements of $\mathrm{GL}_n(E)$, both g and γ have the same characteristic polynomial.
- We have $\mathbf{v}^\top \gamma^i \mathbf{u} = \langle g^i u, u \rangle$ for all $0 \le i \le n-1$, where $\langle -, - \rangle$ is the Hermitian form on $\mathbb{V}_n^{\pm}$.

We have the following analogous criteria for matching.

Proposition 3.3.2 ([Liu21])**.** *This definition of matching gives a bijection of regular semisimple orbits*

$$[S_n(F) \times V_n'(F)]_{\mathrm{rs}} \xrightarrow{\sim} [\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+]_{\mathrm{rs}} \amalg [\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-]_{\mathrm{rs}}.$$

Moreover, we can detect whether $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in S_n(F)_{\mathrm{rs}}$ *matches an orbit of* $(\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+)_{\mathrm{rs}}$ *or* $(\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}}$ *as follows: consider the determinant*

$$\Delta \coloneqq \det \left[\mathbf{v}^\top \gamma^{i+j-2}\mathbf{u}\right]_{i,j=1}^{n} \neq 0.$$

Then

- γ *matches an orbit in* $(\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+)_{\mathrm{rs}}$ *if* $v(\Delta)$ *is even;*
- γ *matches an orbit in* $(\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}}$ *if* $v(\Delta)$ *is odd.*

In this paper, Conjecture 1.2.2 requires that $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ should match an element of $(\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}}$ and consequently we will usually only be interested in the latter case. Accordingly we write the following shorthand:

Definition 3.3.3 ($(S_n(F) \times V_n'(F))_{\mathrm{rs}}^{\pm}$)**.** We let

$$(S_n(F) \times V_n'(F))_{\mathrm{rs}}^{-} \subset (S_n(F) \times V_n'(F))_{\mathrm{rs}}$$

denote the subset of those elements in $(S_n(F) \times V_n'(F))_{\mathrm{rs}}$ that match with an element of $(\mathrm{U}(\mathbb{V}_n^-) \times \mathbb{V}_n^-)_{\mathrm{rs}}$. Define $(S_n(F) \times V_n'(F))_{\mathrm{rs}}^{+}$ analogously. Hence $(S_n(F) \times V_n'(F))_{\mathrm{rs}} = (S_n(F) \times V_n'(F))_{\mathrm{rs}}^{-} \amalg (S_n(F) \times V_n'(F))_{\mathrm{rs}}^{+}$.

Chapter 4

Base change

This section introduces necessary background material on the base change

$$\mathrm{BC}_{S_n}^{\eta^{n-1}} : \mathcal{H}(S_n(F)) \to \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)).$$

Throughout this section we let $\mathrm{Sym}(n)$ denote the symmetric group in n variables with order $n!$ (since $S_n(F) \subseteq \mathrm{GL}_n(E)$ is already reserved for the symmetric space).

4.1 Background on the Satake transform in general

We recall a general form of the Satake transform, which will be used later.

For this subsection, G will denote an arbitrary connected reductive group over some non-Archimedean local field F. We will not distinguish between G and $G(F)$ when there is no confusion.

To simplify things, we will assume G is unramified; but we do *not* assume G is split. Introduce the following notation:

- Let K be a hyperspecial maximal compact subgroup of G (it exists because G is unramified).

- Let A denote a maximal F-split torus in G. All the maximal F-split tori in G are conjugate; let A denote one of them.
- Let M be the centralizer of A; this is itself a maximal torus in G.
- Let ${}^\circ M := M(F) \cap K$ be the maximal compact subgroup of M.
- Let P denote a minimal F-parabolic containing A.
- Let δ denotes the modulus character of P. It can be describes as follows. Let ϖ denote a uniformizer for F and q the residue characteristic. Then if ρ is the Weyl vector and μ is a positive cocharacter, then

$$\delta(\mu(\varpi)) = q^{-\langle \mu, \rho \rangle}.$$

- Let N denote the unipotent radical of P.
- Let W be the relative Weyl group for the pair (G, A), which acts on $\mathcal{H}(M, {}^\circ M)$.

We can now state the Satake isomorphism.

Definition 4.1.1 (Satake transform)**.** The *Satake transform* is a canonical isomorphism of Hecke algebras

$$\mathrm{Sat}\colon \mathcal{H}(G, K) \to \mathcal{H}(M, {}^\circ M)^W$$

which is given by defining

$$(\mathrm{Sat}(f))(t) := \delta(t)^{\frac{1}{2}} \int_N f(nt) \, \mathrm{d}n$$

for each $t \in M$.

We are going to apply this momentarily in two situations: once when G is the general linear group (which is split), and once when G is a unitary group.

4.2 The Satake transform for the particular Hecke algebras $\mathcal{H}(\mathrm{GL}_n(E))$ and $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$

To take the Satake transform of $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$, we define the following abbreviations.

- Let T denote the split diagonal torus of GL_n.
- Let
$$N' := \left\{ \begin{pmatrix} 1 & * & \dots & * \\ & 1 & \dots & * \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n(E)$$
denote the unipotent upper-triangular matrices.

Similarly for $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$:

- Set $m := \lfloor n/2 \rfloor$ for brevity.
- Let
$$A := \left\{ \mathrm{diag}(x_1, \dots, x_m, 1_{n-2m}, x_m^{-1}, \dots, x_1^{-1}) \right\}$$
so that $A(F)$ is a maximal F-split torus of $\mathrm{U}(\mathbb{V}_n^+)$.
- Let $N := N' \cap G$ denote the unipotent upper triangular matrices which are also unitary.
- For brevity, let $W_m := (\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathrm{Sym}(m)$ be the relative Weyl group of (G, A).

We can now introduce the Satake transform for our two *bona fide* Hecke algebras, using the data in Table 4.1.

Hence, the Satake transforms obtained can be viewed as

$$\mathrm{Sat}\colon \mathcal{H}(\mathrm{GL}_n(E)) \xrightarrow{\sim} \mathbb{Q}[T(E)/T(O_E)]^{\mathrm{Sym}(n)}$$

Group	$G' = \mathrm{GL}_n(E)$	$G = \mathrm{U}(\mathbb{V}_n^+)$
Local field	E	F
Hyperspecial compact	$K' = \mathrm{GL}_n(O_E)$	$K = G \cap \mathrm{GL}_n(O_E)$
Max'l split torus	$T(E)$	$A(F)$
Centralizer of split torus	$T(O_E)$	$A(O_F)$
Parabolic (Borel)	Upper tri in G'	Upper tri in G
Unipotent rad. of parabolic	N' (unipot. upper tri)	N (unipot. upper tri)
Relative Weyl group	$\mathrm{Sym}(n)$	$W_m = (\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathrm{Sym}(m)$

Table 4.1: Data needed to run the Satake transform.

$$\mathrm{Sat}\colon \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) \xrightarrow{\sim} \mathbb{Q}[A(F)/A(O_F)]^{W_m}$$

(In both cases, the modular character $\delta^{1/2}$ gives rational values, so it is okay to work over $\mathbb{Q}$.)

To make this further concrete, we remark that the cocharacter groups involved are free abelian groups with known bases. This identification lets us rewrite the right-hand sides above as concrete polynomials. Specifically, we identify

$$\mathbb{Q}[T(E)/T(O_E)]^{\mathrm{Sym}(n)} \xrightarrow{\sim} \mathbb{Q}[X_1^{\pm}, \dots, X_n^{\pm}]^{\mathrm{Sym}(n)}$$

by identifying X_i with the cocharacter corresponding to injection into the i^{th} factor. Similarly, we identify

$$\mathbb{Q}[A(F)/A(O_F)]^{W_m} \xrightarrow{\sim} \mathbb{Q}[Y_1^{\pm}, \dots, Y_m^{\pm}]^{W_m}$$

by identifying $Y_i + Y_i^{-1}$ with the cocharacter corresponding to

$$x \mapsto \mathrm{diag}(1, \dots, x, \dots, x^{-1}, \dots, 1)$$

where x is in the i^{th} position and x^{-1} is in the $(n-i)^{\text{th}}$ position, and all other positions are 1. Here $\mathbb{Q}[Y_1^{\pm}, \dots, Y_m^{\pm}]^{W_m}$ denotes the ring of symmetric polynomials in $Y_i + Y_i^{-1}$.

So, henceforth, we will consider

$$\begin{aligned}\mathrm{Sat}\colon \mathcal{H}(\mathrm{GL}_n(E)) &\xrightarrow{\sim} \mathbb{Q}[X_1^{\pm},\dots,X_n^{\pm}]^{\mathrm{Sym}(n)}\\ \mathrm{Sat}\colon \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) &\xrightarrow{\sim} \mathbb{Q}[Y_1^{\pm},\dots,Y_m^{\pm}]^{W_m}.\end{aligned}$$

4.3 Relation of Satake transform to base change

Let

$$\mathrm{BC}\colon \mathcal{H}(\mathrm{GL}_n(E)) \to \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$$

denote the stable base change morphism from $\mathrm{GL}_n(E)$ to the unitary group U. The relevance of the Satake transform is that (see e.g. [Les23, Proposition 3.4]) it gives a way to make this BC completely explicit: we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}(\mathrm{GL}_n(E)) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d, "\mathrm{BC}"] & \mathbb{Q}[X_1^{\pm},\dots,X_n^{\pm}]^{\mathrm{Sym}(n)} \arrow[d, "\mathrm{BC}"] \\ \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) \arrow[r, "\mathrm{Sat}", "\sim"'] & \mathbb{Q}[Y_1^{\pm},\dots,Y_m^{\pm}]^{W_m} \end{tikzcd}$$

Here the right arrow is also denoted BC following [LRZ24] (although it is denoted ν in [Les23]). This gives a way in which we can concretely calculate the map BC in some situations.

4.4 The map $\mathrm{BC}_{S_n}^{\eta^{n-1}}$

Before we can define the map $\mathrm{BC}_{S_n}^{\eta^{n-1}}$ we need one more piece of notation. Consider the following map.

Definition 4.4.1 (proj)**.** Denote by $\mathrm{proj}\colon \mathrm{GL}_n(E) \twoheadrightarrow S_n(F)$ the projection defined by

$$\mathrm{proj}(g) := g\bar{g}^{-1}.$$

Then proj induces a map

$$\begin{aligned}\mathrm{proj}_*\colon \mathcal{H}(\mathrm{GL}_n(E)) &\to \mathcal{H}(S_n(F))\\ \mathrm{proj}_*(f')\left(g\bar{g}^{-1}\right) &= \int_{\mathrm{GL}_n(F)} f'(gh)\,\mathrm{d}h\end{aligned}$$

by integration on the fibers. A similar twisted version by η

$$\begin{aligned}\mathrm{proj}_*^{\eta}\colon \mathcal{H}(\mathrm{GL}_n(E)) &\to \mathcal{H}(S_n(F))\\ \mathrm{proj}_*^{\eta}(f')\left(g\bar{g}^{-1}\right) &= \int_{\mathrm{GL}_n(F)} f'(gh)\eta(gh)\,\mathrm{d}h\end{aligned}$$

is defined analogously, where as before $\eta(g) = (-1)^{v(\det g)}$ in a slight abuse of notation.

Then Leslie [Les23] shows the following result.

Theorem 4.4.2 ([Les23, Theorem 3.2 and Proposition 3.4])**.** *Both maps* proj_* *and* proj_*^{η} *induce isomorphisms*

$$\begin{aligned}\mathrm{BC}_{S_n}\colon \mathcal{H}(S_n(F)) &\xrightarrow{\sim} \mathcal{H}(\mathrm{GL}_n(E))\\ \mathrm{BC}_{S_n}^{\eta^{n-1}}\colon \mathcal{H}(S_n(F)) &\xrightarrow{\sim} \mathcal{H}(\mathrm{GL}_n(E))\end{aligned}$$

such that

$$\begin{aligned}\mathrm{BC} &= \mathrm{BC}_{S_n} \circ \mathrm{proj}_*\\ \mathrm{BC} &= \mathrm{BC}_{S_n}^{\eta^{n-1}} \circ \mathrm{proj}_*^{\eta^{n-1}}.\end{aligned}$$

We take these isomorphisms promised by this theorem as the definition of BC_{S_n} and $\mathrm{BC}_{S_n}^{\eta^{n-1}}$ in our conjectures (noting when n is odd they coincide, as $\eta^{n-1} = 1$).

When combined with the Satake information we have, we get the following diagram.

$$\begin{tikzcd}
\mathcal{H}(\mathrm{GL}_n(E)) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d, "\mathrm{BC}"] \arrow[dd, bend right=60, "\mathrm{proj}^{\eta^{n-1}}_*"'] & \mathbb{Q}[X_1^\pm, \dots, X_n^\pm]^{\mathrm{Sym}(n)} \arrow[d, "\mathrm{BC}"] \\
\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) \arrow[r, "\mathrm{Sat}", "\sim"'] & \mathbb{Q}[Y_1^\pm, \dots, Y_m^\pm]^{W_m} \\
\mathcal{H}(S_n(F)) \arrow[u, "\mathrm{BC}^{\eta^{n-1}}_{S_n}", "\sim"'] &
\end{tikzcd}$$

4.5 Calculation of BC_{S_n} when $n = 3$

The goal of this section is to make the base change fully known in the special case $n = 3$, where $m = \lfloor n/2 \rfloor = 1$. (In this case $\mathrm{BC}^{\eta^{n-1}}_{S_n} = \mathrm{BC}_{S_n}$ as $\eta^2 = 1$.) The completed result is Proposition 4.5.4.

This calculation parallels the $n = 2$ case that was done in [LRZ24, Lemma 7.1.1]. However, we will not use these results again later on. When it is not more difficult, some of the results will be stated for all n, rather than $n = 3$ specifically.

4.5.1 Overview

Throughout this subsection, we use the shorthand

$$\varpi^{(n_1,n_2,n_3)} := \mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{n_3}).$$

As a $\mathbb{Q}$-module, the spaces $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ and $\mathcal{H}(S_n(F))$ have a canonical basis of indicator functions indexed by $\mathbb{Z}$:

- $\mathcal{H}(S_n(F))$ has $\mathbb{Q}$-module basis $\mathbf{1}_{K'_{S,j}}$ for $j \geq 0$.
- $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ has a $\mathbb{Q}$-module basis given by the indicator functions

$$\mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}(O_E) \cap \mathrm{U}(\mathbb{V}_n^+)}$$

for $r \geq 0$.

On the other hand, the natural $\mathbb{Q}$-module basis for $\mathcal{H}(\mathrm{GL}_n(E))$, namely

$$\mathbf{1}_{K'\varpi^{(n_1,n_2,n_3)}K'}$$

is given by triples of integers $n_1 \geq n_2 \geq n_3 \geq 0$, and is much larger. So explicit calculations for the proj_* or the Satake transforms viewed in $\mathbb{C}[X_1, X_2, X_3]^{\mathrm{Sym}(n)}$ are nontrivial if one works with the entire basis.

Hence the overall strategy, to reduce the amount of work we have to do, is to focus on only the $\mathbb{Z}$-indexed elements

$$\mathbf{1}_{\mathrm{Mat}_3(O_E), v\circ\det=r} = \sum_{\substack{n_1\geq n_2\geq n_3 \\ n_1+n_2+n_3=r}} \mathbf{1}_{K'\varpi^{(n_1,n_2,n_3)}K'} \in \mathcal{H}(\mathrm{GL}_n(E))$$

for $r \geq 0$. This aggregated indicator function is easier to compute, because given an explicit matrix it is somewhat easier to evaluate

$$\mathbf{1}_{\mathrm{Mat}_3(O_E), v\circ\det=r}$$

at it (one only needs to check it has O_E entries and that the determinant has valuation r, rather than determining the exact coset $K'\varpi^{(n_1,n_2,n_3)}K'$).

4.5.2 Satake transform of the determinant characteristic function on the top arrow

This is the easiest calculation, and we do it for all n rather than just $n = 3$.

Proposition 4.5.1 (Satake transform for $v \circ \det = r$)**.** *For every integer $r \geq 0$, we have*

$$\mathrm{Sat}(\mathbf{1}_{\mathrm{Mat}_n(O_E), v\circ\det=r}) = q^{(n-1)r} \sum_{e_1\cdots+e_n=r} X_1^{e_1}\dots X_n^{e_n}.$$

Proof. We evaluate the coefficient $X_1^{e_1} \dots X_n^{e_n}$. Choose a cocharacter μ, and suppose $\mu(\varpi) = \varpi^{(e_1, \dots, e_n)}$ with $n_1 \geq n_2 \geq n_3$. Let $q_E = q^2$ be the residue characteristic of E. Take the upper triangular matrices as our Borel subgroup as usual, so the unipotent radical of this Borel subgroup are the unipotent upper triangulars N' which we describe as

$$N' := \left\{ \begin{pmatrix} 1 & y_{12} & y_{13} & \dots & y_{1n} \\ & 1 & y_{23} & \dots & y_{2n} \\ & & 1 & \dots & y_{3n} \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} \mid y_{12}, \dots, y_{(n-1)n} \in E \right\}$$

and with additive Haar measure is $\mathrm{d}y_{12}\,\mathrm{d}y_{23} \dots \mathrm{d}y_{(n-1)n}$. Recall also the Weyl vector for $\mathrm{GL}_n(E)$ is just

$$\rho_{\mathrm{GL}_n(E)} = \left\langle \frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2} \right\rangle.$$

Compute

$$\begin{aligned}
&\mathrm{Sat}(\mathbf{1}_{\mathrm{Mat}_n(O_E), v \circ \det = r})(\mu(\varpi)) \\
&= \delta(\mu(\varpi))^{\frac{1}{2}} \int_{n' \in N'} \mathbf{1}_{\mathrm{Mat}_n(O_E, v \circ \det = r)}(\mu(\varpi) n') \,\mathrm{d}n' \\
&= q_E^{-\langle \mu, \rho \rangle} \underbrace{\int_{y_{12} \in E} \int_{y_{13} \in E} \dots \int_{y_{(n-1)n} \in E}}_{\binom{n}{2} \text{ integrals}} \\
&\quad \mathbf{1}_{\mathrm{Mat}_3(O_E), v \circ \det = r} \left(\begin{pmatrix} \varpi^{e_1} & \varpi^{e_1} y_{12} & \varpi^{e_1} y_{13} & \dots & \varpi^{e_1} y_{1n} \\ & \varpi^{e_2} & \varpi^{e_2} y_{23} & \dots & \varpi^{e_2} y_{2n} \\ & & \varpi^{e_3} & \dots & \varpi^{e_3} y_{3n} \\ & & & \ddots & \vdots \\ & & & & \varpi^{e_n} \end{pmatrix} \right) \\
&\quad \mathrm{d}y_{12}\,\mathrm{d}y_{23} \dots \mathrm{d}y_{(n-1)n}
\end{aligned}$$

$$= q_E^{-\left(\frac{n-1}{2}e_1+\frac{n-3}{2}e_2+\cdots+-\frac{n-1}{2}e_n\right)} \mathbf{1}_{e_1+\cdots+e_n=r} \underbrace{\int_{y_{12}\in E}\int_{y_{13}\in E}\cdots\int_{y_{(n-1)n}\in E}}_{\binom{n}{2}\text{ integrals}}$$

$$\prod_{1\le i<j\le n} \mathbf{1}_{O_E}(\varpi^{e_i} y_{ij})\,\mathrm{d}y_{ij}$$

$$= q_E^{-\left(\frac{n-1}{2}e_1+\frac{n-3}{2}e_2+\cdots+-\frac{n-1}{2}e_n\right)} \mathbf{1}_{e_1+\cdots+e_n=r} \prod_{1\le i<j\le n} q_E^{e_i}$$

$$= q_E^{-\left(\frac{n-1}{2}e_1+\frac{n-3}{2}e_2+\cdots+-\frac{n-1}{2}e_n\right)} \mathbf{1}_{e_1+\cdots+e_n=r} \prod_{1\le i\le n} q_E^{(n-i)e_i}$$

$$= \mathbf{1}_{e_1+\cdots+e_n=r} \prod_{1\le i\le n}^{n} q_E^{\frac{n-1}{2}e_i}$$

$$= q_E^{\frac{n-1}{2}r}\mathbf{1}_{e_1+\cdots+e_n=r}$$

$$= \begin{cases} q^{\frac{n-1}{2}r} & \text{if } e_1+\cdots+e_n=r \\ 0 & \text{otherwise.} \end{cases}$$

This gives the sum claimed earlier. □

4.5.3 Satake transform of the indicator on the bottom arrow

Proposition 4.5.2 (Satake transform for $\varpi^{-r}\operatorname{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)$). *For each $r\ge 0$ we have*

$$\operatorname{Sat}\left(\mathbf{1}_{\varpi^{-r}\operatorname{Mat}_3(O_E)\cap\mathrm{U}(\mathbb{V}_3^+)}\right) = \sum_{i=0}^{r} q^{2r-\mathbf{1}_{r\equiv i \bmod 2}} \mathrm{Y}_1^{\pm i}$$

where we adopt the shorthand

$$\mathrm{Y}_1^{\pm i} := \begin{cases} \mathrm{Y}_1^{i}+\mathrm{Y}_1^{-i} & i>0 \\ 1 & i=0. \end{cases}$$

Proof. We first need to describe

$$N = N' \cap \mathrm{U}(\mathbb{V}_3^+)$$

a little more carefully. For $n \in N'$ we have

$$n^*\beta n = \begin{pmatrix} 1 & & \\ \bar{y_1} & 1 & \\ \bar{y_2} & \bar{y_3} & 1 \end{pmatrix} \beta \begin{pmatrix} 1 & y_1 & y_2 \\ & 1 & y_3 \\ & & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & y_3 + \bar{y_1} \\ 1 & y_1 + \bar{y_3} & y_2 + \bar{y_2} + y_3\bar{y_3} \end{pmatrix}.$$

So $n \in N$ if and only if the above matrix equals β, which means

$$0 = y_3 + \bar{y_1} = y_2 + \bar{y_2} + y_3\bar{y_3}.$$

Then we can re-parametrize by $z_1, z_2, z_3 \in F$ according to

$$\begin{aligned} y_3 &= z_1 + z_2\sqrt{\varepsilon} \\ y_2 &= -\frac{z_1^2 + z_2^2\varepsilon}{2} + z_3\sqrt{\varepsilon} \\ y_1 &= -z_1 + z_2\sqrt{\varepsilon}. \end{aligned}$$

Back to the original task. For each $i \geq 0$ we can evaluate the Satake transform at the element $\nu(\varpi) = \operatorname{diag}(\varpi^i, 1, \varpi^{-i})$, for the cocharacter ν corresponding to $Y_1^i + Y_1^{-i}$:

$$\begin{aligned} &\operatorname{Sat}\left(\mathbf{1}_{\varpi^{-r}\operatorname{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}\right)(\nu(\varpi)) \\ &= \delta(\nu(\varpi))^{\frac{1}{2}} \int_{n\in N} \mathbf{1}_{\varpi^{-r}\operatorname{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}\left(\nu(\varpi)n'\right) \mathrm{d}n \\ &= \delta(\nu(\varpi))^{\frac{1}{2}} \int_{n\in N} \mathbf{1}_{\varpi^{-r}\operatorname{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}\left(\begin{pmatrix} \varpi^i & \varpi^i y_1 & \varpi^i y_2 \\ & 1 & y_3 \\ & & \varpi^{-i} \end{pmatrix}\right) \mathrm{d}n \end{aligned}$$

The matrix itself is always in $\mathrm{U}(\mathbb{V}_3^+)$, because it's the product of two unitary matrices. So the indicator needs to check whether all the entries have valuation at least $-r$. If we switch characterization to the coordinates z_1, z_2, z_3 we described earlier, we see that the conditions are

$$\begin{aligned}
i &\le r, \\
v(z_1) &\ge -r, \\
v(z_2) &\ge -r, \\
v(z_3) &\ge -(r+i), \\
v(z_1^2 + z_2^2\varepsilon) &\ge -(r+i).
\end{aligned}$$

Assume $i \le r$ henceforth. The condition for z_1 and z_2 then really says

$$\min(v(z_1), v(z_2)) \ge -\left\lfloor \frac{r+i}{2} \right\rfloor.$$

So the integral factors as a triple integral

$$\int_{z_1\in F}\int_{z_2\in F}\int_{z_3\in F} \mathbf{1}_{\varpi^{-\left\lfloor \frac{r+i}{2}\right\rfloor} O_F}(z_1)\mathbf{1}_{\varpi^{-\left\lfloor \frac{r+i}{2}\right\rfloor} O_F}(z_2)\mathbf{1}_{\varpi^{-(r+i)}O_F}(z_3)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3$$

which is equal to

$$q^{2\left\lfloor \frac{r+i}{2}\right\rfloor + r + i}.$$

Meanwhile, $\delta(\nu(\varpi))^{\frac{1}{2}} = q^{-2i}$. In summary,

$$\operatorname{Sat}\left(\mathbf{1}_{\varpi^{-r}\operatorname{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}\right)(\nu(\varpi)) = \begin{cases} q^{2\left\lfloor \frac{r+i}{2}\right\rfloor - i + r} & i \le r \\ 0 & i > r \end{cases}$$

Finally, since

$$2\left\lfloor \frac{r+i}{2}\right\rfloor - i + r = \begin{cases} 2r & r+i \text{ is even} \\ 2r-1 & r+i \text{ is odd} \end{cases}$$

we get the formula claimed. $\square$

4.5.4 Integration over fiber

Proposition 4.5.3 (Integration over fiber)**.** *For every integer $r \geq 0$, we have*

$$\begin{aligned}&\operatorname{proj}_*(\mathbf{1}_{\mathrm{Mat}_3(O_E),v\circ\det=r})\\&=\sum_{j=0}^{r}\left(\sum_{i=0}^{2(r-j)}\min\left(1+\left\lfloor\frac{i}{2}\right\rfloor,1+\left\lfloor\frac{2(r-j)-i}{2}\right\rfloor\right)q^i\right)\mathbf{1}_{K'_{S,j}}.\end{aligned}$$

Proof. The coefficient of $\mathbf{1}_{K'_{S,j}}$ will be equal to the evaluation of the integral at any g such that $g\bar{g} \in K'_{S,j}$. Fixing $j \geq 0$, we are going to take the choice

$$g = \begin{pmatrix}1 & & \varpi^{-j}\sqrt{\varepsilon}\\ & 1 & \\ & & 1\end{pmatrix}.$$

We need to check this choice of g indeed satisfies $g\bar{g}^{-1} \in K'_{S,j}$. This follows as

$$\bar{g} = \begin{pmatrix}1 & & -\varpi^{-j}\sqrt{\varepsilon}\\ & 1 & \\ & & 1\end{pmatrix} \implies \bar{g}^{-1} = \begin{pmatrix}1 & & \varpi^{-j}\sqrt{\varepsilon}\\ & 1 & \\ & & 1\end{pmatrix}$$

and therefore

$$g\bar{g}^{-1} = \begin{pmatrix}1 & & 2\varpi^{-j}\sqrt{\varepsilon}\\ & 1 & \\ & & 1\end{pmatrix} \in K'_{S,j}$$

as needed.

Having chosen the representative g, we aim to calculate the right-hand side of

$$\operatorname{proj}_*(\mathbf{1}_{\mathrm{Mat}_3(O_E),v\circ\det=r})(g\bar{g}) = \int_{h\in\mathrm{GL}_3(F)}\mathbf{1}_{\mathrm{Mat}_3(O_E),v\circ\det=r}(gh)\,\mathrm{d}h.$$

We take (non-Archimedean) Iwasawa decomposition of $h \in \mathrm{GL}_3(F)$ to rewrite it as

$$h = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 \\ & 1 & y_3 \\ & & 1 \end{pmatrix} k$$

for $k \in \mathrm{GL}_3(O_F) \subseteq K'$, which does not affect the indicator function. Here $x_1, x_2, x_3 \in F^\times$ and $y_1, y_2, y_3 \in F$. In that case, note that

$$\begin{aligned} gh &= \begin{pmatrix} 1 & & \varpi^{-j}\sqrt{\varepsilon} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 \\ & 1 & y_3 \\ & & 1 \end{pmatrix} k \\ &= \begin{pmatrix} 1 & & \varpi^{-j}\sqrt{\varepsilon} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_1 y_1 & x_1 y_2 \\ & x_2 & x_2 y_3 \\ & & x_3 \end{pmatrix} k \\ &= \begin{pmatrix} x_1 & x_1 y_1 & x_1 y_2 + x_3 \varpi^{-j}\sqrt{\varepsilon} \\ & x_2 & x_2 y_3 \\ & & x_3 \end{pmatrix} k. \end{aligned}$$

Hence, we can rewrite the $\mathrm{proj}_*(\mathbf{1}_{\mathrm{Mat}_3(O_E), v \circ \det = r})$ as a six-fold integral

$$\begin{aligned} &\mathrm{proj}_*(\mathbf{1}_{\mathrm{Mat}_3(O_E), v \circ \det = r}) \\ &= \int_{x_1 \in F^\times} \int_{x_2 \in F^\times} \int_{x_3 \in F^\times} \int_{y_1 \in F} \int_{y_2 \in F} \int_{y_3 \in F} \\ &\quad \mathbf{1}_{\mathrm{Mat}_3(O_E), v \circ \det = r} \left(\begin{pmatrix} x_1 & x_1 y_1 & x_1 y_2 + x_3 \varpi^{-j}\sqrt{\varepsilon} \\ & x_2 & x_2 y_3 \\ & & x_3 \end{pmatrix} \right) \\ &\quad \mathrm{d}^\times x_1 \, \mathrm{d}^\times x_2 \, \mathrm{d}^\times x_3 \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}y_3. \end{aligned}$$

Apparently the indicator function only depends on the valuations, so accordingly we rewrite the six-fold integral as a discrete sum over the valuations $\alpha_i := v(x_i)$. Then the conditions are that

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_3 \geq j$$
$$v(y_1) \geq -\alpha_1, \quad v(y_2) \geq -\alpha_1, \quad v(y_3) \geq -\alpha_2.$$

We have $\mathrm{Vol}(\varpi^{\alpha_i} O_F^\times) = 1$ and $\mathrm{Vol}(\varpi^{-\alpha_i} O_F) = q^{\alpha_i}$. Hence the integral can be rewritten as the discrete sum

$$\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=r \\ \alpha_1 \geq 0 \\ \alpha_2 \geq 0 \\ \alpha_3 \geq j}} q^{\alpha_1} \cdot q^{\alpha_1} \cdot q^{\alpha_2} = \sum_{\substack{\alpha_1+\alpha_2 \leq r-j \\ \alpha_1 \geq 0 \\ \alpha_2 \geq 0}} q^{2\alpha_1+\alpha_2}$$
$$= \sum_{i=0}^{2(r-j)} \min\left(1 + \left\lfloor \frac{i}{2} \right\rfloor, 1 + \left\lfloor \frac{2(r-j)-i}{2} \right\rfloor\right) q^i$$

as desired. □

4.5.5 Base change from $\mathcal{H}(\mathrm{U}(\mathbb{V}_3^+))$ to $\mathcal{H}(S_3(F))$

We first need to determine an element of $\mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ which is in the pre-image of

$$\mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E) \cap \mathrm{U}(\mathbb{V}_3^+)}$$

under BC: $\mathcal{H}(\mathrm{GL}_3(E)) \to \mathcal{H}(\mathrm{U}(\mathbb{V}_3^+))$.

For convenience, we define the shorthand

$$\mathcal{H}(\mathrm{GL}_3(E)) \ni f'_r := \begin{cases} \mathbf{1}_{\mathrm{Mat}_3(O_E), v \circ \det = r} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

for every integer r. We start with the following intermediate calculation.

$$\begin{aligned}
&\mathrm{BC}\left(\mathrm{Sat}\left(f'_r - q^2 f'_{r-1}\right)\right)\\
&= \mathrm{BC}\left(q^{2r} \sum_{n_1+n_2+n_3=r} X_1^{n_1}X_2^{n_2}X_3^{n_3} - q^2 \cdot q^{2(r-1)} \sum_{n_1+n_2+n_3=(r-1)} X_1^{n_1}X_2^{n_2}X_3^{n_3}\right)\\
&= q^{2r}\left(\sum_{n_1+n_2+n_3=r} Y_1^{n_1-n_3} - \sum_{n_1+n_2+n_3=(r-1)} Y_1^{n_1-n_3}\right)\\
&= q^{2r}\left(\sum_{n_1+n_3=r} Y_1^{n_1-n_3}\right)\\
&= q^{2r}\left(Y_1^r + Y_1^{r-2} + \cdots + Y_1^{-r}\right).
\end{aligned}$$

Replacing r with $r-1$ gives

$$\begin{aligned}
&\mathrm{BC}\left(\mathrm{Sat}\left(f'_{r-1} - q^2 f'_{r-2}\right)\right)\\
&= q^{2r-2}\left(Y_1^{r-1} + Y_1^{r-3} + \cdots + Y_1^{-(r-1)}\right).
\end{aligned}$$

Adding the former equation to q times the latter gives

$$\begin{aligned}
&\mathrm{BC}\left(\mathrm{Sat}\left(f'_r + (q-q^2) f'_{r-1} - q^3 f'_{r-2}\right)\right)\\
&= q^{2r}\left(Y_1^r + Y_1^{r-2} + \cdots + Y_1^{-r}\right) + q^{2r-1}\left(Y_1^{r-1} + Y_1^{r-3} + \cdots + Y_1^{-(r-1)}\right)\\
&= \mathrm{Sat}(\mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}).
\end{aligned}$$

This shows that

$$\mathrm{BC}(f'_r + (q-q^2) f'_{r-1} - q^3 f'_{r-2}) = \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}$$

so indeed $f'_r + (q-q^2) f'_{r-1} - q^3 f'_{r-2}$ lies in the desired pre-image of the map $\mathrm{BC}\colon \mathcal{H}(\mathrm{GL}_3(E)) \to \mathcal{H}(\mathrm{U}(\mathbb{V}_3^+))$.

On the other hand, it is easy to check that

$$
\begin{aligned}
&\operatorname{proj}_*(f'_r - q^2 f'_{r-1}) \\
&= \sum_{j=0}^{r} \left[\sum_{i=0}^{2(r-j)} \min\left(1 + \left\lfloor \frac{i}{2} \right\rfloor, 1 + \left\lfloor \frac{2(r-j)-i}{2} \right\rfloor\right) q^i \right. \\
&\qquad \left. - \sum_{i=0}^{2(r-1-j)} \min\left(1 + \left\lfloor \frac{i}{2} \right\rfloor, 1 + \left\lfloor \frac{2((r-1)-j)-i}{2} \right\rfloor\right) q^{i+2} \right] \mathbf{1}_{K'_{S,j}} \\
&= \sum_{j=0}^{r} \left[1 + q + q^2 + \cdots + q^{r-j}\right] \mathbf{1}_{K'_{S,j}}
\end{aligned}
$$

so

$$
\begin{aligned}
&\operatorname{proj}_*(f'_r - q^2 f'_{r-1} + q\left(f'_{r-1} - q^2 f'_{r-3}\right)) \\
&= \sum_{j=0}^{r} \left[(1 + q + q^2 + \cdots + q^{r-j}) + (q + q^2 + \cdots + q^{r-j})\right] \mathbf{1}_{K'_{S,j}} \\
&= \sum_{j=0}^{r} \left[1 + 2q + 2q^2 + \cdots + 2q^{r-j}\right] \mathbf{1}_{K'_{S,j}}.
\end{aligned}
$$

To summarize, the completed commutative diagram can be written in full as

$$
\begin{tikzcd}
\begin{matrix} f'_r + (q-q^2) f'_{r-1} \\ -q^3 f'_{r-2} \in \mathcal{H}(\mathrm{GL}_3(E)) \end{matrix} \arrow[r, mapsto, "\mathrm{Sat}"] \arrow[d, mapsto, "\mathrm{BC}"] \arrow[dd, bend right=60, "\operatorname{proj}_*"'] & \cdots \in \mathbb{Q}[X_1^{\pm}, X_2^{\pm}, X_3^{\pm}]^{\mathrm{Sym}(3)} \arrow[d, mapsto, "\mathrm{BC}"] \\
\begin{matrix} \mathbf{1}_{\varpi^{-r} \mathrm{Mat}_3(O_E) \cap \mathrm{U}(\mathbb{V}_3^+)} \\ \in \mathcal{H}(\mathrm{U}(\mathbb{V}_3^+)) \end{matrix} \arrow[r, mapsto, "\mathrm{Sat}"] & \begin{matrix} q^{2r}\left(Y_1^{\pm r} + \cdots + Y_1^{\mp r}\right) \\ + q^{2r-1}\left(Y_1^{\pm(r-1)} + \cdots + Y_1^{\mp(r-1)}\right) \\ \in \mathbb{Q}[Y_1^{\pm}]^{W_1} \end{matrix} \\
\begin{matrix} \sum_{j=0}^{r} \left[1 + 2q + 2q^2 \right. \\ \left. + \cdots + 2q^{r-j}\right] \mathbf{1}_{K'_{S,j}} \\ \in \mathcal{H}(S_3(F)) \end{matrix} \arrow[u, mapsto, "\mathrm{BC}_{S_3}", "\sim"'] &
\end{tikzcd}
$$

Thus, we arrive at the following:

Proposition 4.5.4 (Base change BC_{S_3})**.** *For $n = 3$, we have*

$$\mathrm{BC}_{S_3}\left(\sum_{j=0}^{r}\left[1 + 2q + 2q^2 + \cdots + 2q^{r-j}\right]\mathbf{1}_{K'_{S,j}}\right) = \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}$$

$$\mathrm{BC}_{S_3}\left(\mathbf{1}_{K'_{S,r}} + \sum_{j=0}^{r-1} 2q^{r-j}\mathbf{1}_{K'_{S,j}}\right) = \mathbf{1}_{K\varpi^{(r,0,-r)}K}$$

for every integer $r \geq 0$.

Proof. The first equation is the one we just proved. The second one follows by noting that

$$\mathbf{1}_{K\varpi^{(r,0,-r)}K} = \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)} - \mathbf{1}_{\varpi^{-(r-1)}\,\mathrm{Mat}_3(O_E)\cap \mathrm{U}(\mathbb{V}_3^+)}$$

so one merely subtracts the left-hand sides evaluated at r and $r-1$ for $r \geq 1$ to get

$$\begin{aligned}
&\sum_{j=0}^{r}\left[1 + 2q + 2q^2 + \cdots + 2q^{r-j}\right]\mathbf{1}_{K'_{S,j}} - \sum_{j=0}^{r-1}\left[1 + 2q + 2q^2 + \cdots + 2q^{(r-1)-j}\right]\mathbf{1}_{K'_{S,j}} \\
&= \mathbf{1}_{K'_{S,r}} + \sum_{j=0}^{r-1}\left[1 + 2q + 2q^2 + \cdots + 2q^{r-j}\right]\mathbf{1}_{K'_{S,j}} - \sum_{j=0}^{r-1}\left[1 + 2q + 2q^2 + \cdots + 2q^{(r-1)-j}\right]\mathbf{1}_{K'_{S,j}} \\
&= \mathbf{1}_{K'_{S,r}} + \sum_{j=0}^{r-1}\left[2q^{r-j}\mathbf{1}_{K'_{S,j}}\right].
\end{aligned}$$

as claimed. $\square$

Chapter 5

Synopsis of the weighted orbital integral $\mathrm{Orb}(\gamma, \phi, s)$ for $\gamma \in S_3(F)_{\mathrm{rs}}$ and $\phi \in \mathcal{H}(S_3(F))$

This section defines the weighted orbital integral and describes the parameters which we will use to express our answer.

5.1 Initial definition of the weighted orbital integral for general $S_n(F)$

Let $H = \mathrm{GL}_{n-1}(F)$. Then H has a natural embedding into $\mathrm{GL}_n(E)$ by

$$h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$$

which endows it with an action $S_n(F)$. Then our weighted orbital integral is defined as follows.

Definition 5.1.1 ([LRZ24, Equation (3.2.3)]). For brevity let $\eta(h) := \eta(\det h)$ for $h \in H$. For $\gamma \in S_n(F)$, $\phi \in \mathcal{H}(S_n(F))$, and $s \in \mathbb{C}$, we define the *weighted orbital integral* by

$$\mathrm{Orb}(\gamma, \phi, s) := \int_{h \in H} \phi(h^{-1}\gamma h)\eta(h) \left|\det(h)\right|_F^{-s} \mathrm{d}h.$$

Definition 5.1.2 (The abbreviation $\partial \mathrm{Orb}(\gamma, \phi)$). From now on we will abbreviate

$$\partial \mathrm{Orb}(\gamma, \phi) := \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Orb}(\gamma, \phi, s).$$

We remark that this weighted orbital integral is related to an (unweighted) orbital integral on the unitary side by the so-called relative fundamental lemma. Specifically, for $g \in \mathrm{U}(\mathbb{V}_n^+)$ and $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$, we define the (unweighted) orbital integral by

$$\mathrm{Orb}^{\mathrm{U}(\mathbb{V}_n^+)}(g, f) := \int_{\mathrm{U}(\mathbb{V}_n^+)} f(x^{-1}gx)\, \mathrm{d}x.$$

Then the following result is true.

Theorem 5.1.3 (Relative fundamental lemma; [Les23, Theorem 1.1]). *Let $\phi \in \mathcal{H}(S_n(F))$ and $\gamma \in S_n(F)_{\mathrm{rs}}$.*

$$\omega(\gamma)\,\mathrm{Orb}(\phi, \gamma, 0) = \begin{cases} 0 & \text{if } \gamma \in S_n(F)_{\mathrm{rs}}^- \\ \mathrm{Orb}^{\mathrm{U}(\mathbb{V}_n^+)}(g, \mathrm{BC}_{S_n}^{\eta^{n-1}}(\phi)) & \text{if } \gamma \in S_n(F)_{\mathrm{rs}}^+ \end{cases}$$

where the transfer factor ω is defined in Chapter 13.

5.2 Basis for the indicator functions in $\mathcal{H}(S_3(F))$

From now on assume $n = 3$. We have the symmetric space

$$S_3(F) := \{g \in \mathrm{GL}_3(E) \mid g\bar{g} = \mathrm{id}_3\}$$

which has a left action under $\mathrm{GL}_3(E)$ by $g \cdot s \mapsto gs\bar{g}^{-1}$.

Then $S_3(F)$ admits the following decomposition, which we will use:

Lemma 5.2.1 (Cartan decomposition of $S_3(F)$)**.** *For each integer $r \geq 0$ let*

$$K'_{S,r} \coloneqq \mathrm{GL}_3(O_E) \cdot \begin{pmatrix} 0 & 0 & \varpi^r \\ 0 & 1 & 0 \\ \varpi^{-r} & 0 & 0 \end{pmatrix}$$

denote the orbit of $\begin{pmatrix} 0 & 0 & \varpi^r \\ 0 & 1 & 0 \\ \varpi^{-r} & 0 & 0 \end{pmatrix}$ *under the left action of* $\mathrm{GL}_3(O_E)$*. Then we have a decomposition*

$$S_3(F) = \coprod_{r \geq 0} K'_{S,r}.$$

Proof. See [Off04, §3]. $\square$

The $r = 0$ case will be given a special shorthand, and can be expressed in a few equivalent ways:

$$\begin{aligned} K'_S &\coloneqq K'_{S,0} \\ &= \mathrm{GL}_3(O_E) \cdot \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \\ &= \mathrm{GL}_3(O_E) \cdot \mathrm{id}_3 = S_3(F) \cap \mathrm{GL}_3(O_E). \end{aligned}$$

One can equivalently define $K'_{S,r}$ to be the part of $S_3(F)$ for which the most negative valuation among the nine entries is $-r$.

For $r \geq 0$, define

$$K'_{S,\leq r} \coloneqq S_3(F) \cap \varpi^{-r}\,\mathrm{GL}_3(O_E).$$

We can re-parametrize the problem according to the following.

Corollary 5.2.2. *We have a decomposition*

$$K'_{S,\leq r} = K'_{S,0} \sqcup K'_{S,1} \sqcup \cdots \sqcup K'_{S,r}.$$

Then an integral over each $K'_{S,\leq r}$ lets us extract the integrals over $K'_{S,r}$.

Corollary 5.2.3 (Basis for $\mathcal{H}(S_3(F))$)**.** *For $r \geq 0$, the indicator functions $\mathbf{1}_{K'_{S,\leq r}}$ form a basis of $\mathcal{H}(S_3(F))$.*

Then, our goal is to compute for $\partial \operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}})$ at $s = 0$ for any $r > 0$ as well; note that the $r = 0$ case is already done in [Zha12].

5.3 Parametrization of γ

Again, assume $n = 3$. Further assume $\gamma \in S_3(F)_{\mathrm{rs}}$ is regular semisimple. We identify some parameters for the orbit of γ that we can use for our explicit calculations.

5.3.1 Rewriting the weighted orbital integral as a double integral over E via the group $H' \cong \mathrm{GL}_2(F)$

Our weighted orbital integral is at present a quadruple integral over F, owing to $H = \mathrm{GL}_2(F)$ being a four-dimensional F-vector space.

It will be more economical to work with the weighted orbital integral as a double integral with two coefficients in E, in the following sense. As in [Zha12, §4.1] define

$$H' := \left\{ \begin{pmatrix} t_1 & t_2 \\ \bar{t}_2 & \bar{t}_1 \end{pmatrix} \mid t_1, t_2 \in E \right\}$$

which is indeed a four-dimensional F-algebra. As before $H' \hookrightarrow \mathrm{U}(\mathbb{V}_n^+)$ according to the same embedding $\mathrm{GL}_2(E) \hookrightarrow \mathrm{GL}_3(E)$ and so H' also acts on $S_n(E)$ by conjugation.

As an F-algebra, we have an isomorphism (see [Zha12, §4.1])

$$\begin{aligned}
\iota_2 \colon H = \mathrm{GL}_2(F) &\xrightarrow{\cong} H' \\
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &\mapsto \begin{pmatrix} t_1 & t_2 \\ \bar{t}_2 & \bar{t}_1 \end{pmatrix} \\
t_1 &= \frac{1}{2}\left(a_{11} + a_{22} + \frac{a_{12}}{\sqrt{\varepsilon}} + a_{21}\sqrt{\varepsilon}\right) \\
t_2 &= \frac{1}{2}\left(a_{11} - a_{22} + \frac{a_{12}}{\sqrt{\varepsilon}} - a_{21}\sqrt{\varepsilon}\right).
\end{aligned}$$

Under this isomorphism, we have

$$h\gamma h^{-1} = \iota_2(h)\gamma\overline{\iota_2(h)^{-1}}.$$

This allows us to rewrite the weighted orbital integral over H' instead. If we write $h' = \overline{\iota_2(h)^{-1}}$, then the following integral formula is obtained.

Proposition 5.3.1 ([Zha12, §4.2])**.** *For brevity let $\eta(h') := \eta(\det h')$ for $h' \in H'$. For $\gamma \in S_3(F)$, $\phi \in \mathcal{H}(S_3(F))$, and $s \in \mathbb{C}$, the weighted orbital integral can instead be written as*

$$\mathrm{Orb}(\gamma, \phi, s) = \int_{h' \in H'} \phi(\bar{h}'^{-1}\gamma h')\eta(h')\left|\det(h')\right|_F^s \,\mathrm{d}h'$$

where

$$\mathrm{d}h' = \kappa \cdot \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{|t_1\bar{t}_1 - t_2\bar{t}_2|_F^2}$$

for the constant

$$\kappa := \frac{1}{(1-q^{-1})(1-q^{-2})}.$$

5.3.2 Identifying a representative in the H'-orbit

Write $\gamma \overset{H'}{\sim} \gamma'$ to mean that γ and γ' are in the same H'-orbit. Evidently the weighted orbital integral $\mathrm{Orb}(\gamma, \phi, s)$ in Proposition 5.3.1 only depends on such an orbit. So it makes sense to pick a canonical representative for the H'-orbit to compute the weighted orbital integral in terms of.

Since we assumed $\gamma \in S_3(F)_{\mathrm{rs}}$ is regular semisimple, we can invoke [Zha12, Proposition 4.1] to assume that γ that under the orbit of H' we have

$$\gamma \overset{H'}{\sim} \begin{pmatrix} a & 0 & 0 \\ b & -\bar{d} & 1 \\ c & 1 - d\bar{d} & d \end{pmatrix} \in S_3(F)_{\mathrm{rs}}; \quad \text{where } c = -a\bar{b} + bd$$

over all $a \in E^1$, $b \in E$, $d \in E$ for which $(1 - d\bar{d})^2 - c\bar{c} \neq 0$. In other words, the representatives described here cover all the regular H'-orbits in $S_3(F)_{\mathrm{rs}}$.

5.3.3 Simplification due to the matching of non-quasi-split unitary group

In this calculation, we restrict attention to the case where our regular γ matches an element in the non-quasi-split unitary group $\mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$ (rather than $\mathrm{U}(\mathbb{V}_n^+)_{\mathrm{rs}}$). As we described in Proposition 3.2.2, this is controlled by the parity of the invariant

$$v\left((1 - d\bar{d})^2 - c\bar{c}\right)$$

being odd. Hence, we only have to consider this case:

Assumption 5.3.2. *We will assume that*

$$v\left((1 - d\bar{d})^2 - c\bar{c}\right) \equiv 1 \pmod 2.$$

This is the same assumption made in [Zha12, Equation (4.3)].

To summarize everything we've said.

Lemma 5.3.3 (Parametrization for $S_3(F)^-_{\mathrm{rs}}$ via H'-orbit)**.** *Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$. Then there exists $a \in E^1$, $b \in E$, $d \in E$ such that*

$$\gamma \overset{H'}{\sim} \begin{pmatrix} a & 0 & 0 \\ b & -\bar{d} & 1 \\ c & 1-d\bar{d} & d \end{pmatrix} \in S_3(F)_{\mathrm{rs}}; \quad \textit{where } c = -a\bar{b} + bd.$$

Moreover, $v\left((1-d\bar{d})^2 - c\bar{c}\right)$ is odd.

We will mostly be interested in the case where $v(b) = v(d) = 0$. In fact, few other cases even occur at all given Assumption 5.3.2; we will see momentarily that either $v(b) = v(d) < 0$, or one of $\{v(b), v(d)\}$ is zero and the other is nonnegative.

5.4 Parameters used in the calculation of the weighted orbital integral

We now evaluate the orbital integral in terms of

$$a \in E^1, \qquad b, d \in E, \qquad r \geq 0.$$

To simplify the notation in what follows, it will be convenient to define several quantities that reappear frequently. From Assumption 5.3.2, we may define

$$\delta := v(1-d\bar{d}) = v(c) \neq -\infty. \tag{5.1}$$

Following [Zha12, Equation (4.3)] we will also define

$$u \coloneqq \frac{\bar{c}}{1 - d\bar{d}} \in O_E^\times \tag{5.2}$$

so that $\nu(1 - u\bar{u}) \equiv 1 \pmod 2$ and

$$b = -au - \bar{d}\bar{u}. \tag{5.3}$$

Note that this gives us the following repeatedly used identity

$$b^2 - 4a\bar{d} = (au - \bar{d}\bar{u})^2 - 4a\bar{d}(1 - u\bar{u}). \tag{5.4}$$

Finally, define

$$\ell \coloneqq v(b^2 - 4a\bar{d}). \tag{5.5}$$

We will also define one additional parameter which is useful when ℓ is even (but as we will see, redundant for odd ℓ):

$$\lambda \coloneqq v(1 - u\bar{u}) \equiv 1 \pmod 2. \tag{5.6}$$

Just as many pairs $(v(b), v(d))$ do not occur (given Assumption 5.3.2) and $v(b) = v(d) = 0$ is the main case of interest, the parameters (δ, ℓ, λ) satisfy some additional relations. We will now describe them.

Lemma 5.4.1 (Constraints between ℓ, δ, λ)**.** *Let a, b, c, d be as in Lemma 5.3.3, and let δ, ℓ, and λ be as defined in (5.1), (5.5), (5.6). Then exactly one of the following situations is true:*

- *$v(b) = v(d) = 0$, $\ell \geq 1$ is odd, $\ell < 2\delta$, and $\lambda = \ell$.*
- *$v(b) = v(d) = 0$, $\ell \geq 0$ is even, $\ell \leq 2\delta$, and $\lambda > \ell$ is odd.*
- *$v(b) = 0$, $v(d) > 0$, $\ell = \delta = 0$, and $\lambda > 0$ is odd.*

- $v(b) > 0$, $v(d) = 0$, $\ell = 0$, $\delta \geq 0$, *and* $\lambda > 0$ *is odd.*

- $v(b) = v(d) < 0$, $\ell = \delta = 2v(d) < 0$, *and* $\lambda > 0$ *is odd.*

See Table 5.1. Moreover, whenever ℓ *is even, the quantity* $b^2 - 4a\bar{d}$ *is a square of some element in* E.

	$v(b) = 0$	$v(b) > 0$	$v(b) < 0$
$v(d) = 0$	$0 \leq \ell \leq 2\delta$	$\ell = 0,\ \delta \geq 0$	never
$v(d) > 0$	$\ell = \delta = 0$	never	never
$v(d) < 0$	never	never	$v(b) = v(d) = \frac{\ell}{2} = \frac{\delta}{2} < 0$

Table 5.1: A table showing the five cases in Lemma 5.4.1.

Before proving the lemma in full we first prove the following lemmas.

Lemma 5.4.2 ($v(au - \bar{d}\bar{u})$)**.** *Assume* $v(d) \geq 0$. *For odd* ℓ, *we have*

$$2v(au - \bar{d}\bar{u}) > \ell = \lambda$$

while for even ℓ *we instead have*

$$2v(au - \bar{d}\bar{u}) = \ell < \lambda.$$

Proof. If $v(d) = 0$, this follows from (5.4) directly, since $v\left((au - \bar{d}\bar{u})^2\right)$ is even, and hence can never equal $v(4a\bar{d}(1 - u\bar{u})) = \lambda \equiv 1 \pmod{2}$.

Meanwhile, if $v(d) > 0$, then from (5.3) it follows $v(b) = 0$, and hence $\ell = 0$. And $v(au - \bar{d}\bar{u}) = 0$ in this case as well. Since λ is a positive odd integer, the lemma is proved. $\square$

Lemma 5.4.3 ($\ell \leq 2\delta$)**.** *If* $v(b) = v(d) = 0$ *and* $\ell \geq 0$, *then* $\ell \leq 2\delta$.

Proof. If $\ell = 0$ there is nothing to prove so assume $\ell > 0$. Let us write

$$O_E^\times \ni \frac{adu}{\bar{u}} = x + y\sqrt{\varepsilon} \qquad x, y \in F \tag{5.7}$$

which has norm

$$O_F^\times \ni x^2 + y^2\varepsilon = \frac{adu}{\bar{u}} \cdot \frac{\bar{a}\bar{d}\bar{u}}{u} = d\bar{d}. \tag{5.8}$$

Now, according to Lemma 5.4.2 we have that

$$0 < \ell \leq 2v(au - \bar{d}\bar{u}) = 2v\left(\frac{adu}{\bar{u}} - d\bar{d}\right) = 2v\left((x - d\bar{d}) + y\sqrt{\varepsilon}.\right)$$

and since $d\bar{d} \in F$, it follows that

$$v\left(x - d\bar{d}\right) > \frac{\ell}{2} \tag{5.9}$$

$$v(y) > \frac{\ell}{2}. \tag{5.10}$$

In particular, (5.10) implies $v(y) > 0$ which has two consequences:

- From (5.7) we get $v(x) = 0$.
- From $v(y^2) > 0$ and (5.8) we conclude

$$v(x^2 - d\bar{d}) = v(y^2) > \ell.$$

Putting (5.9) together with the previous two bullets,

$$\frac{\ell}{2} \leq v\left(x^2 - d\bar{d} - x(x - d\bar{d})\right) = v(x) + v(1 - d\bar{d}) = 0 + \delta$$

and this proves $\ell \leq 2\delta$. □

Now we can prove Lemma 5.4.1.

Proof of Lemma 5.4.1. It's clear the five bullets above are disjoint.

- First assume ℓ is odd. We assert in this case we have $v(b) = v(d) = 0$. Indeed if $v(d) \neq 0$, then $b = -au - \bar{d}\bar{u}$ is a unit, and hence so is $b^2 - 4a\bar{d}$, causing $\ell = 0$, contradiction.

And if d is a unit, $\ell \neq 0$ means $v(b) = 0$ too. In particular, $\ell > 0$. The rest of the claims follow by Lemma 5.4.2 and Lemma 5.4.3.

For the rest of the proof we only consider even ℓ. Because $b = -au - \bar{d}\bar{u}$, it cannot be the case that $v(b) > 0$ and $v(d) > 0$; moreover if either $v(b) < 0$ or $v(d) < 0$, then in fact $v(b) = v(d)$. We consider each of the four possibilities.

- Suppose and $v(b) = v(d) = 0$. Then Lemma 5.4.2 and Lemma 5.4.3 imply the results.
- If $v(b) = 0$ and $v(d) > 0$, then $b^2 - 4a\bar{d}$ is a unit and $1 - d\bar{d}$ are both units, ergo $\ell = \delta = 0$.
- If $v(b) > 0$ and $v(d) = 0$ then $b^2 - 4a\bar{d}$ is a unit, and there is nothing left to prove.
- Finally suppose $v(b) = v(d) < 0$. Then $v(b^2) < v(4a\bar{d}) < 0$, so indeed $\ell = 2v(d) < 0$. And $v(1 - d\bar{d}) = 2v(d) < 0$ as well.

We now verify the last assertion that $b^2 - 4a\bar{d}$ is a square whenever ℓ is even. The proof in all cases uses (5.4) to show $b^2 - 4a\bar{d}$ is equal to ϖ^ℓ times a quadratic residue in $O_E^\times$. Indeed we need only verify that $v(4a\bar{d}(1 - u\bar{u})) = v(d) + \lambda$ has larger valuation than $v\left((au - \bar{d}\bar{u})^2\right) = \ell$. In the case $v(b) = 0$ this follows from $\lambda > \ell$. Whereas if $v(d) > 0$ we have $\ell = 0$ and if $v(b) = v(d) < 0$ then $\ell = -2v(d)$; so in all these cases the claim is obvious too. □

In the case where ℓ is odd (and hence $\ell \geq 1$ and $v(b) = v(d) = 0$), we get (5.4) implying $\lambda = \ell$ and thus λ will never be used — the weighted orbital will be computed as a function of ℓ and δ (and r).

However for even ℓ these numbers are never equal and our weighted orbital integral will be stated in terms of ℓ, δ, and λ (and r). We just saw that in these situations $b^2 - 4a\bar{d}$ is a square; moving forward, we need to fix the choice of the square root τ. We do so as follows.

Definition 5.4.4 (Fixing the choice of τ). Assuming ℓ is even, using (5.4) in the form

$$(au - \bar{d}\bar{u})^2 - \tau^2 = 4a\bar{d}(1 - u\bar{u})$$

we agree now to fix the choice of the square root of τ such that

$$\begin{aligned} v(au - \bar{d}\bar{u} + \tau) &= \lambda + v(d) - \frac{1}{2}\ell > 0, \\ v(au - \bar{d}\bar{u} - \tau) &= \frac{1}{2}\ell. \end{aligned} \tag{5.11}$$

Here $\lambda + v(d) - \frac{1}{2}\ell > 0$ is obvious when $v(d) \geq 0$ (since $\lambda = \ell > 0$ for odd ℓ and otherwise $\lambda > \ell$), and for $v(d) < 0$ we have $v(d) = \frac{1}{2}\ell$ anyway.

Lemma 5.4.5 ($v(4 - \mathrm{N}(b \pm \tau))$). *With this choice of τ, we have*

$$\begin{aligned} v\left(4 - \mathrm{N}(b + \tau)\right) &= \lambda + \delta - \ell \\ v\left(4 - \mathrm{N}(b - \tau)\right) &= \delta. \end{aligned} \tag{5.12}$$

Proof. We consider several cases.

- If $v(b) = v(d) = 0$ then from (5.4) we have

  $$\ell = 2v(\tau) = 2v(au - \bar{d}\bar{u}) < \lambda$$

  and thus [Zha12, Lemma 4.7] applies to give (5.12), verabtim.

- Now suppose $v(d) > 0$ but still $v(b) = v(\tau) = 0$. We begin with the observation that

  $$4a\bar{d} = b^2 - \tau^2 = (b + \tau)(b - \tau) \tag{5.13}$$

  and so $\{v(b+\tau), v(b-\tau)\} = \{0, v(d)\}$. We need to determine which is which. However, note that we may write

  $$au - \bar{d}\bar{u} - \tau = -(b + \tau) - 2\bar{d}\bar{u}.$$

  Since $v(au - \bar{d}\bar{u} - \tau) = 0$ and $v(d) > 0$, it follows we must have $v(b + \tau) = 0$. And thus

$v(b-\tau) = v(d)$. Hence $v(4 - \mathrm{N}(b-\tau)) = 0 = \delta$, and we have obtained the bottom equation of (5.12).

It remains to show that $v(4 - \mathrm{N}(b+\tau)) = \lambda$ to complete the proof. We quote [Zha12, Lemma 4.6] which states more generally that

$$\begin{aligned} 2\delta + \lambda = v(4 - \mathrm{N}(b+\tau)) + v(4 - \mathrm{N}(b-\tau)) \\ + v(16 + 16d\bar{d} - 8b\bar{b} + 8\tau\bar{\tau}). \end{aligned}$$

In our case $\delta = 0$, $v(4-\mathrm{N}(b-\tau)) = 0$. Moreover since $v(\tau - b) > 0$ we get $v(\tau\bar{\tau} - b\bar{b}) > 0$. (Indeed, if $\tau = x_\tau + \varepsilon y_\tau$ and $b = x_b + \varepsilon y_b$, then $\tau\bar{\tau} - b\bar{b} = (x_\tau^2 - x_b^2 + \varepsilon(y_\tau^2 - y_b^2)) + \varepsilon(x_\tau y_\tau - y_\tau y_b)$. Since $x_\tau \equiv x_b \pmod{\varpi}$ and $y_\tau \equiv y_b \pmod{\varpi}$, the conclusion is immediate.) Hence the final term on the right-hand side is 0 too.

- Consider $v(b) > 0$. As mentioned on [Zha12, p. 242], the identity (5.12) is still true in this situation too.

- Finally assume $v(\tau) = v(b) = v(d) < 0$. Again from (5.13) we know $\{v(b+\tau), v(b-\tau)\} = \{0, v(d)\}$ and need to determine which is which. This time we write

$$au - \bar{d}\bar{u} + \tau = -(b-\tau) - 2\bar{d}\bar{u}.$$

Since $v(au - \bar{d}\bar{u} + \tau) = \lambda > 0$, but $v(2\bar{d}\bar{u}) = v(d) < 0$, it follows we must have $v(b-\tau) < 0$, so in fact $v(b-\tau) = v(d)$ and $v(b+\tau) = 0$. So, in this case, we get $v(4 - \mathrm{N}(b-\tau)) = 2v(b) = \delta$, which is the bottom equation of (5.12).

Then it remains to show that $v(4 - \mathrm{N}(b+\tau)) = \lambda$, which is done in the same way as $v(d) > 0$ earlier. $\square$

5.5 Statement of the full orbital integral

We now show the full orbital integrals that were previously only summarized as Theorem 1.3.9. The proof of these formulas is carried out in Chapters 6 and 7.

5.5.1 Arches

We introduce one piece of notation to compress the particular shape our formulas are about to take.

Definition 5.5.1. Suppose $\{a_0, a_0+1, \dots, a_1\}$ is an interval of integers for some $a_0 \le a_1$, and consider two more integers w_1 and w_2 such that $w_1 + w_2 \le \frac{a_1 - a_0}{2}$. Then we can define a piecewise linear function

$$\underset{[a_0,a_1]}{\mathrm{ARCH}}(w_1, w_2) \colon \{a_0, a_0+1, \dots, a_1\} \to \mathbb{Z}_{\ge 0}$$

according to the following definition:

$$k \mapsto \begin{cases} k - a_0 & \text{if } a_0 \le k \le a_0 + w_1 \\ w_1 + \left\lfloor \frac{k-(a_0+w_1)}{2} \right\rfloor & \text{if } a_0 + w_1 \le k \le a_0 + w_1 + w_2 \\ w_1 + \left\lfloor \frac{w_2}{2} \right\rfloor & \text{if } a_0 + w_1 + w_2 \le k \le a_1 - (w_1 + w_2) \\ w_1 + \left\lfloor \frac{(a_1-w_1)-k}{2} \right\rfloor & \text{if } a_1 - (w_1 + w_2) \le k \le a_1 - w_1 \\ a_1 - k & \text{if } a_1 - w_1 \le k \le a_1. \end{cases}$$

The nomenclature is meant to be indicative of the shape of the graph, which looks a little bit like an arch. It is a function symmetric around $\frac{a_0+a_1}{2}$ defined piecewise. The function grows linearly with slope 1 at the far left for w_1 steps, then changes to slope $1/2$ for w_2 steps (rounding down), before stabilizing, then doing the symmetric descent on the right half.

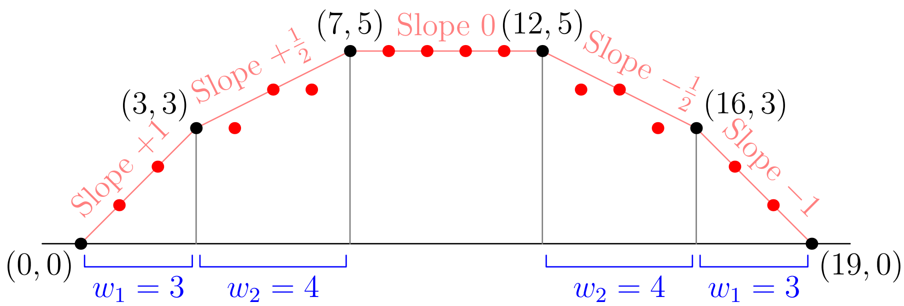

Figure 5.1: A plot of $\mathrm{ARCH}_{[0,19]}(3, 4)$.

5.5.2 Full explicit weighted orbital integral for the case where ℓ odd

If ℓ is odd, so $\lambda = \ell$, then the weighted orbital integral can be expressed succinctly in the following way.

Theorem 5.5.2 (Weighted orbital integral for odd ℓ)**.** *Let $r \geq 0$. Let $\gamma \in S_3(F)^{-}_{\mathrm{rs}}$ and let b, d, δ, ℓ, be as in Lemma 5.3.3 and Lemma 5.4.1. If ℓ is odd, define*

$$\mathbf{n}_\gamma := \underset{[-2r,\ell+2\delta+2r]}{\mathrm{ARCH}}(r, \ell).$$

Then for any $r \geq 0$ we have the formula:

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = \sum_{k=-2r}^{\ell+2\delta+2r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right) (q^s)^k.$$

Remark 5.5.3. To make $\mathbf{n}_\gamma$ fully explicit, one could also expand the arch shorthand to

$$\mathbf{n}_\gamma(k) := \begin{cases} k + 2r & \text{if } -2r \leq k \leq -r \\ \left\lfloor \frac{k+r}{2} \right\rfloor + r & \text{if } -r \leq k \leq \ell - r \\ \frac{\ell-1}{2} + r & \text{if } \ell - r \leq k \leq 2\delta + r \\ \left\lfloor \frac{(\ell+2\delta+r)-k}{2} \right\rfloor + r & \text{if } 2\delta + r \leq k \leq \ell + 2\delta + r \\ (\ell + 2\delta + 2r) - k & \text{if } \ell + 2\delta + r \leq k \leq \ell + 2\delta + 2r. \end{cases}$$

From the identity

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,r}}, s) = \mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) - \mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq (r-1)}}, s)$$

we can then write the following equivalent formulation.

Corollary 5.5.4. *Retaining the setting of the previous theorem, we have for any $r \geq 1$ the formula*

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,r}}, s) = \sum_{k=-2r}^{\ell+2\delta+2r} (-1)^k q^{\mathbf{n}_\gamma(k)} (1+q^{-1})^{\mathbf{1}[k \in \mathcal{I}_{\gamma,r}]} (q^s)^k$$

where $\mathcal{I}_{\gamma,r}$ is the set of indices defined by

$$\begin{aligned} \mathcal{I}_{\gamma,r} \coloneqq & \{-(2r-1), -(2r-2), -(2r-3), \dots, -(r+1)\} \\ & \sqcup \{-r, -r+2, -r+4 \dots, -r+\ell-1\} \\ & \sqcup \{2\delta+r+1, 2\delta+r+3, \dots, 2\delta+r+1, 2\delta+r+3, \dots, 2\delta+\ell+r\} \\ & \sqcup \{\ell+2\delta+r+1, \ell+2\delta+r+2, \dots, \ell+2\delta+2r-1\}. \end{aligned}$$

Example 5.5.5. If $r = 3$, $\ell = 5$, and $\delta = 100$, the formulas above read

$$\begin{aligned} \mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq 3}}, s) = & \, q^{-6s} \\ & - (q+1) \cdot q^{-5s} \\ & + (q^2+q+1) \cdot q^{-4s} \\ & - (q^3+q^2+q+1) \cdot q^{-3s} \\ & + (q^3+q^2+q+1) \cdot q^{-2s} \\ & - (q^4+q^3+q^2+q+1) \cdot q^{-s} \\ & + (q^4+q^3+q^2+q+1) \cdot q^{0} \\ & - (q^5+q^4+q^3+q^2+q+1) \cdot q^{s} \\ & + (q^5+q^4+q^3+q^2+q+1) \cdot q^{2s} \end{aligned}$$

$$\begin{aligned}
&\vdots \\
&+ (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{204s} \\
&- (q^4 + q^3 + q^2 + q + 1) \cdot q^{205s} \\
&+ (q^4 + q^3 + q^2 + q + 1) \cdot q^{206s} \\
&- (q^3 + q^2 + q + 1) \cdot q^{207s} \\
&+ (q^3 + q^2 + q + 1) \cdot q^{208s} \\
&- (q^2 + q + 1) \cdot q^{209s} \\
&+ (q + 1) \cdot q^{210s} \\
&- q^{211s}.
\end{aligned}$$

In the ellipses, all the omitted terms have the same coefficient $q^5 + q^4 + q^3 + q^2 + q + 1$ and alternate sign.

Example 5.5.6. Continuing the previous example with $r = 3$, $\ell = 5$, and $\delta = 100$, we have

$$\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,3}}, s) = q^{-6s} \\
&- (q + 1) \cdot q^{-5s} \\
&+ (q^2 + q) \cdot q^{-4s} \\
&- (q^3 + q^2) \cdot q^{-3s} \\
&+ q^3 \cdot q^{-2s} \\
&- (q^4 + q^3) \cdot q^{-s} \\
&+ q^4 \cdot q^0 \\
&- (q^5 + q^4) \cdot q^{s} \\
&+ q^5 \cdot q^{2s} \\
&- q^5 \cdot q^{3s} \\
&+ q^5 \cdot q^{4s}
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
&+ q^5 \cdot q^{202s} \\
&- q^6 \cdot q^{203s} \\
&+ (q^5 + q^4) \cdot q^{204s} \\
&- q^4 \cdot q^{205s} \\
&+ (q^4 + q^3) \cdot q^{206s} \\
&- q^3 \cdot q^{207s} \\
&+ (q^3 + q^2) \cdot q^{208s} \\
&- (q^2 + q) \cdot q^{209s} \\
&+ (q + 1) \cdot q^{210s} \\
&- q^{211s}.
\end{aligned}$$

5.5.3 Full explicit weighted orbital integral for the case where $\ell \geq 0$ is even

When ℓ is even the formula has a change to the leading coefficient as well.

Theorem 5.5.7 (Weighted orbital integral for even $\ell \geq 0$)**.** *Let $r \geq 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let b, d, δ, ℓ, λ be as in Lemma 5.3.3 and Lemma 5.4.1. Suppose also $\ell \geq 0$ is even. Define*

$$\begin{aligned}
\mathbf{n}_\gamma &:= \operatorname*{ARCH}_{[-2r, \lambda+2\delta+2r]}(r, \ell) \\
\mathbf{c}_\gamma &:= \operatorname*{ARCH}_{[\ell-r, \lambda-\ell+2\delta+r]}(\delta - \ell/2, \min(2r, \lambda - \ell)).
\end{aligned}$$

Then for any $r \geq 0$ we have:

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = \sum_{k=-2r}^{\lambda+2\delta+2r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right) (q^s)^k$$

$$+\sum_{k=\ell-r}^{2\delta+\lambda-\ell+r} \mathbf{c}_\gamma(k)(-1)^k q^{\frac{\ell}{2}+r}(q^s)^k.$$

Example 5.5.8. We provide an example for $r=2$, $\ell=2$, $\delta=4$, $\lambda=9$ for concreteness, In this case $\min(2r, \lambda-\ell) = \min(4,7) = 4$ and we have

$$\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) &= q^{-4s} \\
&- (q+1)q^{-3s} \\
&+ (q^2+q+1)q^{-2s} \\
&- (q^2+q+1)q^{-s} \\
&+ (q^3+q^2+q+1)q^{0} \\
&- (2q^3+q^2+q+1)q^{s} \\
&+ (3q^3+q^2+q+1)q^{2s} \\
&- (4q^3+q^2+q+1)q^{3s} \\
&+ (4q^3+q^2+q+1)q^{4s} \\
&- (5q^3+q^2+q+1)q^{5s} \\
&+ (5q^3+q^2+q+1)q^{6s} \\
&- (6q^3+q^2+q+1)q^{7s} \\
&+ (6q^3+q^2+q+1)q^{8s} \\
&- (6q^3+q^2+q+1)q^{9s} \\
&+ (6q^3+q^2+q+1)q^{10s} \\
&- (5q^3+q^2+q+1)q^{11s} \\
&+ (5q^3+q^2+q+1)q^{12s} \\
&- (4q^3+q^2+q+1)q^{13s} \\
&+ (4q^3+q^2+q+1)q^{14s} \\
&- (3q^3+q^2+q+1)q^{15s}
\end{aligned}$$

$$
\begin{aligned}
&+ (2q^3 + q^2 + q + 1)q^{16s} \\
&- (q^3 + q^2 + q + 1)q^{17s} \\
&+ (q^2 + q + 1)q^{18s} \\
&- (q^2 + q + 1)q^{19s} \\
&+ (q + 1)q^{20s} \\
&- q^{21s}.
\end{aligned}
$$

Example 5.5.9. We provide an example for $r = 5$, $\ell = 2$, $\delta = 4$, $\lambda = 9$ for concreteness, where $\min(2r, \lambda - \ell) = \min(10, 7) = 7$:

$$
\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) &= q^{-10s} \\
&- (q+1) \cdot q^{-9s} \\
&+ (q^2 + q + 1) \cdot q^{-8s} \\
&- (q^3 + q^2 + q + 1) \cdot q^{-7s} \\
&+ (q^4 + q^3 + q^2 + q + 1) \cdot q^{-6s} \\
&- (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-5s} \\
&+ (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-4s} \\
&- (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-3s} \\
&+ (2q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-2s} \\
&- (3q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-s} \\
&+ (4q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{0} \\
&- (4q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{s} \\
&+ (5q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2s} \\
&- (5q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{3s} \\
&+ (6q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{4s}
\end{aligned}
$$

$$
\begin{aligned}
&- (6q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{5s} \\
&+ (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{6s} \\
&- (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{7s} \\
&+ (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{8s} \\
&- (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{9s} \\
&+ (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{10s} \\
&- (7q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{11s} \\
&+ (6q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{12s} \\
&- (6q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{13s} \\
&+ (5q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{14s} \\
&- (5q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{15s} \\
&+ (4q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{16s} \\
&- (4q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{17s} \\
&+ (3q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{18s} \\
&- (2q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{19s} \\
&+ (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{20s} \\
&- (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{21s} \\
&+ (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{22s} \\
&- (q^4 + q^3 + q^2 + q + 1) \cdot q^{23s} \\
&+ (q^3 + q^2 + q + 1) \cdot q^{24s} \\
&- (q^2 + q + 1) \cdot q^{25s} \\
&+ (q + 1) \cdot q^{26s} \\
&- q^{27s}.
\end{aligned}
$$

5.5.4 Full explicit weighted orbital integral for the case $\ell < 0$

In this case, $\ell = \delta = 2v(b) = 2v(d) < 0$. We will just state the relevant theorem in terms of $v(b)$ and $v(d)$, omitting ℓ and δ.

Theorem 5.5.10 (Weighted orbital integral when $v(b) = v(d) < 0$)**.** *Let $r \geq 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let b, d, λ be as in Lemma 5.3.3 and Lemma 5.4.1. Suppose also $v(b) = v(d) < 0$.*

Then if $|v(d)| > r$, the entire orbital integral is zero. Otherwise define

$$\mathbf{n}_\gamma := \underset{[-2r, \lambda+2r-4|v(d)|]}{\mathrm{ARCH}}(r - |v(d)|, 0)$$

$$\mathbf{c}_\gamma := \underset{[-r-|v(d)|, \lambda+r-3|v(d)|]}{\mathrm{ARCH}}(0, \min(2r - 2|v(d)|, \lambda)).$$

Then for any $r \geq 0$ we have the formula:

$$\begin{aligned}\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = & \sum_{k=-2r}^{\lambda+2r-4|v(d)|} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right) (q^s)^k \\ & + \sum_{k=-r-|v(d)|}^{\lambda+r-3|v(d)|} \mathbf{c}_\gamma(k)(-1)^k q^{r-|v(d)|}(q^s)^k.\end{aligned}$$

Example 5.5.11. In the case where $r = |v(d)|$, the orbital simplifies to just

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = \sum_{k=-2r}^{\lambda-2r} (-1)^k (q^s)^k.$$

Example 5.5.12. If $\lambda = 7$, $v(d) = -5$ and $r = 12$ we have

$$\begin{aligned}\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = & \, q^{-24s} \\ & - (q+1) \cdot q^{-23s} \\ & + (q^2 + q + 1) \cdot q^{-22s} \\ & - (q^3 + q^2 + q + 1) \cdot q^{-21s}\end{aligned}$$

$$
\begin{aligned}
&+ (q^4 + q^3 + q^2 + q + 1) \cdot q^{-20s} \\
&- (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-19s} \\
&+ (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-18s} \\
&- (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-17s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-16s} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-15s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-14s} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-13s} \\
&+ (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-12s} \\
&- (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-11s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-10s} \\
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-9s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-8s} \\
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-7s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-6s} \\
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-5s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-4s} \\
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-3s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-2s} \\
&- (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-s} \\
&+ (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{0} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2s} \\
&- (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{3s}
\end{aligned}
$$

$$
\begin{aligned}
&+ (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{4s} \\
&- (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{5s} \\
&+ (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{6s} \\
&- (q^4 + q^3 + q^2 + q + 1) \cdot q^{7s} \\
&+ (q^3 + q^2 + q + 1) \cdot q^{8s} \\
&- (q^2 + q + 1) \cdot q^{9s} \\
&+ (q + 1) \cdot q^{10s} \\
&- q^{11s}.
\end{aligned}
$$

Example 5.5.13. If $\lambda = 2025$, $v(d) = -5$ and $r = 12$ we have

$$
\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) &= q^{-24s} \\
&- (q + 1) \cdot q^{-23s} \\
&+ (q^2 + q + 1) \cdot q^{-22s} \\
&- (q^3 + q^2 + q + 1) \cdot q^{-21s} \\
&+ (q^4 + q^3 + q^2 + q + 1) \cdot q^{-20s} \\
&- (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-19s} \\
&+ (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-18s} \\
&- (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-17s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-16s} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-15s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-14s} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-13s} \\
&+ (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-12s} \\
&- (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-11s}
\end{aligned}
$$

$$
\begin{aligned}
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-10s} \\
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-9s} \\
&+ (5q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-8s} \\
&- (5q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-7s} \\
&+ (6q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-6s} \\
&- (6q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-5s} \\
&+ (7q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-4s} \\
&- (7q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-3s} \\
&+ (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-2s} \\
&- (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{-s} \\
&+ (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{0} \\
&- (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{s} \\
&+ (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2s} \\
&- (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{3s} \\
&+ (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{4s} \\
&\vdots \\
&- (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2007s} \\
&+ (8q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2008s} \\
&- (7q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2009s} \\
&+ (7q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2010s} \\
&- (6q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2011s} \\
&+ (6q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2012s} \\
&- (5q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2013s} \\
&+ (5q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2014s}
\end{aligned}
$$

$$
\begin{aligned}
&- (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2015s} \\
&+ (4q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2016s} \\
&- (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2017s} \\
&+ (3q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2018s} \\
&- (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2019s} \\
&+ (2q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2020s} \\
&- (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2021s} \\
&+ (q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2022s} \\
&- (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2023s} \\
&+ (q^5 + q^4 + q^3 + q^2 + q + 1) \cdot q^{2024s} \\
&- (q^4 + q^3 + q^2 + q + 1) \cdot q^{2025s} \\
&+ (q^3 + q^2 + q + 1) \cdot q^{2026s} \\
&- (q^2 + q + 1) \cdot q^{2027s} \\
&+ (q + 1) \cdot q^{2028s} \\
&- q^{2029s}.
\end{aligned}
$$

5.6 Derivatives of the orbital integrals

We now state the derivative of the orbital integral in the three cases we described before.

Theorem 5.6.1. *Let $r \geq 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let ℓ, δ, λ be as in Lemma 5.3.3 and Lemma 5.4.1. Then*

$$\frac{1}{\log q} \partial \operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}})$$

is given by the following three expressions:

- *If ℓ is odd, the derivative is*

$$(-1)^{r+1}\sum_{j=r+1}^{r+\frac{\ell-1}{2}}\left(\frac{\ell+2\delta+1}{2}+3r-2j\right)\cdot q^j$$
$$+\sum_{j=0}^{r}(-1)^{j+1}\left(\frac{\ell+2\delta+1}{2}+2r-j\right)\cdot q^j.$$

- *If $\ell \geq 0$ is even, the derivative is*

$$(-1)^{r+1}\sum_{j=r+1}^{r+\frac{\ell}{2}}\left(\frac{\lambda+2\delta+1}{2}+3r-2j\right)\cdot q^j$$
$$+\sum_{j=0}^{r}(-1)^{j+1}\left(\frac{\lambda+2\delta+1}{2}+2r-j\right)\cdot q^j$$
$$+(-1)^{r+\delta-\frac{\ell}{2}}q^{r+\frac{\ell}{2}}\cdot\begin{cases}\frac{\delta-\frac{\ell}{2}}{2}-\frac{\lambda-\ell-1}{2}\cdot r & \text{if } \delta\equiv\frac{\ell}{2} \pmod 2\\ -\frac{\delta-\frac{3}{2}\ell+\lambda}{2}-\frac{\lambda-\ell+1}{2}\cdot r & \text{if } \delta\not\equiv\frac{\ell}{2} \pmod 2.\end{cases}$$

- *If $\ell < 0$ is even (so $\ell = -2|v(d)|$), the derivative is*

$$\sum_{j=0}^{r-|v(d)|}(-1)^{j+1}\left(\frac{\lambda+1}{2}+2(r-|v(d)|)-j\right)\cdot q^j$$
$$+(-1)^{r-|v(d)|+1}\cdot\frac{\lambda-1}{2}\cdot\max\left(r-|v(d)|,0\right)\cdot q^{r-|v(d)|}.$$

We prove Theorem 5.6.1 in Section 7.7.

Chapter 6

Support for the weighted orbital integral for $S_3(F)$

In this section we set up the framework of the weighted orbital integral based on the definitions in the previous section. This involves rewriting the integral as an infinite discrete double sum over two parameters (n, m) that we will introduce later, and determining the volume of the supports of the indicator function.

We always retain the setting of Lemma 5.3.3 throughout this chapter.

6.1 Reparametrization in terms of valuations

6.1.1 Computation of value in indicator function

We are integrating over $t_1 \in E$ and $t_2 \in E$. Regarding $h' \in H'$ as an element of $\mathrm{GL}_3(E)$ as described before, we have

$$h' = \begin{pmatrix} t_1 & t_2 & 0 \\ \bar{t}_2 & \bar{t}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We therefore have

$$\bar{h'}^{-1} = \begin{pmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} \bar{h'}^{-1}\gamma h' &= \begin{pmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & -\bar{d} & 1 \\ c & 1 - d\bar{d} & d \end{pmatrix} \begin{pmatrix} t_1 & t_2 & 0 \\ \bar{t}_2 & \bar{t}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} at_1 & at_2 & 0 \\ bt_1 - \bar{d}\bar{t}_2 & bt_2 - \bar{d}\bar{t}_1 & 1 \\ ct_1 + (1 - d\bar{d})\bar{t}_2 & ct_2 + (1 - d\bar{d})\bar{t}_1 & d \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{at_1^2 - bt_1\bar{t}_2 + d\bar{t}_2^2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{at_1t_2 - bt_2\bar{t}_2 + \bar{d}\bar{t}_1\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} \\ \dfrac{-at_1t_2 + bt_1\bar{t}_1 - \bar{d}\bar{t}_1\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{-at_2^2 + b\bar{t}_1t_2 - d\bar{t}_1^2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} \\ ct_1 + (1 - d\bar{d})\bar{t}_2 & ct_2 + (1 - d\bar{d})\bar{t}_1 & d \end{pmatrix} \end{aligned}$$

At this point, note that the entry d appears by itself at the bottom-right.

Let us define

$$t = t_2\bar{t}_1^{-1} \iff t_2 = t\bar{t}_1.$$

This lets us rewrite everything in terms of the ratio t and $t_1 \in E$:

$$\bar{h'}^{-1}\gamma h' = \begin{pmatrix} \dfrac{t_1^2(a - b\bar{t} + \bar{d}\bar{t}^2)}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{t_1\bar{t}_1(at - bt\bar{t} + \bar{d}\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{t_1 \cdot (-\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} \\ \dfrac{t_1\bar{t}_1(-at + b - \bar{d}\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{\bar{t}_1^2(-at^2 + bt - \bar{d})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{-\bar{t}_1}{t_1\bar{t}_1(1 - t\bar{t})} \\ t_1(c + (1 - d\bar{d})\bar{t}) & \bar{t}_1(ct + (1 - d\bar{d})) & d \end{pmatrix}$$

This new parametrization is better because t_1 only plays the role of a scale factor on the outside, with "interesting" terms only involving t. To make this further explicit, we write

$$t_1 = \varpi^{-m}\epsilon$$

for $m \in \mathbb{Z}$ and $\epsilon \in O_E^\times$. Then

$$\begin{pmatrix} \bar{\epsilon} & & \\ & \epsilon & \\ & & 1 \end{pmatrix} \bar{h'}^{-1}\gamma h' \begin{pmatrix} \epsilon^{-1} & & \\ & \bar{\epsilon}^{-1} & \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} & \dfrac{at - bt\bar{t} + \bar{d}\bar{t}}{1 - t\bar{t}} & \dfrac{-\varpi^m \bar{t}}{1 - t\bar{t}} \\ \dfrac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} & \dfrac{-at^2 + bt - \bar{d}}{1 - t\bar{t}} & \dfrac{-\varpi^m}{1 - t\bar{t}} \\ \dfrac{c + (1 - d\bar{d})\bar{t}}{\varpi^m} & \dfrac{ct + (1 - d\bar{d})}{\varpi^m} & d \end{pmatrix}.$$

For brevity, we will let $\Gamma(\gamma, t, m)$ denote the right-hand matrix. The conjugation by $\left(\begin{smallmatrix} \epsilon^{-1} & & \\ & \bar{\epsilon}^{-1} & \\ & & 1 \end{smallmatrix}\right)$ has no effect on any of the $K'_{S,\le r}$, so that we can simply use

$$\mathbf{1}_{K'_{S,\le r}}(\bar{h'}^{-1}\gamma h') = \mathbf{1}_{K'_{S,\le r}}(\Gamma(\gamma, t, m))$$

in the work that follows. For brevity, we abbreviate

$$\mathbf{1}_{\le r}(\gamma, t, m) \coloneqq \mathbf{1}_{K'_{S,\le r}}(\Gamma(\gamma, t, m)).$$

6.1.2 Reparametrizing the integral in terms of t and m

From now on, following [Zha12, §4] we always fix the notation

$$m = m(t_1) \coloneqq -v(t_1)$$
$$n = n(t) \coloneqq v(1 - t\bar{t}).$$

(At the risk of stating the obvious, this n is not the same n in GL_n, etc.) We need to rewrite the integral, phrased originally via $\mathrm{d}h'$, in terms of the parameters t (hence n), m, and γ. We start by observing that

$$\det h' = t_1\bar{t}_1 - t_2\bar{t}_2 = t_1\bar{t}_1(1 - t\bar{t})$$

which means that

$$v(\det h') = -2m + n$$

ergo

$$\begin{aligned} \left|\det h'\right|_F &= q^{-v(\det h')} = q^{2m-n} \\ \eta(h') &= (-1)^{v(\det h')} = (-1)^n. \end{aligned}$$

Meanwhile, from $t_2 = t\bar{t}_1$ we derive

$$\mathrm{d}t_2 = \left|t_1\right|_E \mathrm{d}t = q^{2m}\,\mathrm{d}t.$$

Bringing this all into the weighted orbital integral gives

$$\begin{aligned} \mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\le r}} s) &= \kappa \int_{t,t_1 \in E} \mathbf{1}_{\le r}(\gamma, t, m)(-1)^n \left(q^{2m-n}\right)^{s-2} \mathrm{d}t_1 \cdot (q^{2m}\,\mathrm{d}t) \\ &= \kappa \int_{t,t_1 \in E} \mathbf{1}_{\le r}(\gamma, t, m)(-1)^n q^{s(2m-n)} \cdot q^{2n-2m}\,\mathrm{d}t\,\mathrm{d}t_1. \end{aligned}$$

6.2 Description of the support of $\mathbf{1}_{\leq r}$ when $n \leq 0$

Proposition 6.2.1 (Support of $\mathbf{1}_{\leq r}$ for $n = 0$)**.** *Whenever $n = 0$ (this requires $v(t) \geq 0$),*

$$\mathbf{1}_{\leq r}(\gamma, t, m) = \begin{cases} 1 & \text{if } -r \leq m \leq \delta + r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have to consider the nine entries of $\Gamma(\gamma, t, m)$ in tandem.

The upper 2×2 matrix is always in $\omega^{-r} O_E$, because $v(t) \geq 0$, $v(d) \geq -r$, $v(b) \geq -r$, and $v(a) = 0$ suffices.

In the right column, since $v(t) \geq 0$ and $n = 0$, the condition is simply $m \geq -r$.

In the bottom row, we need

$$v\left(c + (1 - d\bar{d})\bar{t}\right) - m \geq -r,$$
$$v\left(ct + (1 - d\bar{d})\right) - m \geq -r.$$

If $v(t) > 0$ this is equivalent to $m - r \leq \delta$. In the case where $v(t) = 0$ we instead use the observation that

$$\left[c + (1 - d\bar{d})\bar{t}\right] - \bar{t}\left[ct + (1 - d\bar{d})\right] = (1 - t\bar{t})c \tag{6.1}$$

which forces at least one of $ct + (1 - d\bar{d})$ and $c + (1 - d\bar{d})\bar{t}$ to have valuation δ. So the claim follows now. $\square$

Proposition 6.2.2 (Support of $\mathbf{1}_{\leq r}$ for $n < 0$)**.** *Suppose $n = -2k < 0$, equivalently, $v(t) = -k < 0$, for some k.*

$$\mathbf{1}_{\leq r}(\gamma, t, m) = \begin{cases} 1 & \text{if } -r \leq m + k \leq \delta + r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the previous claim, but simpler.

Since $k > 0$, the fraction $\frac{t^2}{1-t\bar{t}}$ has positive valuation, so the upper 2×2 submatrix of $\Gamma(\gamma, t, m)$ is always in $\varpi^{-r}O_E$. Turning to the right column, the condition reads exactly $m + k \geq -r$. Finally, in the bottom row, from $v(t) > 0$ and $v(c) = \delta$ the condition is simply $-k + \delta - m \geq -r$. $\square$

Note all the results in this section hold for any b and d with $v(b) \geq -r$ and $v(d) \geq -r$, i.e. we do not yet need to consider cases based on whether b and d are units or not.

6.3 Description of the support of $\mathbf{1}_{\leq r}$ when $n > 0$

In this situation we evaluate over $n > 0$ only. In this case t is automatically a unit.

Consider the upper 2×2 submatrix of $\Gamma(\gamma, t, m)$. Using the identities

$$\frac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} - \bar{t} \cdot \frac{at - bt\bar{t} + \bar{d}\bar{t}}{1 - t\bar{t}} = a - b\bar{t} \in \varpi^{-r}O_E$$

$$\frac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} + \bar{t} \cdot \frac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} = a \in \varpi^{-r}O_E$$

$$\frac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} - \bar{t} \cdot \frac{-at^2 + bt - \bar{d}}{1 - t\bar{t}} = -a + b \in \varpi^{-r}O_E,$$

it follows that as soon as one entry in the upper 2×2 submatrix is in $\varpi^{-r}O_E$, then all four are. Focusing on the fraction with numerator $-at^2 + bt - \bar{d}$ we use the identity

$$(2at - b)^2 - (b^2 - 4a\bar{d}) = -4a(-at^2 + bt - \bar{d})$$

to rewrite $v(-at^2 + bt - \bar{d}) \geq n - r$ as

$$v\left((2at - b)^2 - (b^2 - 4a\bar{d})\right) \geq n - r.$$

This takes care of all four of the entries in the upper 2×2 submatrix.

Meanwhile, the requirements on the other entries amount to

$$m \geq n - r \tag{6.2}$$

$$v\left(c + (1 - d\bar{d})\bar{t}\right) \geq m - r \tag{6.3}$$

$$v\left(ct + (1 - d\bar{d})\right) \geq m - r \tag{6.4}$$

$$v(d) \geq -r. \tag{6.5}$$

According to the earlier identity (6.1), if (6.3) is assumed true, then (6.4) is equivalent to

$$\delta + v(1 - t\bar{t}) \geq m - r.$$

Meanwhile, since $v(c + (1 - d\bar{d})\bar{t}) = v(\bar{c} + (1 - d\bar{d})t)$, (6.3) is itself equivalent to

$$v(t + u) + \delta \geq m - r$$

by reading the definition of (5.2).

In summary:

Proposition 6.3.1 (Support of $\mathbf{1}_{\leq r}$ when $n > 0$)**.** *Assume t is such that $n = v(1 - t\bar{t}) > 0$. Then $\mathbf{1}_{\leq r}(\gamma, t, m) = 1$ if and only if $v(d) \geq -r$,*

$$n - r \leq m \leq n + \delta + r$$

and t lies in the set specified by

$$\begin{aligned} v\left((2at - b)^2 - (b^2 - 4a\bar{d})\right) &\geq n - r \\ v(t + u) &\geq m - \delta - r. \end{aligned} \tag{6.6}$$

6.4 Rewriting the quadratic constraint on the valuation of t in the $n > 0$ situation

We now analyze the inequality

$$v\left((2at-b)^2 - (b^2 - 4a\bar{d})\right) \geq n - r \tag{6.7}$$

and divide it into several (disjoint) possibilities. Recalling that $\ell = b^2 - 4a\bar{d}$, there are several possibilities:

- If $\ell \geq n - r$, then (6.7) is equivalent to

$$2v(2at - b) \geq n - r \iff v\left(t - \frac{b}{2a}\right) \geq \left\lceil \frac{n-r}{2} \right\rceil. \tag{6.8}$$

  We will further subdivide this into two cases.

  - **Case 1** is the situation where $\left\lceil \frac{n-r}{2} \right\rceil \geq m - \delta - r$.
  - **Case 2** is the situation where $\left\lceil \frac{n-r}{2} \right\rceil < m - \delta - r$.

- If $\ell < n - r$, then (6.7) could only hold if $v(2at - b) = \frac{\ell}{2}$. Note that in particular, this requires ℓ to be even.

  If this happens, then as we saw in Lemma 5.4.1 the quantity $b^2 - 4a\bar{d}$ must be a square and we denote it τ^2. Thus, (6.7) then reads

$$v(2at - b + \tau) + v(2at - b - \tau) \geq n - r. \tag{6.9}$$

  Since we are assuming $n > \ell + r$, it must be the case that one of the two factors $v(2at - b \mp \tau)$ is equal to $v(\tau) = \ell/2$ exactly, and the other is at least $n - r - \frac{\ell}{2} > \frac{\ell}{2}$.

Hence, there is one situation where $v(2at - b + \tau) = \frac{\ell}{2}$ and

$$v\left(t - \frac{b+\tau}{2a}\right) = v(2at - b - \tau) \geq n - \frac{\ell}{2} - r \tag{6.10}$$

Note that conversely (6.10) implies (6.9). We further subdivide this into two cases:

– **Case 3$^+$** is the situation where $n - \frac{\ell}{2} - r > m - \delta - r$.

– **Case 4$^+$** is the situation where $n - \frac{\ell}{2} - r \leq m - \delta - r$.

Replacing τ with $-\tau$ above in (6.10) yields the other situation where $v(2at - b - \tau) = \frac{\ell}{2}$ and

$$v\left(t - \frac{b-\tau}{2a}\right) = v(2at - b + \tau) \geq n - \frac{\ell}{2} - r \tag{6.11}$$

which gives us two additional cases. We denote them **Case 3$^-$** and **Case 4$^-$**.

This gives us six cases, with each $t \in E$ satisfying at most one of them. (If ℓ is odd, only **Case 1** and **Case 2** are used.) In each case, for a given pair (n, m) we are interested in the volume of t such that two disk inequalities hold together with the assumption $n = v(1 - t\bar{t})$.

We rewrite these six cases in the format specified by Lemma 2.2.3, noting that each possibility will actually split into two sub-cases (although the lemma will only apply in the cases where the centers ξ_i are actually in $O_E^\times$; this which will be true in the main case $v(b) = v(d) = 0$). This gives Table 6.1.

Note that in generating Table 6.1, we did the calculations

$$v\left(u + \frac{b}{2a}\right) = v\left(\frac{au - \bar{d}\bar{u}}{2a}\right) = v(au - \bar{d}\bar{u})$$
$$v\left(u + \frac{b \pm \tau}{2}\right) = v(au - \bar{d}\bar{u} \pm \tau).$$

to populate the entries for $v(\xi_1 - \xi_2)$, as well as the identity

$$1 - \frac{b \pm \tau}{2a} \cdot \frac{\bar{b} \pm \bar{\tau}}{2\bar{a}} = \frac{4 - \mathrm{N}(b \pm \tau)}{4}$$

to calculate $v(1-\xi_1\bar{\xi}_1)$ entries in the latter four cases.

6.5 Case analysis

Up until now the analysis has been valid for all five cases of Lemma 5.4.1. However, starting from now on we will have to be a little more careful and think about which situation of Lemma 5.4.1 we are in.

6.5.1 Analysis of Case 1 and 2 assuming $n > 0$ and $v(b) = 0$

We analyze Case 1 and 2 assuming $v(b) = 0$.

Considering $n > 0$ and $n - r \leq m \leq n + \delta + r$ as fixed, we compute the volume of the set of t for which $n = v(1 - t\bar{t})$ and $\mathbf{1}_{\leq r}(\gamma, t, m) = 1$.

In addition to the constraint $n \leq \ell + r$, we see that the two cases have the following additional requirements:

Case 1 If $m < \left\lceil \frac{n-r}{2} \right\rceil + \delta + r$ then we need

$$v(4 - b\bar{b}) \geq \left\lceil \frac{n-r}{2} \right\rceil \tag{6.12}$$

$$v(au - \bar{d}\bar{u}) \geq m - \delta - r. \tag{6.13}$$

Case 2 If $m \geq \left\lceil \frac{n-r}{2} \right\rceil + \delta + r$ then we need

$$\lambda = v(1 - u\bar{u}) \geq m - \delta - r \tag{6.14}$$

$$v(au - \bar{d}\bar{u}) \geq \left\lceil \frac{n-r}{2} \right\rceil. \tag{6.15}$$

We will now show that some of these inequalities are redundant and can be ignored.

Lemma 6.5.1. *If $v(b) = 0$, then (6.13) and (6.15) are redundant i.e. they are automatically true for $0 < n \leq \ell + r$.*

Proof. This is immediate from Lemma 5.4.2. □

Lemma 6.5.2. *If $v(b) = 0$, then (6.12) is redundant.*

Proof. Regardless of whether $v(d) = 0$ or $v(d) > 0$, the equation

$$(4 - b\bar{b}) = -4au(1 - d\bar{d}) - \bar{b}(b^2 - 4a\bar{d})$$

always implies

$$v(4 - b\bar{b}) \geq \min(\ell, \delta) \text{ with equality if } \ell \neq \delta \tag{6.16}$$

sinec $-4au$ and $\bar{b}$ are units. Hence, a priori (6.16) suggests that we have a condition $n \leq r+2\delta$ in addition to $n \leq r + \ell$. However, by Lemma 5.4.1, we always have $\ell \leq 2\delta$, and consequently (6.12) is redundant as well. □

Putting all of this together, we find that the valid pairs (n, m) come in two cases.

Double sum for Case 1 We sum over (m, n) such that

$$\begin{aligned} 1 \leq n \leq \ell + r, \\ n - r \leq m \leq \left\lceil \frac{n-r}{2} \right\rceil + \delta + r - 1 \end{aligned} \tag{6.17}$$

where each (m, n) gives a volume contribution of

$$\begin{cases} q^{-n-\left\lceil \frac{n-r}{2} \right\rceil}\left(1 - q^{-1}\right) & \text{if } n > r \\ q^{-n}\left(1 - q^{-2}\right) & \text{if } n \leq r. \end{cases} \tag{6.18}$$

Double sum for Case 2 We sum over (m, n) such that

$$\begin{aligned} 1 \leq n \leq \ell + r, \\ \max\left(n - r, \left\lceil \frac{n-r}{2} \right\rceil + \delta + r\right) \leq m \leq \min(n, \lambda) + \delta + r \end{aligned} \tag{6.19}$$

where each (m, n) gives a volume contribution of

$$\begin{cases} q^{-n-(m-\delta-r)}\left(1-q^{-1}\right) & \text{if } m > \delta + r \\ q^{-n}\left(1-q^{-2}\right) & \text{if } m \leq \delta + r. \end{cases} \tag{6.20}$$

Notice that $m \leq \delta + r$ could only occur when $n \leq r$.

6.5.2 Analysis of merged Case 1 and 2 assuming $n > 0$ and $v(b) > 0$

If $v(b) > 0$ instead, so that $\ell = 0$ but still $\delta \geq 0$. Then (6.8) becomes independent of the unit t because we must have

$$v\left(t - \frac{b}{2a}\right) = 0.$$

Consequently, in this situation, it is not necessary to distinguish between Cases 1 and 2. Instead, (6.6) merely requires that

$$\begin{aligned} 0 &\geq n - r \\ v(t+u) &\geq m - \delta - r \end{aligned}$$

In this situation, by Lemma 2.2.1, we get a nonzero contribution in the merged case if and only if

$$\begin{aligned} 1 &\leq n \leq r \\ n - r &\leq m \leq n + \delta + r \\ \lambda &\geq m - \delta - r. \end{aligned} \tag{6.21}$$

In that case the volume contribution is given by the same expression as (6.20).

To avoid having to consider the situation $v(b) > 0$ separately, we make the following observation:

Lemma 6.5.3 (Reducing $v(b) > 0$ to before)**.** *Consider the two (disjoint) ranges* (6.17) *and* (6.19) *in the special case $\ell = 0$. These two ranges collectively cover exactly the same elements*

(m, n) *as* (6.21). *Moreover, the volume contribution* (6.18) *equals that of* (6.20) *for pairs* (m, n) *in* (6.17).

Proof. If we combine (6.17) and (6.19), they say that

$$1 \leq n \leq \ell + r = r$$
$$n - r \leq m \leq \min(n, \lambda) + \delta + r = \min(n, \lambda) + r$$

which matches (6.21) exactly. So it remains to verify that the volume contributions match.

Now suppose that $m < \lceil \frac{n-r}{2} \rceil + \delta + r$. Since $n \leq r$, it follows that $m \leq \delta + r$. So on the one hand we have the $n \leq r$ case in (6.18) and on the other hand we have the $m \leq \delta + r$ case in (6.20), which both equal $q^{-n}(1 - q^{-2})$. Hence the proof is complete. □

Because of this proposition then we can fold the $v(b) > 0$ result into the case $v(b) = 0$ too in future calculations. In other words the double sums in Case 1 and 2 mentioned in the previous subsection will work verbatim even in the $v(b) > 0$ situation, which is convenient.

Remark 6.5.4 (Comparison to [Zha12])**.** Note that in the original paper [Zha12] only $r = 0$ is considered and in this case all three ranges (6.17), (6.19) and (6.21) are empty. Therefore the corresponding step in the calculation of [Zha12] is much simpler, consisting only of the one-line observation that Case 1 and Case 2 cannot occur at all if $r = 0$. In contrast once $r > 0$ then the ranges are not necessarily empty and therefore one needs to ensure that the terms arising actually match.

6.5.3 Analysis of Case 3$^+$, 3$^-$, 4$^+$, 4$^-$ assuming $n > 0$ and $v(b) \geq 0$ and $v(d) \geq 0$

Suppose $\ell \geq 0$ is even. As before we consider $n > 0$ and $n - r \leq m \leq n + \delta + r$ as fixed, and seek to compute the volume of the set of t for which $n = v(1 - t\bar{t})$ and $\mathbf{1}_{\leq r}(\gamma, t, m) = 0$.

Double sum for Case 3$^+$ Suppose $n > \ell + r$, $m < n - \frac{\ell}{2} + \delta$, and we choose $\frac{b+\tau}{2a}$. Then Lemma 2.2.3 gives a nonzero contribution if and only if

$$\lambda + \delta - \ell = v(4 - \mathrm{N}(b+\tau)) \geq n - \frac{\ell}{2} - r$$
$$\lambda + v(d) - \frac{\ell}{2} = v(au - \bar{d}\bar{u} + \tau) \geq m - \delta - r.$$

Compiling all seven constraints gives that the valid pairs (m, n) are those for which

$$\max(1, \ell + r + 1) \leq n \leq -\frac{\ell}{2} + \delta + \lambda + r,$$
$$n - r \leq m \leq \min\left(n + \delta + r, n - \frac{\ell}{2} + \delta - 1, \lambda + v(d) - \frac{\ell}{2} + \delta + r\right)$$

which from $\ell, \delta, r \geq 0$ can be simplified to just

$$\ell + r + 1 \leq n \leq -\frac{\ell}{2} + \delta + \lambda + r,$$
$$n - r \leq m \leq \min(n - 1, \lambda + v(d) + r) - \frac{\ell}{2} + \delta$$

Moreover, in any situation where $v(d) > 0$, we have $\ell = \delta = 0$, and hence $n - 1 < \lambda + r < \lambda + v(d) + r$ is obvious. Thus we may drop the $v(d)$ term from the inequality and obtain

$$\begin{aligned} \ell + r + 1 \leq n \leq \lambda - \frac{\ell}{2} + \delta + r, \\ n - r \leq m \leq \min(n - 1, \lambda + r) - \frac{\ell}{2} + \delta \end{aligned} \tag{6.22}$$

Each (m, n) gives a volume contribution of

$$q^{-n-(n-\frac{\ell}{2}-r)}\left(1 - q^{-1}\right).$$

Double sum for Case 3$^-$ Suppose $n > \ell + r$, $m < n - \frac{\ell}{2} + \delta$, and we choose $\frac{b-\tau}{2a}$. Then

Lemma 2.2.3 gives a nonzero contribution if and only if

$$\delta = v(4 - \mathrm{N}(b-\tau)) \geq n - \frac{\ell}{2} - r$$
$$\frac{\ell}{2} = v(au - \bar{d}\bar{u} - \tau) \geq m - \delta - r.$$

Compiling all seven constraints gives that the valid pairs (m, n) are those for which

$$\ell + r + 1 \leq n \leq \frac{\ell}{2} + \delta + r,$$
$$n - r \leq m \leq \min\left(n - \frac{\ell}{2} + \delta - 1, \frac{\ell}{2} + \delta + r, n + \delta + r\right)$$

This simplifies to

$$\begin{aligned} \ell + r + 1 \leq n &\leq \frac{\ell}{2} + \delta + r, \\ n - r \leq m &\leq \frac{\ell}{2} + \delta + r. \end{aligned} \tag{6.23}$$

As in the previous case, (m, n) gives a volume contribution of

$$q^{-n-(n-\frac{\ell}{2}-r)}\left(1 - q^{-1}\right).$$

Double sum for Case 4$^+$ Suppose $n > \ell + r$, $m \geq n - \frac{\ell}{2} + \delta$ (which implies $m \geq n - r$ since $r \geq 0$ and $0 \leq \ell \leq 2\delta$), and we choose $\frac{b+\tau}{2a}$. Then Lemma 2.2.3 gives a nonzero contribution if and only if

$$\lambda \geq m - \delta - r$$
$$\lambda + v(d) - \frac{\ell}{2} = v(au - \bar{d}\bar{u} + \tau) \geq n - \frac{\ell}{2} - r.$$

Rearranging gives that the valid pairs (m, n) are those for which

$$\ell + r + 1 \leq n \leq \lambda + v(d) + r$$

$$n - \frac{\ell}{2} + \delta \le m \le \min(n, \lambda) + \delta + r.$$

However, in any situation where $v(d) > 0$ and hence $\ell = \delta = 0$, the inequality on m already implies that $n \le \lambda + r$. Hence we may drop the $v(d)$ term to get instead the inequalities

$$\begin{aligned} \ell + r + 1 \le n \le \lambda + r \\ n - \frac{\ell}{2} + \delta \le m \le \min(n, \lambda) + \delta + r. \end{aligned} \tag{6.24}$$

Here, each (m, n) gives a volume contribution of

$$q^{-n-(m-\delta-r)}\left(1 - q^{-1}\right).$$

Double sum for Case 4⁻ Suppose $n > \ell + r$, $m \ge n - \frac{\ell}{2} + \delta$, and we choose $\frac{b-\tau}{2a}$. Then Lemma 2.2.3 gives a nonzero contribution if and only if

$$\begin{aligned} \lambda \ge m - \delta - r \\ \frac{\ell}{2} = v(au - \bar{d}\bar{u} - \tau) \ge n - \frac{\ell}{2} - r. \end{aligned}$$

The latter inequality contradicts the assumption that $n > \ell + r$, so in fact this case can never occur.

6.6 Case analysis when $v(b) = v(d) < 0$

We need to handle now the situation where $v(b) = v(d) < 0$. In that case we have $\ell = \delta = 2v(d) < 0$.

For **Case 3⁺**, **Case 3⁻**, **Case 4⁺**, **Case 4⁻**, we still agree to fix the choice of square root τ such that (5.11) holds. (Note that in the right-hand side of the first inequality, we still have $\lambda + v(d) - \frac{1}{2}\ell = \lambda > 0$.)

6.6.1 Analysis of merged Case 1 and 2 assuming $v(b) = v(d) < 0$

In the situation where $\ell \geq n - r$, then for any unit t we have $v(t - \frac{b}{2a}) = v(b) = -|v(d)| < 0$, so (6.8) becomes simply

$$v(b) \geq \left\lceil \frac{n-r}{2} \right\rceil \iff n \leq r + 2v(b) = r + \ell.$$

Thus, the only constraint on the unit t is that

$$v(t+u) \geq m - \delta - r.$$

So, by applying Lemma 2.2.1, we find that the nonzero contributions occur exactly when

$$\begin{aligned} 1 \leq n &\leq -2|v(d)| + r \\ n - r \leq m &\leq n - 2|v(d)| + r \\ \lambda &\geq m + 2|v(d)| - r \end{aligned} \tag{6.25}$$

with the same volume contribution given by (6.20), that is.

$$\begin{cases} q^{-n-(m+2|v(d)|-r)} \left(1 - q^{-1}\right) & \text{if } m > -2|v(d)| + r \\ q^{-n} \left(1 - q^{-2}\right) & \text{if } m \leq -2|v(d)| + r. \end{cases} \tag{6.26}$$

So an analysis almost identical to the one in Lemma 6.5.3 applies here to show that (6.17) and (6.19) match the above range, with the same volume contributions. This means that the same expression can be used for the case $v(b) = v(d) < 0$ as well.

6.6.2 Analysis of Case 3^+ and Case 4^+ assuming $v(b) = v(d) < 0$

Now suppose $n > \ell + r$ and $v(b) = v(d) < 0$. In this situation, we have $\frac{b+\tau}{2a}$ is a unit, and

$$\lambda = v\,(1 - u\bar{u}) = v\left(1 - \frac{\mathrm{N}(b+\tau)}{4}\right).$$

Hence when we apply Lemma 2.2.3 we see that we are simply requiring that

$$\lambda \geq \max\left(n - \frac{\ell}{2} - r, m - \delta - r\right)$$

with **Case 3^+** corresponding to $n - \frac{\ell}{2} - r > m - \delta - r$ and **Case 4^+** corresponding to $n - \frac{\ell}{2} - r \geq m - \delta - r$; we do not separate these. As before we immediately replace $\ell = \delta = -2|v(d)|$ above.

Consequently, we will need to sum over the range

$$\begin{aligned} \max(1, -2|v(d)| + r + 1) \leq n \leq \lambda - |v(d)| + r \\ n - r \leq m \leq \min(n, \lambda) - 2|v(d)| + r. \end{aligned} \tag{6.27}$$

The volume contribution is now given by

$$\begin{cases} q^{-n-\max(n+|v(d)|-r, m+2|v(d)|-r)}(1-q^{-1}) & \text{if } \max(n+|v(d)|-r, m+2|v(d)|-r) > 0 \\ q^{-n}(1-q^{-2}) & \text{if } \max(n+|v(d)|-r, m+2|v(d)|-r) \leq 0. \end{cases} \tag{6.28}$$

since $\max(n + |v(d)| - r, m + 2|v(d)| - r)$ could actually be non-positive, in contrast to the earlier cases.

6.6.3 Analysis of Case 3- and Case 4^+ assuming $v(b) = v(d) < 0$

Again suppose $n > \ell + r$ and $v(b) = v(d) < 0$. In this case, (6.11) becomes simply

$$\frac{\ell}{2} = v\left(t - \frac{b - \tau}{2a}\right) \geq n - \frac{\ell}{2} - r$$

which implies $n \leq \ell + r$. So actually neither **Case 3-** nor **Case 4-** could happen.

	Assume	ξ_1 and ξ_2	ρ_1 and ρ_2
1	$n \leq \ell + r$	$\xi_1 = \frac{b}{2a}$ $\xi_2 = -u$ $v(1-\xi_1\bar{\xi}_1) = v(4 - b\bar{b})$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u})$	$\lceil \frac{n-r}{2} \rceil$ $\geq m - \delta - r$
2	$n \leq \ell + r$	$\xi_1 = -u$ $\xi_2 = \frac{b}{2a}$ $v(1-\xi_1\bar{\xi}_1) = \lambda$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u})$	$m - \delta - r$ $> \lceil \frac{n-r}{2} \rceil$
3$^+$	$n > \ell + r$	$\xi_1 = \frac{b+\tau}{2a}$ $\xi_2 = -u$ $v(1-\xi_1\bar{\xi}_1) = v(1 - \frac{\mathrm{N}(b+\tau)}{4})$ $= \lambda + \delta - \ell$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u} + \tau)$ $= \lambda + v(d) - \frac{\ell}{2}$	$n - \dfrac{\ell}{2} - r$ $> m - \delta - r$
3$^-$	$n > \ell + r$	$\xi_1 = \frac{b-\tau}{2a}$ $\xi_2 = -u$ $v(1-\xi_1\bar{\xi}_1) = v(1 - \frac{\mathrm{N}(b-\tau)}{4})$ $= \delta$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u} - \tau)$ $= \frac{\ell}{2}$	$n - \dfrac{\ell}{2} - r$ $> m - \delta - r$
4$^+$	$n > \ell + r$	$\xi_1 = -u$ $\xi_2 = \frac{b+\tau}{2a}$ $v(1-\xi_1\bar{\xi}_1) = \lambda$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u} + \tau)$ $= \lambda + v(d) - \frac{\ell}{2}$	$m - \delta - r$ $\geq n - \dfrac{\ell}{2} - r$
4$^-$	$n > \ell + r$	$\xi_1 = -u$ $\xi_2 = \frac{b-\tau}{2a}$ $v(1-\xi_1\bar{\xi}_1) = \lambda$ $v(\xi_1 - \xi_2) = v(au - \bar{d}\bar{u} - \tau)$ $= \frac{\ell}{2}$	$m - \delta - r$ $\geq n - \dfrac{\ell}{2} - r$

Table 6.1: The six cases for calculating the weighted orbital integral for $S_3(F)$, in the inhomogeneous group version of the AFL.

Chapter 7

Evaluation of the weighted orbital integral for $S_3(F)$

We now put together the sums we found in the previous section to come up with the expression for the weighted orbital integral. We continue to assume Lemma 5.3.3 and Lemma 5.4.1 in this chapter.

7.1 Region where $n \leq 0$ for all values of ℓ

Proposition 7.1.1 ($I_{n\leq 0}$). *The contribution to the integral* $\operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s)$ *over* $n \leq 0$ *is exactly*

$$I_{n\leq 0} := q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} q^{-2js} = q^{-2rs} + \cdots + q^{2(\delta+r)s}.$$

Proof. For $n = 0$ we get a contribution of

$$\begin{aligned}
&\kappa \int_{t,t_1 \in E} \mathbf{1}[n=0] \mathbf{1}_{\leq r}(\gamma, t, m) q^{2s\cdot m} q^{-2m} \, \mathrm{d}t \, \mathrm{d}t_1 \\
&= \kappa \operatorname{Vol}(t : n = 0) \sum_{m=-r}^{\delta+r} \operatorname{Vol}(t_1 : -v(t_1) = m) q^{2m(s-1)}
\end{aligned}$$

$$= \kappa \left(1 - \frac{q+1}{q^2}\right) \sum_{m=-r}^{\delta+r} \left(q^{2m}\left(1-q^{-2}\right)\right) q^{2m(s-1)}$$
$$= \kappa \left(1 - \frac{q+1}{q^2}\right) \left(1-q^{-2}\right) \sum_{m=-r}^{\delta+r} q^{2ms}.$$

For the region where $v(t) = -k < 0$, for each individual $k > 0$,

$$\kappa \int_{t,t_1 \in E} \mathbf{1}[v(t) = -k] \mathbf{1}_{\leq r}(\gamma, t, m) q^{s(2m-n)} q^{2n-2m} \, \mathrm{d}t \, \mathrm{d}t_1$$
$$= \kappa \operatorname{Vol}(t : v(t) = -k) \sum_{m=-r-k}^{\delta+r-k} \operatorname{Vol}(t_1 : -v(t_1) = m) q^{s(2m+2k)-4k-2m}$$
$$= \kappa q^{2k} \left(1-q^{-2}\right) \sum_{m=-r-k}^{\delta+r-k} \left(q^{2m}\left(1-q^{-2}\right)\right) q^{s(2m+2k)-4k-2m}$$
$$= \kappa q^{-2k} \left(1-q^{-2}\right)^2 \sum_{m=-r-k}^{\delta+r-k} q^{2(m+k)s}$$
$$= \kappa q^{-2k} \left(1-q^{-2}\right)^2 \sum_{i=-r}^{\delta+r} q^{2is}.$$

Since $\sum_{k>0} q^{-2k} = \frac{q^{-2}}{1-q^{-2}}$, we find that the total contribution across both the $n = 0$ case and the $k > 0$ case is

$$\left(\left(1 - \frac{q+1}{q^2}\right)\left(1-q^{-2}\right) + q^{-2}(1-q^{-2})\right) \kappa \sum_{i=-r}^{\delta+r} q^{2is}$$
$$= \left(1-q^{-1}\right)\left(1-q^{-2}\right) \kappa \sum_{i=-r}^{\delta+r} q^{2is}$$
$$= \sum_{i=-r}^{\delta+r} q^{2is}.$$

This equals the claimed sum above. (We write it over $0 \leq j \leq \delta + 2r$ for consistency with a later part.) $\square$

7.2 Contribution from Case 1 and Case 2 assuming $\boldsymbol{v(b)} \geq 0$ and $\boldsymbol{v(d)} \geq 0$

Again using $\mathrm{Vol}(t_1 : -v(t_1) = m) = q^{2m}(1-q^{-2})$, summing all the cases gives the following contribution within $\kappa \int_{t,t_1 \in E} \mathbf{1}[n>0]\mathbf{1}_{\leq r}(\gamma, t, m)$:

$$
\begin{aligned}
I^{1+2}_{n>0} \coloneqq \kappa \sum_{n=1}^{r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} & q^{-n}\left(1-q^{-2}\right) \\
& \cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big) \\
+ \kappa \sum_{n=r+1}^{\ell+r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} & q^{-n-\lceil \frac{n-r}{2} \rceil}\left(1-q^{-1}\right) \\
& \cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big) \\
+ \kappa \sum_{n=1}^{r} \sum_{m=\max\left(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r\right)}^{\delta + r} & q^{-n}\left(1-q^{-2}\right) \\
& \cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big) \\
+ \kappa \sum_{n=1}^{\ell+r} \sum_{m=\max\left(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r, \delta + r + 1\right)}^{\min(n,\lambda)+\delta+r} & q^{-n-(m-\delta-r)}\left(1-q^{-1}\right) \\
& \cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big) \\
= \sum_{n=1}^{r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} & q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(2m-n)} \\
+ \sum_{n=r+1}^{\ell+r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} & q^{\lfloor \frac{n+r}{2} \rfloor} \cdot (-1)^n q^{s(2m-n)} \\
+ \sum_{n=1}^{r} \sum_{m=\max\left(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r\right)}^{\delta + r} & q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(2m-n)}
\end{aligned}
$$

$$+\sum_{n=1}^{\ell+r} \sum_{m=\max\left(n-r,\left\lceil\frac{n-r}{2}\right\rceil+\delta+r,\delta+r+1\right)}^{\min(n,\lambda)+\delta+r} q^{n-(m-\delta-r)} \cdot (-1)^n q^{s(2m-n)}.$$

To simplify the expressions, we replace the summation variable m with

$$j \coloneqq (n+\delta+r) - m \geq 0.$$

In that case,

$$2m - n = 2(\delta + n + r - j) - n = n + 2\delta + 2r - 2j.$$

Then the expression rewrites as

$$\begin{aligned}
I_{n>0}^{1+2} = &\sum_{n=1}^{r} \sum_{j=\left\lfloor\frac{n+r}{2}\right\rfloor+1}^{\delta+2r} q^n\left(1+q^{-1}\right) \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&+ \sum_{n=r+1}^{\ell+r} \sum_{j=\left\lfloor\frac{n+r}{2}\right\rfloor+1}^{\delta+2r} q^{\left\lfloor\frac{n+r}{2}\right\rfloor} \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&+ \sum_{n=1}^{r} \sum_{j=n}^{\min\left(\delta+2r,\left\lfloor\frac{n+r}{2}\right\rfloor\right)} q^n\left(1+q^{-1}\right) \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&+ \sum_{n=1}^{\ell+r} \sum_{j=\max(0,n-\lambda)}^{\min(\delta+2r,\left\lfloor\frac{n+r}{2}\right\rfloor,n-1)} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)}.
\end{aligned}$$

We interchange the order of summation so that it is first over j and then n. There are four double sums to interchange.

- The first double sum runs from $j = \left\lfloor\frac{r+1}{2}\right\rfloor + 1$ to $j = \delta + 2r$. In addition to $1 \leq n \leq r$, we need $\left\lfloor\frac{n+r}{2}\right\rfloor + 1 \leq j$, which solves to $\frac{n+r}{2} \leq j - \frac{1}{2}$ or $n \leq 2j - 1 - r$. Thus the condition on n is

$$1 \leq n \leq \min(2j-1-r, r).$$

- The second double sum runs from $j = r+1$ to $\delta + 2r$. We also need $r + 1 \leq n \leq \ell + r$

and $n \leq 2j-1-r$. Hence, the desired condition on n is

$$r+1 \leq n \leq \min(2j-1-r, \ell+r).$$

- The third double sum runs from $j=1$ to $j=r$. Meanwhile, the values of n need to satisfy $1 \leq n \leq r$, $n \leq j$ and $j \leq \left\lfloor \frac{n+r}{2} \right\rfloor \implies n \geq 2j-r$, consequently we just obtain

$$\max(1, 2j-r) \leq n \leq j.$$

- The fourth double sum runs $j=0$ to

$$j = \min\left(\delta+2r, \left\lfloor \frac{\ell}{2} \right\rfloor + r, \ell+r-1\right) = \left\lfloor \frac{\ell}{2} \right\rfloor + r - \mathbf{1}[\ell=0]$$

again because of $\ell < 2\delta$. Meanwhile, we require $1 \leq n \leq \ell+r$, $j \geq n-\lambda$, $j \leq n-1$, as well as $j \leq \left\lfloor \frac{n+r}{2} \right\rfloor \iff n \geq 2j-r$. Putting these four conditions together gives

$$\max(j+1, 2j-r) \leq n \leq \min(j+\lambda, \ell+r).$$

Hence we get

$$\begin{aligned}
I_{n>0}^{1+2} = & \sum_{j=\left\lfloor \frac{r+1}{2} \right\rfloor+1}^{\delta+2r} \sum_{n=1}^{\min(2j-1-r,r)} q^n\left(1+q^{-1}\right) \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
& + \sum_{j=r+1}^{\delta+2r} \sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\left\lfloor \frac{n+r}{2} \right\rfloor} \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
& + \sum_{j=1}^{r} \sum_{n=\max(1,2j-r)}^{j} q^n\left(1+q^{-1}\right) \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
& + \sum_{j=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor+r-\mathbf{1}[\ell=0]} \sum_{n=\max(j+1,2j-r)}^{\min(j+\lambda,\ell+r)} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)}.
\end{aligned}$$

At this point, we can unify the sum over j by noting that for j outside of the summation range, the inner sum is empty anyway. Specifically, note that:

- In the first and second double sum, the inner sum over n is empty anyway when $j < r$.
- In the third double sum, adding $j = 0$ does not introduce new terms. Moreover, when $j > r$ the inner sum over n is also empty anyway.
- In the fourth double sum,
  - If $\ell = 0$ and $j \geq r$, then $j + 1 \geq 0 + r$; and
  - If $\ell > 0$ and $j > \frac{\ell}{2} + r$, then $2j - r \geq \ell + r$.

  So no new terms are introduced in this case either.

So we can unify all four double sums to run over $0 \leq j \leq \delta + 2r$, simplifying the expression to just

$$
\begin{aligned}
I_{n>0}^{1+2} = q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} \Bigg(& \sum_{n=1}^{\min(2j-1-r,r)} q^n \left(1 + q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\
& + \sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\left\lfloor \frac{n+r}{2} \right\rfloor} \cdot (-1)^n q^{s(n-2j)} \\
& + \sum_{n=\max(1,2j-r)}^{j} q^n \left(1 + q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\
& + \sum_{n=\max(j+1,2j-r)}^{\min(j+\lambda,\ell+r)} q^j \cdot (-1)^n q^{s(n-2j)} \Bigg).
\end{aligned}
$$

7.3 Merging of $I_{n\le 0}$ with $I^{1+2}_{n>0}$ (and proof of Theorem 5.5.2)

We continue assuming $v(b) = v(d) = 0$. It turns out that $I_{n\le 0} + I^{1+2}_{n>0}$ can be rewritten more compactly (giving a simple answer especially when ℓ is odd). Then

$$I_{n\le 0} + I^{1+2}_{n>0}$$

can be rewritten to the collation

$$\begin{aligned}
= q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} \Bigg(q^{-2js} &+ \sum_{n=1}^{\min(2j-1-r,r)} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\
&+ \sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\left\lfloor \frac{n+r}{2} \right\rfloor} \cdot (-1)^n q^{s(n-2j)} \\
&+ \sum_{n=\max(1,2j-r)}^{j} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\
&+ \sum_{n=\max(j+1,2j-r)}^{\min(j+\lambda,\ell+r)} q^j \cdot (-1)^n q^{s(n-2j)} \Bigg).
\end{aligned}$$

Note that when $r = 0$ and $\ell = \lambda \equiv 1 \pmod 2$ we recover [Zha12, Equation (4.13)].

The above expression can be considered as a Laurent polynomial in $-q^s$, whose coefficients are nonnegative polynomials in q (note that $(-1)^n = (-1)^{n-2j}$). Now we are going to extract the coefficient of $(-q^s)^k$, for each integer k. First, note that

- The initial term before the sums adds 1 if k is even and $-2r \le k \le 2\delta + 2r$, and 0 otherwise.

We move on to the inner sums and calculate their contributions. For a fixed $k \in \mathbb{Z}$, we want to consider (n, j) with $n - 2j + 2(\delta + r) = k$, that is, $2j = n + 2\delta + 2r - k$, or $n = 2j + k - 2\delta - 2r$. The condition that $j \in \mathbb{Z}$ and $0 \le j \le \delta + 2r$ is then equivalent to

$$k - 2\delta - 2r \le n \le k + 2r \qquad \text{and} \qquad n \equiv k \pmod 2. \tag{7.1}$$

We note also that

$$n < 2j - r \iff n < n + 2\delta + r - k \iff k < 2\delta + r \tag{7.2}$$

which needs to hold for the first two sums to contribute. Conversely, in the latter two sums, we will assume that

$$n \geq 2j - r \iff k \geq 2\delta + r. \tag{7.3}$$

Now we are ready for the main calculation. In what follows $i\%2 \in \{0, 1\}$ means the remainder when i is divided by 2. Moreover, any ellipses of the form

$$q^i + \cdots + q^{i'}$$

will be an abbreviation for $q^i + q^{i-1} + \cdots + q^{i'}$ (i.e. within any ellipses, the exponents are understood to decrease by 1, and the sums are always nonempty, meaning $i \geq i'$).

In the region where $k < 2\delta + r$, the first two sums contribute:

- The first sum contributes if and only if (7.1) holds, $1 \leq n \leq r$ and (7.2) is true. Hence, the contribution only occurs when $k < 2\delta + r$. In that case, all $1 \leq n \leq \min(r, k + 2r)$ with $n \equiv k \pmod 2$ appear. Since the contribution of a given n is $q^n + q^{n-1}$ and the n are incrementing by 2, our final total is

$$\begin{cases} q^{k+2r} + \cdots + q^{(k-1)\%2} & \text{if } -2r < k \leq -r \\ q^{r-(k-r)\%2} + \cdots + q^{(k-1)\%2} & \text{if } -r \leq k < 2\delta + r \\ 0 & \text{otherwise.} \end{cases}$$

- The second sum contributes if and only if (7.1) holds, (7.2) holds and $r + 1 \leq n \leq \ell + r$. The hypothesis $n > r$ means we need $k \geq -r$. Since $\ell < 2\delta$, the upper bound for n is $n \leq \min(\ell + r, k + 2r)$ which we split into two cases.

In the case where $k+2r \le \ell + r$, then since $\ell < 2\delta$, the inequality $k < 2\delta + r$ holds automatically. We have the largest term $n = k+2r \equiv k \pmod 2$, so the largest exponent q that appears is $\left\lfloor \frac{(k+2r)+r}{2} \right\rfloor$.

In the other case $\ell + r \le k + 2r$, we obtain largest exponents of

$$\left\lfloor \frac{(\ell + r - (\ell + r - k)\%2) + r}{2} \right\rfloor = r + \left\lfloor \frac{\ell - (\ell + r - k)\%2}{2} \right\rfloor.$$

Thus, we obtain

$$\begin{cases} q^{\left\lfloor \frac{k+3r}{2} \right\rfloor} + \cdots + q^{r+(r+1-k)\%2} & \text{if } -r \le k \le \ell - r \\ q^{r+\left\lfloor \frac{\ell-(\ell+r-k)\%2}{2} \right\rfloor} + \cdots + q^{r+(r+1-k)\%2} & \text{if } \ell - r \le k < 2\delta + r \\ 0 & \text{otherwise.} \end{cases}$$

In the region where $k \ge 2\delta + r$, the latter two sums are in play:

- In the third sum, we assume (7.3); then the other constraints on n are

$$n \ge 1$$
$$n \le j \iff 2n \le n + 2\delta + 2r - k \iff n \le 2\delta + 2r - k$$

  which implies $k < 2\delta + 2r$ for this range to be nonempty. In this case, (7.1) is actually redundant already. That means our contribution can be described as

$$\begin{cases} q^{2\delta+2r-k} + \cdots + q^{(k-1)\%2} & \text{if } 2\delta + r \le k < 2\delta + 2r \\ 0 & \text{otherwise.} \end{cases}$$

- Unlike the other sums, the j is in the exponent in the fourth sum, so (7.1) will not be useful to us. Instead our goal is to detect the values of j for which the corresponding value of

$$n = 2j - 2\delta - 2r + k$$

lies in the desired interval. That is, we get a contribution of q^j if and only if (7.3) holds and

$$\begin{aligned} 0 &\leq j \leq \delta + 2r \\ j < 2j - 2\delta - 2r + k &\implies 2\delta + 2r - k < j \\ 2j - 2\delta - 2r + k \leq j + \lambda &\iff j \leq 2\delta + 2r + \lambda - k \\ 2j - 2\delta - 2r + k \leq \ell + r &\iff j \leq \delta + \frac{3r + \ell - k}{2} \end{aligned}$$

The values of k for which there is any valid index j is given by

$$2\delta + r \leq k \leq 2\delta + \min\left(\lambda + 2r, \ell + 3r\right).$$

The breakpoint for the two upper bounds on j occurs when

$$2\delta + 2r - k + \lambda \leq \delta + \frac{3r + \ell - k}{2} \iff k \geq 2\delta + 2\lambda - \ell + r.$$

Comparing all the bounds, we find there are three possible scenarios.

- If $\lambda \leq \frac{\ell + r}{2}$, then we get

$$\begin{cases} q^{\delta + \left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^{2\delta+2r-k+1} & \text{if } 2\delta + r \leq k \leq 2\delta + 2r \\ q^{\delta + \left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^{0} & \text{if } 2\delta + 2r < k \leq 2\delta + 2\lambda - \ell + r \\ q^{2\delta+2r+\lambda-k} + \cdots + q^{0} & \text{if } 2\delta + 2\lambda - \ell + r \leq k \leq 2\delta + \lambda + 2r \\ 0 & \text{otherwise.} \end{cases}$$

– If $\frac{\ell+r}{2} < \lambda \leq \ell + r$, then we get

$$\begin{cases} q^{\delta+\left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^{2\delta+2r-k+1} & \text{if } 2\delta + r \leq k \leq 2\delta + 2\lambda - \ell + r \\ q^{\delta+\left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^{2\delta+2r-k+1} & \text{if } 2\delta + 2\lambda - \ell + r \leq k \leq 2\delta + 2r \\ q^{2\delta+2r+\lambda-k} + \cdots + q^0 & \text{if } 2\delta + 2r < k \leq 2\delta + \lambda + 2r \\ 0 & \text{otherwise.} \end{cases}$$

– If $\ell + r < \lambda$, then we get

$$\begin{cases} q^{\delta+\left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^{2\delta+2r-k+1} & \text{if } 2\delta + r \leq k \leq 2\delta + 2r \\ q^{\delta+\left\lfloor \frac{3r+\ell-k}{2} \right\rfloor} + \cdots + q^0 & \text{if } 2\delta + 2r < k \leq 2\delta + \ell + 3r \\ 0 & \text{otherwise.} \end{cases}$$

This completes the analysis of the four sums above. For later purposes, it will be more symmetric to rewrite the exponent as

$$\delta + \left\lfloor \frac{3r + \ell - k}{2} \right\rfloor = r + \left\lfloor \frac{(2\delta + \ell + r) - k}{2} \right\rfloor .$$

Now we can piece together all the parts below. It turns out that for every value of k, the coefficient of $(-q^s)^k$ is an expression of the form $1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}$ for some k. Indeed,

- When $k = -2r$ the only term is q^0.
- For $-2r < k < r$, only the first sum contributes $q^{k+2r} + \cdots + q^{(k-1)\%2}$, which is then completed by the q^0 contribution from $I_{n\leq 0}$ when k is even with a possible q^0.
- For $-r \leq k < 2\delta + r$, the first and second sum actually fit together with a "seam" near q^r, which is for even k then completed by the single q^0 contribution from $I_{n\leq 0}$ (only when k is even).

- For $2\delta+r \le k \le 2\delta+2r$, the same holds with piecing the third and fourth sum together (where the seam is near q^k this time).

- Finally, only the fourth sum contributes for $k \ge 2\delta+2r$, and it is of the desired form.

This gives us a succinct description of the weighted orbital integral.

If ℓ is even, we now have the following intermediate result.

Proposition 7.3.1 (Case 1 and 2 for $\ell \ge 0$ even)**.** *Suppose $\ell \ge 0$ is even. Then we have the intermediate result*

$$I_{n\le 0} + I^{1+2}_{n>0} = \sum_{k=-2r}^{2\delta+\min(\lambda+2r,\ell+3r)} (-1)^k \left(1+q+q^2+\cdots+q^{\mathbf{n}^{1+2}_\gamma(k)}\right)(q^s)^k$$

where the piecewise function $\mathbf{n}_\gamma \colon \mathbb{Z} \to \mathbb{Z}_{\ge 0}$ is defined by

$$\mathbf{n}^{1+2}_\gamma(k) = \begin{cases} k+2r & \text{if } -2r \le k \le -r \\ \left\lfloor \frac{k+r}{2} \right\rfloor + r & \text{if } -r \le k \le \ell - r \\ \frac{\ell}{2} + r - (k-r)\%2 & \text{if } \ell - r \le k \le 2\delta + r \\ \left\lfloor \frac{(\ell+2\delta+r)-k}{2} \right\rfloor + r & \text{if } 2\delta + r \le k \le \ell + 2\delta + 3r \end{cases}$$

in the case $\lambda \ge \ell + r$, and

$$\mathbf{n}^{1+2}_\gamma(k) = \begin{cases} k+2r & \text{if } -2r \le k \le -r \\ \left\lfloor \frac{k+r}{2} \right\rfloor + r & \text{if } -r \le k \le \ell - r \\ \frac{\ell}{2} + r - (k-r)\%2 & \text{if } \ell - r \le k \le 2\delta + r \\ \left\lfloor \frac{(\ell+2\delta+r)-k}{2} \right\rfloor + r & \text{if } 2\delta + r \le k \le 2\lambda - \ell + 2\delta + r \\ (\lambda+2\delta+2r) - k & \text{if } 2\delta + 2\lambda - \ell + r \le k \le \lambda + 2\delta + 2r \end{cases}$$

in the case $\lambda \le \ell + r$.

On the other hand, when ℓ is odd, we get Theorem 5.5.2:

Theorem 5.5.2 (Weighted orbital integral for odd ℓ)**.** *Let $r \geq 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let b, d, δ, ℓ, be as in Lemma 5.3.3 and Lemma 5.4.1. If ℓ is odd, define*

$$\mathbf{n}_\gamma := \underset{[-2r,\ell+2\delta+2r]}{\mathrm{ARCH}}(r,\ell).$$

Then for any $r \geq 0$ we have the formula:

$$\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = \sum_{k=-2r}^{\ell+2\delta+2r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right)(q^s)^k.$$

7.4 Contribution from Case 3^+, 3-, 4^+ assuming $v(b) \geq 0$ and $v(d) \geq 0$

These cases only appear when ℓ is even and we assume this for this subsection. We consider the contribution of these cases within $\kappa \int_{t,t_1 \in E} \mathbf{1}[n > 0]\mathbf{1}_{\leq r}(\gamma, t, m)$ (using (6.22), (6.23), (6.24)) and put $\mathrm{Vol}(t_1 : -v(t_1) = m) = q^{2m}(1 - q^{-2})$ to get:

$$\begin{aligned}
I^{3+4}_{n>0} := \kappa &\sum_{n=\ell+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\min(n-1,\lambda+r)-\frac{\ell}{2}+\delta} q^{-n-(n-\frac{\ell}{2}-r)}\left(1-q^{-1}\right) \\
&\quad\cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
+ \kappa &\sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} q^{-n-(n-\frac{\ell}{2}-r)}\left(1-q^{-1}\right) \\
&\quad\cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
+ \kappa &\sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-\frac{\ell}{2}+\delta}^{\min(n,\lambda)+\delta+r} q^{-n-(m-\delta-r)}\left(1-q^{-1}\right) \\
&\quad\cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=\ell+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\min(n-1,\lambda+r)-\frac{\ell}{2}+\delta} q^{-n-(n-\frac{\ell}{2}-r)} \cdot \Big((-1)^n q^{s(2m-n)} q^{2n}\Big) \\
&\quad + \sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} q^{-n-(n-\frac{\ell}{2}-r)} \cdot \Big((-1)^n q^{s(2m-n)} q^{2n}\Big) \\
&\quad + \sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-\frac{\ell}{2}+\delta}^{\min(n,\lambda)+\delta+r} q^{-n-(m-\delta-r)} \cdot \Big((-1)^n q^{s(2m-n)} q^{2n}\Big) \\
&= \sum_{n=\ell+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\min(n-1,\lambda+r)-\frac{\ell}{2}+\delta} q^{\frac{\ell}{2}+r} \cdot (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} q^{\frac{\ell}{2}+r} \cdot (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-\frac{\ell}{2}+\delta}^{\min(n,\lambda)+\delta+r} q^{n-m+\delta+r} \cdot (-1)^n q^{s(2m-n)} \\
&= q^{\frac{\ell}{2}+r} \cdot \sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-r}^{n-1-\frac{\ell}{2}+\delta} (-1)^n q^{s(2m-n)} \\
&\quad + q^{\frac{\ell}{2}+r} \cdot \sum_{n=\lambda+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\lambda-\frac{\ell}{2}+\delta+r} (-1)^n q^{s(2m-n)} \\
&\quad + q^{\frac{\ell}{2}+r} \cdot \sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-\frac{\ell}{2}+\delta}^{\min(n,\lambda)+\delta+r} q^{n-m+\delta+r} \cdot (-1)^n q^{s(2m-n)};
\end{aligned}
$$

We now analyze each double sum.

7.4.1 Analysis of the coefficient of the top-degree term $q^{\frac{\ell}{2}+r}$

We start by analyzing just the first three double sums: Fix an index k; we collect the coefficient of $(-q^s)^k$.

- For the first double sum, change the summation for m to

$$m = n - r + j \iff j = m - n + r$$

so that $2m - n = n - 2r + 2j$ and the first double sum rewrites as

$$\begin{aligned}\sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-r}^{n-\frac{\ell}{2}+\delta-1} (-q^s)^{2m-n} &= \sum_{n=\ell+r+1}^{\lambda+r} \sum_{j=0}^{r-\frac{\ell}{2}+\delta-1} (-q^s)^{n-2r+2j} \\ &= \sum_{j=0}^{r-\frac{\ell}{2}+\delta-1} \sum_{n=\ell+r+1}^{\lambda+r} (-q^s)^{n-2r+2j} \\ &= \sum_{j=0}^{r-\frac{\ell}{2}+\delta-1} \sum_{k=\ell-r+2j+1}^{\lambda-r+2j} (-q^s)^k\end{aligned}$$

Now we collect the coefficient of $(-q^s)^k$. The value of k runs from the lowest value $k = \ell - r + 1$ to the highest value $k = \lambda - \ell + 2\delta + r - 2$. We seek the number of $0 \le j \le r - \frac{\ell}{2} + \delta - 1$ such that

$$\ell - r + 2j + 1 \le k \le \lambda - r + 2j \iff \frac{k + r - \lambda}{2} \le j \le \frac{k + r - (\ell + 1)}{2}$$

so the number of terms that appear is

$$\min\left(\left\lfloor \frac{k - \ell + r - 1}{2} \right\rfloor, -\frac{\ell}{2} + \delta + r - 1\right) - \max\left(\left\lceil \frac{k + r - \lambda}{2} \right\rceil, 0\right) + 1$$

which is nonnegative (but could be zero). We add in one more term $k = \lambda - \ell + 2\delta + r - 1$ and $k = \lambda - \ell + 2\delta + r$ for simplicity; this is okay because the above display equals zero at this value anyway.

- For the second double sum, change the order of summation to

$$\sum_{n=\lambda+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\lambda-\frac{\ell}{2}+\delta+r} (-q^s)^{2m-n} = \sum_{m=\lambda+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{n=\lambda+r+1}^{\min\left(\lambda-\frac{\ell}{2}+\delta+r,m+r\right)} (-q^s)^{2m-n}$$
$$= \sum_{m=\lambda+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{k=\max\left(m-r,2m-\lambda+\frac{\ell}{2}-\delta-r\right)}^{2m-\lambda-r-1} (-q^s)^k.$$

Now we collect the coefficient of $(-q^s)^k$. The value of k runs from the lowest value $k = \lambda - r + 1$ to the highest value $k = \lambda - \ell + 2\delta + r - 1$. For k in this interval, we seek values of $\lambda + 1 \leq m \leq \lambda - \frac{\ell}{2} + \delta + r$ and

$$k \leq 2m - \lambda - r - 1 \implies m \geq \frac{k + \lambda + r + 1}{2}$$
$$k \geq m - r \implies m \leq k + r$$
$$k \geq 2m - \lambda + \frac{\ell}{2} - \delta - r \implies m \leq \frac{k + \lambda - \frac{\ell}{2} + \delta + r}{2}.$$

When $k \geq \lambda - r + 1$ we already have $\frac{k+\lambda+r+1}{2} \geq \lambda + 1$, so the number of terms that appear is

$$\min\left(k + r, \left\lfloor \frac{k + \lambda - \frac{\ell}{2} + \delta + r}{2} \right\rfloor, \lambda - \frac{\ell}{2} + \delta + r\right) - \left\lceil \frac{k + \lambda + r + 1}{2} \right\rceil + 1$$

which is nonnegative (but could be zero). We add in two extra term at $k = \lambda - r$ and $k = \lambda - \ell + 2\delta + r$; for simplicity; this is okay because the above display equals zero at this value anyway.

- The third double sum

$$\sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} (-q^s)^{2m-n}$$

happens to coincide with the previous one if one replaces λ by ℓ everywhere. Hence the number of terms that appear is

$$\min\left(k+r, \left\lfloor \frac{k+\frac{\ell}{2}+\delta+r}{2} \right\rfloor, +\frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\frac{\ell}{2}+r+1}{2} \right\rceil + 1$$

running from $k=\frac{\ell}{2}-r$ to $k=\frac{\ell}{2}+2\delta+r$.

Now we collate the contribution of the first three double sums:

$$\begin{aligned}
&\sum_{k=\ell-r}^{\lambda-\ell+2\delta+r}\left(\min\left(\left\lfloor \frac{k-\ell+r-1}{2}\right\rfloor, -\frac{\ell}{2}+\delta+r-1\right) - \max\left(\left\lceil \frac{k+r-\lambda}{2}\right\rceil, 0\right)+1\right)(-q^s)^k \\
&+\sum_{k=\lambda-r}^{\lambda-\ell+2\delta+r}\left(\min\left(k+r, \left\lfloor \frac{k+\lambda-\frac{\ell}{2}+\delta+r}{2}\right\rfloor, \lambda-\frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\lambda+r+1}{2}\right\rceil+1\right)(-q^s)^k \\
&+\sum_{k=\ell-r}^{2\delta+r}\left(\min\left(k+r, \left\lfloor \frac{k+\frac{\ell}{2}+\delta+r}{2}\right\rfloor, \frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\ell+r+1}{2}\right\rceil+1\right)(-q^s)^k \\
&=\sum_{k=\ell-r}^{\lambda-r-1}\left(\min\left(\left\lfloor \frac{k-\ell+r-1}{2}\right\rfloor, -\frac{\ell}{2}+\delta+r-1\right)+1\right)(-q^s)^k \\
&+\sum_{k=\lambda-r}^{\lambda-\ell+2\delta+r}\left(\min\left(\left\lfloor \frac{k-\ell+r-1}{2}\right\rfloor, -\frac{\ell}{2}+\delta+r-1\right) - \left\lceil \frac{k+r-\lambda}{2}\right\rceil+1\right)(-q^s)^k \\
&+\sum_{k=\lambda-r}^{\lambda-\ell+2\delta+r}\left(\min\left(k+r, \left\lfloor \frac{k+\lambda-\frac{\ell}{2}+\delta+r}{2}\right\rfloor, \lambda-\frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\lambda+r+1}{2}\right\rceil+1\right)(-q^s)^k \\
&+\sum_{k=\ell-r}^{2\delta+r}\left(\min\left(k+r, \left\lfloor \frac{k+\frac{\ell}{2}+\delta+r}{2}\right\rfloor, \frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\ell+r+1}{2}\right\rceil+1\right)(-q^s)^k.
\end{aligned}$$

Hence, for each k the coefficient of $(-q^s)^k$ is given by a sum of a subset of the following four coefficients:

- For $\ell-r \le k \le \lambda-r-1$, we get

$$C_1(k) := \begin{cases} \left\lfloor \frac{k-\ell+r+1}{2} \right\rfloor & \text{if } k \le 2\delta+r \\ -\frac{\ell}{2}+\delta+r & \text{if } k \ge 2\delta+r. \end{cases}$$

- For $\lambda - r \leq k \leq \lambda - \ell + 2\delta + r$, we get

$$C_2(k) \coloneqq \begin{cases} \left\lfloor \frac{k-\ell+r+1}{2} \right\rfloor - \left\lceil \frac{k+r-\lambda}{2} \right\rceil & \text{if } k \leq 2\delta + r \\ -\frac{\ell}{2} + \delta + r - \left\lceil \frac{k+r-\lambda}{2} \right\rceil & \text{if } k \geq 2\delta + r \end{cases}$$
$$= \begin{cases} \frac{\lambda-\ell+1}{2} - (k+r+1)\%2 & \text{if } k \leq 2\delta + r \\ \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor & \text{if } k \geq 2\delta + r. \end{cases}$$

- For $\lambda - r \leq k \leq \lambda - \ell + 2\delta + r$, we get

$$C_3(k) \coloneqq \begin{cases} k + r - \left\lceil \frac{k+\lambda+r+1}{2} \right\rceil + 1 & \text{if } k \leq \lambda - \frac{\ell}{2} + \delta - r \\ \left\lfloor \frac{k+\lambda-\frac{\ell}{2}+\delta+r}{2} \right\rfloor - \left\lceil \frac{k+\lambda+r+1}{2} \right\rceil + 1 & \text{if } \lambda - \frac{\ell}{2} + \delta - r \leq k \leq \lambda - \frac{\ell}{2} + \delta + r \\ \left(\lambda - \frac{\ell}{2} + \delta + r\right) - \left\lceil \frac{k+\lambda+r+1}{2} \right\rceil + 1 & \text{if } k \geq \lambda - \frac{\ell}{2} + \delta + r \end{cases}$$
$$= \begin{cases} \left\lfloor \frac{k-\lambda+r+1}{2} \right\rfloor & \text{if } k \leq \lambda - \frac{\ell}{2} + \delta - r \\ \frac{-\frac{\ell}{2}+\delta+1-(k+\frac{\ell}{2}+\delta+r+1)\%2-(k+r)\%2}{2} & \text{if } \lambda - \frac{\ell}{2} + \delta - r \leq k \leq \lambda - \frac{\ell}{2} + \delta + r \\ \left\lfloor \frac{\lambda-\ell+2\delta+r-k+1}{2} \right\rfloor & \text{if } k \geq \lambda - \frac{\ell}{2} + \delta + r. \end{cases}$$

- For $\ell - r \leq k \leq 2\delta + r$, we get

$$C_4(k) \coloneqq \begin{cases} k + r - \left\lceil \frac{k+\ell+r+1}{2} \right\rceil + 1 & \text{if } k \leq \frac{\ell}{2} + \delta - r \\ \left\lfloor \frac{k+\frac{\ell}{2}+\delta+r}{2} \right\rfloor - \left\lceil \frac{k+\ell+r+1}{2} \right\rceil + 1 & \text{if } \frac{\ell}{2} + \delta - r \leq k \leq \frac{\ell}{2} + \delta + r \\ \left(\frac{\ell}{2} + \delta + r\right) - \left\lceil \frac{k+\ell+r+1}{2} \right\rceil + 1 & \text{if } k \geq \frac{\ell}{2} + \delta + r \end{cases}$$
$$= \begin{cases} \left\lfloor \frac{k-\ell+r+1}{2} \right\rfloor & \text{if } k \leq \frac{\ell}{2} + \delta - r \\ \frac{-\frac{\ell}{2}+\delta+1-(k+\frac{\ell}{2}+\delta+r)\%2-(k+r+1)\%2}{2} & \text{if } \frac{\ell}{2} + \delta - r \leq k \leq \frac{\ell}{2} + \delta + r \\ \left\lfloor \frac{2\delta+r-k+1}{2} \right\rfloor & \text{if } k \geq \frac{\ell}{2} + \delta + r. \end{cases}$$

The coefficient we need is then

$$C(k) \coloneqq C_1(k) + C_2(k) + C_3(k) + C_4(k)$$

where we consider $C_i(k) = 0$ if k falls outside the required range for C_i.

Section 7.4.1 gives a sketch of the regions covered by C_1, C_2, C_3, C_4 in one situation. In the general situation, some of the points in the middle could have different orders relative to each other. (In particular, $2\delta + r$ could occur in either C_1 or C_2.)

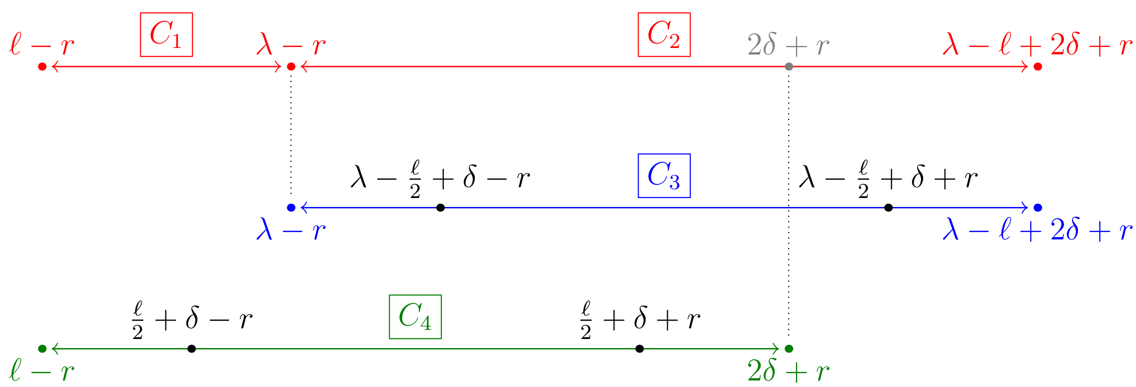

Figure 7.1: Rough illustration of one possibility for the shapes of regions for C_1, C_2, C_3, C_4, in the case where $\lambda - \frac{\ell}{2} + \delta - r \leq \frac{\ell}{2} + \delta + r$. The breaking points within each C_i are marked in black, and grey dotted lines join values of k that overlap between the regions. The four black breaking points show up in the collated formula later.

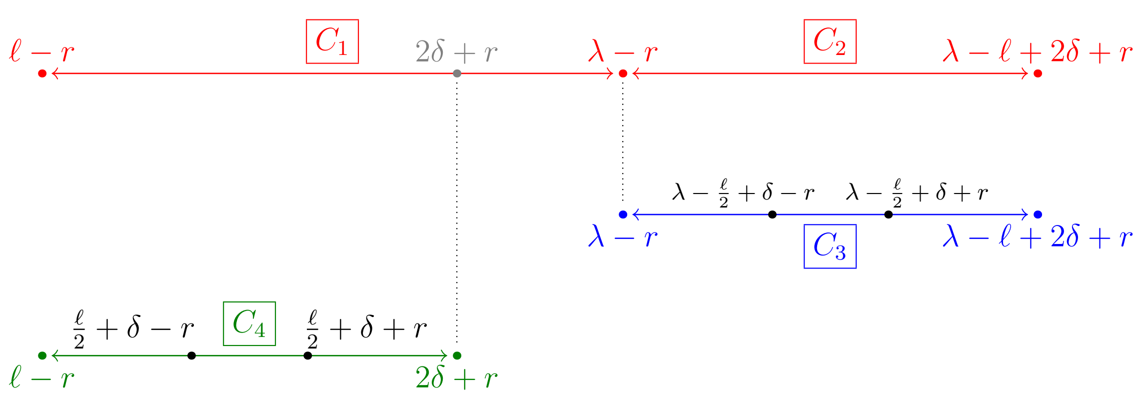

Figure 7.2: An illustration of another possibility for the shapes of regions for C_1, C_2, C_3, C_4, this time in the case where $\frac{\ell}{2} + \delta + r \leq \lambda - \frac{\ell}{2} + \delta - r$.

To make the casework a bit more efficient, we prove the following lemma, which effectively lets us fold together cases on $\lambda - r$ or $2\delta + r$.

Lemma 7.4.1 (Folding cases)**.** *The following two identities are true:*

- *For* $k \leq \min\left(\lambda - \frac{\ell}{2} + \delta - r, 2\delta + r\right)$*,*

$$\left\lfloor \frac{k - \ell + r + 1}{2} \right\rfloor = \begin{cases} C_1(k) & \text{if } k \leq \lambda - r \\ C_2(k) + C_3(k) & \text{if } k \geq \lambda - r. \end{cases}$$

- *For* $k \geq \max\left(\lambda - \frac{\ell}{2} + \delta - r, \lambda - r\right)$*,*

$$\left\lfloor \frac{\lambda - \ell + 2\delta + r - k}{2} \right\rfloor = \begin{cases} C_2(k) & \text{if } k \geq 2\delta + r \\ C_2(k) + C_4(k) - (k + r)\%2 & \text{if } k \leq 2\delta + r. \end{cases}$$

Proof. For the first part, it's trivial for $k \leq \lambda - r$, whilst for $k \geq \lambda - r$ we have

$$\begin{aligned} C_2(k) + C_3(k) &= \frac{\lambda - \ell + 1}{2} - (k + r - \lambda)\%2 + \left\lfloor \frac{k - \lambda + r + 1}{2} \right\rfloor \\ &= \left\lfloor \frac{k - \ell + r + 2}{2} \right\rfloor - (k + r - \lambda)\%2 \end{aligned}$$

which also equals the claimed result.

For the second part, it's again trivial for $k \geq 2\delta + r$ whilst for $k \leq 2\delta + r$ we have

$$\begin{aligned} C_2(k) + C_4(k) &= \frac{\lambda - \ell + 1}{2} - (k + r + 1)\%2 + \left\lfloor \frac{2\delta + r - k + 1}{2} \right\rfloor \\ &= \left\lfloor \frac{\lambda - \ell + 2\delta + r - k}{2} \right\rfloor + (1 - (k + r + 1)\%2) \end{aligned}$$

which also matches. $\square$

We consider the following five cases now:

1. Suppose $\ell - r \le k \le \frac{\ell}{2} + \delta - r$ (in particular, also $k \le 2\delta + r$ and $k \le \lambda - \frac{\ell}{2} = \delta$). Then from Lemma 7.4.1 we have

$$C(k) = \left\lfloor \frac{k - \ell + r + 1}{2} \right\rfloor + C_4(k) = k - (\ell - r) + (k + r)\%2.$$

2. Next suppose that $\frac{\ell}{2} + \delta - r \le k \le \min(\delta + \frac{\ell}{2} + r, \lambda - \frac{\ell}{2} + \delta - r)$. Again from Lemma 7.4.1 we have

$$\begin{aligned} C(k) &= \left\lfloor \frac{k - \ell + r + 1}{2} \right\rfloor + C_4(k) \\ &= \left\lfloor \frac{k - \ell + r + 1}{2} \right\rfloor + \frac{-\frac{\ell}{2} + \delta + 1 - (k + \frac{\ell}{2} + \delta + r)\%2 - (k + r + 1)\%2}{2} \\ &= \frac{k - \ell + r + 1}{2} + \frac{-\frac{\ell}{2} + \delta + 1 - (k + \frac{\ell}{2} + \delta + r)\%2}{2} - (k + r + 1)\%2 \\ &= \left(\delta - \frac{\ell}{2}\right) + \left\lfloor \frac{k - (\frac{\ell}{2} + \delta - r)}{2} \right\rfloor + (k + r)\%2. \end{aligned}$$

3. The next case is split into two possibilities.

   - First suppose that
   $$\lambda - \frac{\ell}{2} + \delta - r \le k \le \frac{\ell}{2} + \delta + r$$
   (which matches Section 7.4.1). Then

$$\begin{aligned} C(k) &= C_2(k) + C_3(k) + C_4(k) \\ &= \frac{\lambda - \ell + 1}{2} - (k + r + 1)\%2 \\ &\quad + \frac{-\frac{\ell}{2} + \delta + 1 - (k + \frac{\ell}{2} + \delta + r + 1)\%2 - (k + r)\%2}{2} \\ &\quad + \frac{-\frac{\ell}{2} + \delta + 1 - (k + \frac{\ell}{2} + \delta + r)\%2 - (k + r + 1)\%2}{2} \\ &= \frac{\lambda - 2\ell + 2\delta + 1}{2} + (k + r)\%2. \end{aligned}$$

- Instead suppose that

$$\frac{\ell}{2}+\delta+r \le k \le \lambda-\frac{\ell}{2}+\delta-r.$$

We consider four more sub-possibilities.

- If $k \le \min(2\delta+r, \lambda-r)$ we get

$$\begin{aligned} C(k) &= C_1(k)+C_4(k) \\ &= \left\lfloor \frac{k-\ell+r+1}{2} \right\rfloor + \left\lfloor \frac{2\delta+r-k+1}{2} \right\rfloor \\ &= -\frac{\ell}{2}+\delta+r+(k+r)\%2. \end{aligned}$$

- Similarly $k \ge \min(2\delta+r, \lambda-r)$ we get

$$\begin{aligned} C(k) &= C_2(k)+C_3(k) \\ &= \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor + \left\lfloor \frac{k-\lambda+r+1}{2} \right\rfloor \\ &= -\frac{\ell}{2}+\delta+r. \end{aligned}$$

- If $2\delta+r \le k \le \lambda-r$ we simply get

$$C(k) = C_1(k) = -\frac{\ell}{2}+\delta+r.$$

- Finally if $\lambda-r \le k \le 2\delta+r$ we get instead

$$\begin{aligned} C(k) &= C_1(k)+C_2(k)+C_3(k) \\ &= \frac{\lambda-\ell+1}{2}-(k+r+1)\%2+\left\lfloor \frac{k-\lambda+r+1}{2} \right\rfloor \\ &\quad + \left(\frac{\ell}{2}+\delta+r\right) - \left\lceil \frac{k+\ell+r+1}{2} \right\rceil + 1 \end{aligned}$$

$$= -\frac{\ell}{2} + \delta + r + (k+r)\%2.$$

4. Moving on, suppose that $\max(\frac{\ell}{2}+\delta+r, \lambda-\frac{\ell}{2}+\delta-r) \leq k \leq \lambda - \frac{\ell}{2}+\delta+r$. Applying Lemma 7.4.1 we get that

$$\begin{aligned} C(k) &= \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor + C_3(k) + \mathbf{1}_{k\leq 2\delta+r}(k+r)\%2 \\ &= \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor + \frac{-\frac{\ell}{2}+\delta+1-(k+\frac{\ell}{2}+\delta+r+1)\%2-(k+r)\%2}{2} \\ &\quad + \mathbf{1}_{k\leq 2\delta+r}(k+r)\%2 \\ &= \left(\delta - \frac{\ell}{2}\right) + \left\lfloor \frac{\left(\lambda-\frac{\ell}{2}+\delta+r\right)-k}{2} \right\rfloor + \mathbf{1}_{k\leq 2\delta+r}(k+r)\%2. \end{aligned}$$

5. Finally, if $\lambda-\frac{\ell}{2}+\delta+r \leq k \leq \lambda-\ell+2\delta+r$, use Lemma 7.4.1 to get just

$$\begin{aligned} C(k) &= \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor + C_3(k) \\ &= \left\lfloor \frac{\lambda-\ell+2\delta+r-k}{2} \right\rfloor + \left\lfloor \frac{\lambda-\ell+2\delta+r-k+1}{2} \right\rfloor + \mathbf{1}_{k\leq 2\delta+r}(k+r)\%2 \\ &= (\lambda-\ell+2\delta+r)-k+\mathbf{1}_{k\leq 2\delta+r}(k+r)\%2. \end{aligned}$$

Having exhausted all five cases, we compile them into the following:

Proposition 7.4.2 ($\mathbf{c}_\gamma$)**.** *Let us define*

$$\mathbf{c}_\gamma \coloneqq \underset{[\ell-r,\lambda-\ell+2\delta+r]}{\mathrm{ARCH}}(\delta-\ell/2, \min(\lambda-\ell-1, 2r)).$$

Then

$$q^{\frac{\ell}{2}+r} \cdot \sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-r}^{n-1-\frac{\ell}{2}+\delta} (-1)^n q^{s(2m-n)} + q^{\frac{\ell}{2}+r} \cdot \sum_{n=\lambda+r+1}^{\lambda-\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\lambda-\frac{\ell}{2}+\delta+r} (-1)^n q^{s(2m-n)}$$

$$+ q^{\frac{\ell}{2}+r} \cdot \sum_{n=\ell+r+1}^{\frac{\ell}{2}+\delta+r} \sum_{m=n-r}^{\frac{\ell}{2}+\delta+r} (-1)^n q^{s(2m-n)}$$

$$= \sum_{k=\ell-r}^{\lambda-\ell+2\delta+r} \mathbf{c}_\gamma(k)(-1)^k (q^s)^k + \sum_{k\equiv\ell-r}^{2\delta+r} (k+r)\%2 \cdot (-1)^k (q^s)^k.$$

Proof. After removing the $\mathbf{1}_{k\leq 2\delta+r} \cdot (k+r)\%2$ term, we see that we need to verify

$$\mathbf{c}_\gamma(k) = \begin{cases} k-(\ell-r) & \ell-r \leq k \leq \frac{\ell}{2}+\delta-r \\ \left(\delta-\frac{\ell}{2}\right)+\left\lfloor \frac{k-\left(\frac{\ell}{2}+\delta-r\right)}{2} \right\rfloor & \frac{\ell}{2}+\delta-r \leq k \leq \min(\delta+\frac{\ell}{2}+r, \lambda-\frac{\ell}{2}+\delta-r) \\ \frac{\lambda-2\ell+2\delta+1}{2} & \lambda-\frac{\ell}{2}+\delta-r \leq k \leq \frac{\ell}{2}+\delta+r \\ -\frac{\ell}{2}+\delta+r & \frac{\ell}{2}+\delta+r \leq k \leq \lambda-\frac{\ell}{2}+\delta-r \\ \left(\delta-\frac{\ell}{2}\right)+\left\lfloor \frac{\left(\lambda-\frac{\ell}{2}+\delta+r\right)-k}{2} \right\rfloor & \max(\frac{\ell}{2}+\delta+r, \lambda-\frac{\ell}{2}+\delta-r) \leq k \leq \lambda-\frac{\ell}{2}+\delta+r \\ (\lambda-\ell+2\delta+r)-k & \lambda-\frac{\ell}{2}+\delta+r \leq k \leq \lambda-\ell+2\delta+r. \end{cases}$$

which follows directly from the definition of ARCH that we proposed. □

7.4.2 Analysis of the remaining double sum

It remains to evaluate

$$\sum_{n=\ell+r+1}^{\lambda+r} \sum_{m=n-\frac{\ell}{2}+\delta}^{\min(n,\lambda)+\delta+r} q^{n-m+\delta+r} \cdot (-1)^n q^{s(2m-n)}.$$

As before if we define

$$j := (n+\delta+r)-m \geq 0$$

$$k := n+2\delta+2r-2j.$$

so the sum becomes

$$
\begin{aligned}
&\sum_{n=\ell+r+1}^{\lambda+r} \sum_{j=\max(0,n-\lambda)}^{\frac{\ell}{2}+r} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&= \sum_{j=\max(r-(\lambda-\ell-1),0)}^{\frac{\ell}{2}+r} \sum_{n=\ell+r+1}^{\lambda+\min(j,r)} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&= \sum_{j=\max(r-(\lambda-\ell-1),0)}^{r} \sum_{n=\ell+r+1}^{j+\lambda} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)} + \sum_{j=r+1}^{\frac{\ell}{2}+r} \sum_{n=\ell+r+1}^{\lambda+r} q^j \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&= \sum_{j=\max(r-(\lambda-\ell-1),0)}^{r} \sum_{k=\ell+2\delta+3r-2j+1}^{\lambda+2\delta+2r-j} q^j \cdot (-q^s)^k + \sum_{j=r+1}^{\frac{\ell}{2}+r} \sum_{k=\ell+2\delta+3r-2j+1}^{\lambda+2\delta+3r-2j} q^j \cdot (-q^s)^k.
\end{aligned}
$$

Now we collect the coefficient of $(-q^s)^k$. In the first double sum the value of k runs from the lowest value $k = \ell + 2\delta + r + 1$ to the highest value $\lambda + 2\delta + 2r - \max(r - (\lambda - \ell - 1), 0)$. The range of j being summed is given by solving

$$
\ell + 2\delta + 3r - 2j + 1 \leq k \leq \lambda + 2\delta + 2r - j \iff \frac{\ell + 2\delta + 3r + 1 - k}{2} \leq j \leq \lambda + 2\delta + 2r - k
$$

to get

$$
\max\left(\left\lceil \frac{\ell + 2\delta + 3r + 1 - k}{2} \right\rceil, 0, r - (\lambda - \ell - 1)\right) \leq j \leq \max\left(\lambda + 2\delta + 2r - k, r\right).
$$

In the second double sum the value of k runs from the lowest value $k = 2\delta + r + 1$ to the highest value $k = \lambda + 2\delta + r - 2$ and j now needs to satisfy

$$
\ell + 2\delta + 3r - 2j + 1 \leq k \leq \lambda + 2\delta + 3r - 2j \iff \frac{\ell + 2\delta + 3r + 1 - k}{2} \leq j \leq \frac{\lambda + 2\delta + 3r - k}{2}
$$

so we get the range to this time be

$$
\max\left(\left\lceil \frac{\ell + 2\delta + 3r + 1 - k}{2} \right\rceil, r + 1\right) \leq j \leq \min\left(\left\lfloor \frac{\lambda + 2\delta + 3r - k}{2} \right\rfloor, \frac{\ell}{2} + r\right).
$$

In this case the max resolves to the ceiling because $\frac{\lambda+2\delta+3r+1-k}{2} \geq \frac{\lambda+2\delta+3r+1-(\lambda+2\delta+r-2)}{2} \geq r+1$ for all k in the desired range, but the min does not resolve immediately. For convenience, we add on two extra values $k = \lambda + 2\delta + r - 1$ and $k = \lambda + 2r + r$ for which the range above for j is void anyway, to make it easier to merge the sums momentarily.

In summary, our rewritten sum equals

$$\sum_{k=\ell+2\delta+r+1}^{\lambda+2\delta+2r-\max(r-(\lambda-\ell-1),0)} \sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,0,r-(\lambda-\ell-1)\right)}^{\min(\lambda+2\delta+2r-k,r)} q^j(-q^s)^k$$
$$+ \sum_{k=2\delta+r+1}^{\lambda+2\delta+r} \sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,r+1\right)}^{\min\left(\left\lfloor\frac{\lambda+2\delta+3r-k}{2}\right\rfloor,\frac{\ell}{2}+r\right)} q^j(-q^s)^k.$$

Notice we always have

$$2\delta + r + 1 \leq \ell + 2\delta + r + 1 \leq \lambda + 2\delta + r \leq \lambda + 2\delta + 2r - \max\left(r - (\lambda - \ell - 1), 0\right)$$

so the values of k in the second double sum are contained inside those of the first. Moreover, the two breaking points are

$$\begin{aligned} \lambda + 2\delta + 2r - k \leq r &\iff k \geq \lambda + 2\delta + r \\ \left\lceil\frac{\ell + 2\delta + 3r + 1 - k}{2}\right\rceil \geq r + 1 &\iff k \leq \ell + 2\delta + r - 1 \end{aligned}$$

which miraculously line up with the boundaries of the sum. Hence for these overlapped values the two sums over j fit together neatly with a "seam" at the value $j = r$. Then rewriting the sum with ascending values of k we have

$$\sum_{k=2\delta+r+1}^{\ell+2\delta+r} \sum_{j=\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil}^{\frac{\ell}{2}+r} q^j(-q^s)^k$$

$$+\sum_{k=\ell+2\delta+r+1}^{\lambda+2\delta+r}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,0,r-(\lambda-\ell-1)\right)}^{\frac{\ell}{2}+r} q^j(-q^s)^k$$
$$+\sum_{k=\lambda+2\delta+r+1}^{\lambda+2\delta+2r-\max(r-(\lambda-\ell-1),0)}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,0,r-(\lambda-\ell-1)\right)}^{\lambda+2\delta+2r-k} q^j(-q^s)^k.$$

where we have used $\frac{\lambda+2\delta+3r-k}{2}\geq\frac{\ell}{2}+r \iff \lambda+2\delta+r+\ell\geq k$ to resolve the minimums in the first two sums.

To proceed further we split this into cases based on whether $\lambda>\ell+r$ or not.

- If $\lambda>\ell+r$, then we have

$$\begin{aligned}&\sum_{k=2\delta+r+1}^{\ell+2\delta+r}\sum_{j=\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil}^{\frac{\ell}{2}+r} q^j(-q^s)^k\\&+\sum_{k=\ell+2\delta+r+1}^{\lambda+2\delta+r}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,0\right)}^{\frac{\ell}{2}+r} q^j(-q^s)^k\\&+\sum_{k=\lambda+2\delta+r+1}^{\lambda+2\delta+2r}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,0\right)}^{\lambda+2\delta+2r-k} q^j(-q^s)^k.\end{aligned}\tag{7.4}$$

- If $\lambda\leq\ell+r$, then we instead have

$$\begin{aligned}&\sum_{k=2\delta+r+1}^{\ell+2\delta+r}\sum_{j=\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil}^{\frac{\ell}{2}+r} q^j(-q^s)^k\\&+\sum_{k=\ell+2\delta+r+1}^{\lambda+2\delta+r}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,r-(\lambda-\ell-1)\right)}^{\frac{\ell}{2}+r} q^j(-q^s)^k\\&+\sum_{k=\lambda+2\delta+r+1}^{2\lambda-\ell+2\delta+r-1}\sum_{j=\max\left(\left\lceil\frac{\ell+2\delta+3r+1-k}{2}\right\rceil,r-(\lambda-\ell-1)\right)}^{\lambda+2\delta+2r-k} q^j(-q^s)^k.\end{aligned}\tag{7.5}$$

7.5 Proof of Theorem 5.5.7

We now put the finishing touches to deduce Theorem 5.5.7.

Theorem 5.5.7 (Weighted orbital integral for even $\ell \geq 0$). *Let $r \geq 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let b, d, δ, ℓ, λ be as in Lemma 5.3.3 and Lemma 5.4.1. Suppose also $\ell \geq 0$ is even. Define*

$$\mathbf{n}_\gamma := \operatorname*{ARCH}_{[-2r,\lambda+2\delta+2r]}(r, \ell)$$
$$\mathbf{c}_\gamma := \operatorname*{ARCH}_{[\ell-r,\lambda-\ell+2\delta+r]}(\delta - \ell/2, \min(2r, \lambda - \ell)).$$

Then for any $r \geq 0$ we have:

$$\begin{aligned}\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = &\sum_{k=-2r}^{\lambda+2\delta+2r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right) (q^s)^k \\ &+ \sum_{k=\ell-r}^{2\delta+\lambda-\ell+r} \mathbf{c}_\gamma(k)(-1)^k q^{\frac{\ell}{2}+r}(q^s)^k.\end{aligned}$$

Proof of Theorem 5.5.7. The coefficient $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ was already introduced in Proposition 7.4.2, where there is an extra $(k+r)\%2$ term for $\ell - r \leq k \leq 2\delta + r$ that matches the corresponding term in Proposition 7.3.1.

Then, when $\lambda > \ell + r$ the terms in Proposition 7.3.1 fit together with the sum in (7.4) to give $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$. The same is true with $\lambda \leq \ell + r$ when one uses (7.5) instead. This completes the proof. □

Remark 7.5.1 (Expanding $\mathbf{n}_\gamma$)**.** One can expand the Arch notation for $\mathbf{n}_\gamma$ to obtain

$$\mathbf{n}_\gamma(k) = \begin{cases} k + 2r & \text{if } -2r \le k \le -r \\ \left\lfloor \frac{k+r}{2} \right\rfloor + r & \text{if } -r \le k \le \ell - r \\ \frac{\ell}{2} + r & \text{if } \ell - r \le k \le \lambda - \ell + 2\delta + r \\ \left\lfloor \frac{(2\delta+\lambda+r)-k}{2} \right\rfloor + r & \text{if } \lambda - \ell + 2\delta + r \le k \le \lambda + 2\delta + r \\ (\lambda + 2\delta + 2r) - k & \text{if } \lambda + 2\delta + r \le k \le \lambda + 2\delta + 2r. \end{cases}$$

Then $\mathbf{c}_\gamma$ can be similarly expanded, but the result is so notationally dense that it is hardly worth including. If one defines the shorthands

$$\begin{aligned} \mathsf{B}_\gamma &:= \frac{\ell}{2} + \delta - r \\ \mathsf{T}_\gamma &:= \lambda - \frac{\ell}{2} + \delta + r \\ \mathsf{w}_\gamma &:= \min(\lambda - \ell - 1, 2r) \end{aligned}$$

then it could be written out more fully as

$$\mathbf{c}_\gamma(k) = \begin{cases} k - (\ell - r) & \text{if } \ell - r \le k \le \mathsf{B}_\gamma \\ \left\lfloor \frac{k - \mathsf{B}_\gamma}{2} \right\rfloor - \frac{\ell}{2} + \delta & \text{if } \mathsf{B}_\gamma \le k \le \mathsf{B}_\gamma + \mathsf{w}_\gamma \\ -\frac{\ell}{2} + \delta + \frac{1}{2}\mathsf{w}_\gamma & \text{if } \mathsf{B}_\gamma + \mathsf{w}_\gamma \le k \le \mathsf{T}_\gamma - \mathsf{w}_\gamma \\ \left\lfloor \frac{\mathsf{T}_\gamma - k}{2} \right\rfloor - \frac{\ell}{2} + \delta & \text{if } \mathsf{T}_\gamma - \mathsf{w}_\gamma \le k \le \mathsf{T}_\gamma \\ (\lambda - \ell + 2\delta + r) - k & \text{if } \mathsf{T}_\gamma \le k \le 2\delta + \lambda - \ell + r. \end{cases}$$

7.6 Contribution of cases for the $v(b) = v(d) < 0$ case (and proof of Theorem 5.5.10)

We now deal with the edge case $v(b) = v(d) < 0$. The region $I_{n\leq 0}$ remains the same, but we need to alter the calculation of the other parts. We assume $|v(d)| \leq r$ henceforth since otherwise the entire orbital integral vanishes.

7.6.1 Contribution of Case 1 and Case 2

This situation only occurs if $r > 2|v(d)|$. In that case we get

$$\begin{aligned}
I^{1+2}_{n>0} &= \kappa \sum_{n=1}^{-2|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^{-n}\left(1-q^{-2}\right) \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&\quad + \kappa \sum_{n=1}^{-2|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n-2|v(d)|+r,\lambda-|v(d)|+r)} q^{-n-(m+2|v(d)|-r)}\left(1-q^{-1}\right) \\
&\qquad \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&= \sum_{n=1}^{-2|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^{n}\left(1+q^{-1}\right)(-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=1}^{-2|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)} \cdot (-1)^n q^{s(2m-n)}.
\end{aligned}$$

7.6.2 Contribution of Case 3^+ and Case 4^+

We split the sum into several cases.

$$\begin{aligned}
I^{3+4}_{n>0} &= \kappa \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^{-n}\left(1-q^{-2}\right) \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&\quad + \kappa \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n,\lambda)-2|v(d)|+r} q^{-n-(m+2|v(d)|-r)}\left(1-q^{-1}\right)
\end{aligned}$$

$$
\cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big)
$$

$$
+ \kappa \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-r}^{n-|v(d)|} q^{-2n+r-|v(d)|}\left(1-q^{-1}\right) \cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big)
$$

$$
+ \kappa \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-|v(d)|+1}^{\min(n,\lambda)-2|v(d)|+r} q^{-n-(m+2|v(d)|-r)}\left(1-q^{-1}\right)
$$

$$
\cdot \Big((-1)^n q^{s(2m-n)} q^{2n-2m}\Big)\Big(q^{2m}(1-q^{-2})\Big)
$$

$$
= \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(2m-n)}
$$

$$
+ \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)}(-1)^n q^{s(2m-n)}
$$

$$
+ \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-r}^{n-|v(d)|} q^{r-|v(d)|}(-1)^n q^{s(2m-n)}
$$

$$
+ \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-|v(d)|+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)}(-1)^n q^{s(2m-n)}.
$$

7.6.3 Merging the contributions

When we put together these sums, the first two double sums fold together and we get just the following five double sums:

$$
\begin{aligned}
I^{1+2}_{n>0} + I^{3+4}_{n>0} = &\sum_{n=1}^{-|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^{n}\left(1+q^{-1}\right)(-1)^n q^{s(2m-n)} \\
&+ \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-r}^{n-|v(d)|} q^{r-|v(d)|}(-1)^n q^{s(2m-n)} \\
&+ \sum_{n=1}^{-2|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)} \cdot (-1)^n q^{s(2m-n)} \\
&+ \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{m=-2|v(d)|+r+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)}(-1)^n q^{s(2m-n)}
\end{aligned}
$$

$$+\sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r}\ \sum_{m=n-|v(d)|+1}^{\min(n,\lambda)-2|v(d)|+r} q^{n-(m+2|v(d)|-r)}(-1)^n q^{s(2m-n)}.$$

The first double sum, combined with $I_{n\leq 0}$

We start with

$$\sum_{n=1}^{-|v(d)|+r}\ \sum_{m=n-r}^{-2|v(d)|+r} q^n\left(1+q^{-1}\right)(-1)^n q^{s(2m-n)}$$

We'd like to write an analogous sum over j. For $1<j<-|v(d)|+r$ we get a contribution of $q^j(-q^s)^k$ for $k=2m-n$ if and only if for $n=j+(k-n)\%2=j+(k-j)\%2$ we have

$$\begin{aligned} n-r\leq m\leq -2|v(d)|+r &\iff 2n-2r\leq k+n\leq -4|v(d)|+2r\\ &\iff j+(k-j)\%2-2r\leq k\leq -j-(k-j)\%2-4|v(d)|+2r\\ &\iff j-2r\leq k\leq -j-4|v(d)|+2r. \end{aligned}$$

When $j=0$, the contribution is analogous except we must have $n=1$ so that k must be odd; and when $j=-|v(d)|+r$, the contribution is analogous except we must have $n=-|v(d)|+r$ so that k must have the same parity as $-|v(d)|+r$. In other words, the double sum can be written as

So this double sum can be rewritten as

$$\sum_{j=1}^{-|v(d)|+r-1}\ \sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j(-q^s)^k+\sum_{\substack{-2r\leq k\leq -4|v(d)|+2r\\ k\equiv 1 \pmod 2}}(-q^s)^k+\sum_{\substack{-|v(d)|-r\leq k\leq -3|v(d)|+3r\\ k\equiv -|v(d)|+r \bmod 2}} q^{-|v(d)|+r}(-q^s)^k.$$

However, we also have

$$I_{n\leq 0}=\sum_{\substack{-2r\leq k\leq -4|v(d)|+2r\\ k\equiv 0 \pmod 2}}(-q^s)^k.$$

Hence once we incorporate $I_{n\leq 0}$ in we have

$$\begin{aligned}
I_{n\leq 0} + &\sum_{n=1}^{-|v(d)|+r} \sum_{m=n-r}^{-2|v(d)|+r} q^n \left(1+q^{-1}\right)(-1)^n q^{s(2m-n)} \\
= &\sum_{j=0}^{-|v(d)|+r-1} \sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j (-q^s)^k \\
+ &\sum_{\substack{-|v(d)|-r\leq k\leq -3|v(d)|+r \\ k\equiv -|v(d)|+r \bmod 2}} q^{-|v(d)|+r}(-q^s)^k.
\end{aligned}$$

The second double sum

We turn to the second double sum

$$\sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{m=n-r}^{n-|v(d)|} (-1)^n q^{s(2m-n)}.$$

Working with $k = 2m - n$, the values of k run from the lowest value $k = -|v(d)| - r + 1$ to the highest value $k = \lambda - 3|v(d)| + r$. The coefficient of k is the number of integers m such that

$$n-r \leq m \leq n-|v(d)| \iff 2m-k-r \leq m \leq 2m-k-|v(d)| \iff k+|v(d)| \leq m \leq k+r$$

and

$$-|v(d)|+r+1 \leq 2m-k \leq \lambda-|v(d)|+r \iff \frac{k-|v(d)|+r+1}{2} \leq m \leq \frac{k+\lambda-|v(d)|+r}{2}.$$

In other words, the double sum in question equals

$$\begin{aligned}
\sum_{k=-|v(d)|-r+1}^{\lambda-3|v(d)|+r} &\left(\min\left(\left\lfloor \frac{k+\lambda-|v(d)|+r}{2}\right\rfloor, k+r\right)\right. \\
&\left. - \max\left(\left\lceil \frac{k-|v(d)|+r+1}{2}\right\rceil, k+|v(d)|\right)+1\right)(-q^s)^k.
\end{aligned}$$

The third, fourth, and fifth double sum

Now we evaluate the final three double sums. Replacing $j = n - (m + 2|v(d)| - r)$ gives $2m - n = 2(n - (2|v(d)| - r) - j) - n = n - 4|v(d)| + 2r - 2j$ and the last three double sums transform into

$$\sum_{n=1}^{-2|v(d)|+r} \sum_{j=\max(0,n-\lambda)}^{n-1} q^j \cdot (-1)^n q^{s(n-4|v(d)|+2r-2j)}$$
$$+ \sum_{n=\max(1,-2|v(d)|+r+1)}^{-|v(d)|+r} \sum_{j=\max(0,n-\lambda)}^{n-1} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)}$$
$$+ \sum_{n=-|v(d)|+r+1}^{\lambda-|v(d)|+r} \sum_{j=\max(0,n-\lambda)}^{-|v(d)|+r-1} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)}.$$

The range of n across the three double sums is disjoint and runs from $n = 1$ up to $n = \lambda - |v(d)| + r$.

We now interchange the order of summation so that j is on the outside to get and then split the second double sum and re-collate:

$$\begin{aligned}
&\sum_{j=0}^{-2|v(d)|+r-1} \sum_{n=j+1}^{-2|v(d)|+r} q^j \cdot (-1)^n q^{s(n-4|v(d)|+2r-2j)} \\
&+ \sum_{j=0}^{-|v(d)|+r-1} \sum_{n=\max(j+1,-2|v(d)|+r+1)}^{\min(j+\lambda,-|v(d)|+r)} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)} \\
&+ \sum_{j=\max(0,-\lambda-|v(d)|+r+1)}^{-|v(d)|+r-1} \sum_{n=-|v(d)|+r+1}^{j+\lambda} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)} \\
&= \sum_{j=0}^{-2|v(d)|+r-1} \sum_{n=j+1}^{-2|v(d)|+r} q^j \cdot (-1)^n q^{s(n-4|v(d)|+2r-2j)} \\
&+ \sum_{j=0}^{-\lambda-|v(d)|+r} \sum_{n=\max(j+1,-2|v(d)|+r+1)}^{j+\lambda} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)} \\
&+ \sum_{j=\max(0,-\lambda-|v(d)|+r+1)}^{-|v(d)|+r-1} \sum_{n=\max(j+1,-2|v(d)|+r+1)}^{-|v(d)|+r} q^j (-1)^n q^{s(n-4|v(d)|+2r-2j)}
\end{aligned}$$

$$+\sum_{j=\max(0,-\lambda-|v(d)|+r+1)}^{-|v(d)|+r-1}\sum_{n=-|v(d)|+r+1}^{j+\lambda} q^j(-1)^n q^{s(n-4|v(d)|+2r-2j)}$$

$$=\sum_{j=0}^{-2|v(d)|+r-1}\sum_{n=j+1}^{-2|v(d)|+r} q^j\cdot(-1)^n q^{s(n-4|v(d)|+2r-2j)}$$

$$+\sum_{j=0}^{-\lambda-|v(d)|+r}\sum_{n=\max(j+1,-2|v(d)|+r+1)}^{j+\lambda} q^j(-1)^n q^{s(n-4|v(d)|+2r-2j)}$$

$$+\sum_{j=\max(0,-\lambda-|v(d)|+r+1)}^{-|v(d)|+r-1}\sum_{n=\max(j+1,-2|v(d)|+r+1)}^{j+\lambda} q^j(-1)^n q^{s(n-4|v(d)|+2r-2j)}$$

$$=\sum_{j=0}^{-2|v(d)|+r-1}\sum_{n=j+1}^{-2|v(d)|+r} q^j\cdot(-1)^n q^{s(n-4|v(d)|+2r-2j)}$$

$$+\sum_{j=0}^{-|v(d)|+r-1}\sum_{n=\max(j+1,-2|v(d)|+r+1)}^{j+\lambda} q^j(-1)^n q^{s(n-4|v(d)|+2r-2j)}.$$

Let $k=n-2j-4|v(d)|+2r$ and transform this into

$$\sum_{j=0}^{-2|v(d)|+r-1}\sum_{k=-j-4|v(d)|+2r+1}^{-2j-6|v(d)|+3r} q^j\cdot(-q^s)^k$$
$$+\sum_{j=0}^{-|v(d)|+r-1}\sum_{k=\max(j+1,-2|v(d)|+r+1)-2j-4|v(d)|+2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k.$$

7.6.4 Collation

Altogether, we have arrived at the following formula

$$\mathrm{Orb}(\gamma,\mathbf{1}_{K'_{S,\le r}},s)=\sum_{j=0}^{-|v(d)|+r-1}\sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j(-q^s)^k+q^{-|v(d)|+r}\sum_{\substack{-|v(d)|-r\le k\le -3|v(d)|+r\\ k\equiv -|v(d)|+r \bmod 2}}(-q^s)^k$$
$$+q^{-|v(d)|+r}\sum_{k=-|v(d)|-r+1}^{\lambda-3|v(d)|+r}\Bigg(\min\left(\left\lfloor\frac{k+\lambda-|v(d)|+r}{2}\right\rfloor,k+r\right)$$
$$-\max\left(\left\lceil\frac{k-|v(d)|+r+1}{2}\right\rceil,k+|v(d)|\right)+1\Bigg)(-q^s)^k$$

$$\sum_{j=0}^{-2|v(d)|+r-1} \sum_{k=-j-4|v(d)|+2r+1}^{-2j-6|v(d)|+3r} q^j \cdot (-q^s)^k$$
$$+ \sum_{j=0}^{-|v(d)|+r-1} \sum_{k=\max(j+1,-2|v(d)|+r+1)-2j-4|v(d)|+2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k.$$

We collapse the sums

$$q^{-|v(d)|+r} \sum_{\substack{-|v(d)|-r\leq k\leq -3|v(d)|+r \\ k\equiv -|v(d)|+r \bmod 2}} (-q^s)^k$$
$$+ q^{-|v(d)|+r} \sum_{k=-|v(d)|-r+1}^{\lambda-3|v(d)|+r} \left(\min\left(\left\lfloor \frac{k+\lambda-|v(d)|+r}{2} \right\rfloor, k+r \right) \right.$$
$$\left. - \max\left(\left\lceil \frac{k-|v(d)|+r+1}{2} \right\rceil, k+|v(d)| \right) + 1 \right) (-q^s)^k$$

by noting that the coefficient of the second sum can be written as

$$\begin{cases} \left\lfloor \frac{k+\lambda-|v(d)|+r}{2} \right\rfloor - \left\lceil \frac{k-|v(d)|+r+1}{2} \right\rceil + 1 & \text{if } \lambda-|v(d)|-r \leq k \leq -3|v(d)|+r \\ r-|v(d)|+1 & \text{if } -3|v(d)|+r \leq k \leq \lambda-|v(d)| \\ \left\lfloor \frac{k+\lambda-|v(d)|+r}{2} \right\rfloor - (k+|v(d)|)+1 & \text{if } k \geq \max(-3|v(d)|+r, \lambda-|v(d)|-r) \\ (k+r) - \left\lceil \frac{k-|v(d)|+r+1}{2} \right\rceil + 1 & \text{if } k \leq \min(-3|v(d)|+r, \lambda-|v(d)|-r) \end{cases}$$
$$= \begin{cases} \frac{\lambda-1}{2} + (k+|v(d)|+r)\%2 & \text{if } \lambda-|v(d)|-r \leq k \leq -3|v(d)|+r \\ r-|v(d)|+1 & \text{if } -3|v(d)|+r \leq k \leq \lambda-|v(d)| \\ \left\lfloor \frac{\lambda-3|v(d)|+r-k}{2} \right\rfloor + 1 & \text{if } k \geq \max(-3|v(d)|+r, \lambda-|v(d)|-r) \\ \left\lfloor \frac{k+|v(d)|+r}{2} \right\rfloor + (k+|v(d)|+r)\%2 & \text{if } k \leq \min(-3|v(d)|+r, \lambda-|v(d)|-r) \end{cases}$$

Thus when we add $q^{-|v(d)|+r}\sum_{\substack{-|v(d)|-r\le k\le -3|v(d)|+r \\ k\equiv -|v(d)|+r \bmod 2}}(-q^s)^k$ back in we conveniently arrive at a coefficient of

$$1+\begin{cases} \frac{\lambda-1}{2} & \text{if } \lambda-|v(d)|-r\le k\le -3|v(d)|+r \\ r-|v(d)| & \text{if } -3|v(d)|+r\le k\le \lambda-|v(d)| \\ \left\lfloor \frac{\lambda-3|v(d)|+r-k}{2}\right\rfloor & \text{if } k\ge \max(-3|v(d)|+r,\lambda-|v(d)|-r) \\ \left\lfloor \frac{k+|v(d)|+r}{2}\right\rfloor & \text{if } k\le \min(-3|v(d)|+r,\lambda-|v(d)|-r). \end{cases}$$

If we now define

$$\mathbf{c}_\gamma(k) \coloneqq \underset{[-|v(d)|-r,\lambda-3|v(d)|+r]}{\mathrm{ARCH}}(0,\min(2r-2|v(d)|,\lambda))$$

then the coefficient can be written instead as $1+\mathbf{c}_\gamma(k)$.

Similarly, we unify the three remaining double sums by repeatedly splitting into five double sums and re-collating everything. The entire expression delightfully collapses into a single double sum as follows:

$$\begin{aligned} &\sum_{j=0}^{-|v(d)|+r-1}\sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j(-q^s)^k \\ &+\sum_{j=0}^{-2|v(d)|+r-1}\sum_{k=-j-4|v(d)|+2r+1}^{-2j-6|v(d)|+3r} q^j\cdot(-q^s)^k \\ &+\sum_{j=0}^{-|v(d)|+r-1}\sum_{k=\max(j+1,-2|v(d)|+r+1)-2j-4|v(d)|+2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k \\ &=\sum_{j=0}^{-2|v(d)|+r-1}\sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j(-q^s)^k \\ &+\sum_{j=\max(-2|v(d)|+r,0)}^{-|v(d)|+r-1}\sum_{k=j-2r}^{-j-4|v(d)|+2r} q^j(-q^s)^k \\ &+\sum_{j=0}^{-2|v(d)|+r-1}\sum_{k=-j-4|v(d)|+2r+1}^{-2j-6|v(d)|+3r} q^j\cdot(-q^s)^k \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=0}^{-2|v(d)|+r-1} \sum_{k=-2j-6|v(d)|+3r+1}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k \\
&+ \sum_{j=\max(-2|v(d)|+r,0)}^{-|v(d)|+r-1} \sum_{k=-j-4|v(d)|+2r+1}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k \\
&= \sum_{j=0}^{-2|v(d)|+r-1} \sum_{k=j-2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k + \sum_{j=\max(-2|v(d)|+r,0)}^{-|v(d)|+r-1} \sum_{k=j-2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k \\
&= \sum_{j=0}^{-|v(d)|+r-1} \sum_{k=j-2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k.
\end{aligned}
$$

Thus, at this point we now have

$$
\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\le r}}, s) &= \sum_{k=-|v(d)|-r}^{\lambda-3|v(d)|+r} q^{-|v(d)|+r}(1+\mathbf{c}_\gamma(k))(-q^s)^k \\
&\quad + \sum_{j=0}^{-|v(d)|+r-1} \sum_{k=j-2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k \\
&= \sum_{k=-|v(d)|-r}^{\lambda-3|v(d)|+r} q^{-|v(d)|+r}\mathbf{c}_\gamma(k)(-q^s)^k \\
&\quad + \sum_{j=0}^{-|v(d)|+r} \sum_{k=j-2r}^{-j+\lambda-4|v(d)|+2r} q^j(-q^s)^k.
\end{aligned}
$$

Interchange the order of the latter sum to be k first:

$$
\begin{aligned}
\mathrm{Orb}(\gamma, \mathbf{1}_{K'_{S,\le r}}, s) &= \sum_{k=-|v(d)|-r}^{\lambda-3|v(d)|+r} q^{-|v(d)|+r}\mathbf{c}_\gamma(k)(-q^s)^k \\
&\quad + \sum_{k=-2r}^{\lambda-4|v(d)|+2r} \sum_{j=0}^{\min(k+2r,-|v(d)|+r,\lambda-4|v(d)|+2r-k)} q^j(-q^s)^k.
\end{aligned}
$$

Putting this all together gives Theorem 5.5.10:

Theorem 5.5.10 (Weighted orbital integral when $v(b) = v(d) < 0$)**.** *Let $r \ge 0$. Let $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let b, d, λ be as in Lemma 5.3.3 and Lemma 5.4.1. Suppose also $v(b) = v(d) < 0$.*

Then if $|v(d)| > r$, the entire orbital integral is zero. Otherwise define

$$\mathbf{n}_\gamma := \operatorname*{ARCH}_{[-2r, \lambda+2r-4|v(d)|]}(r - |v(d)|, 0)$$

$$\mathbf{c}_\gamma := \operatorname*{ARCH}_{[-r-|v(d)|, \lambda+r-3|v(d)|]}(0, \min(2r - 2|v(d)|, \lambda)).$$

Then for any $r \geq 0$ we have the formula:

$$\begin{aligned} \operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\leq r}}, s) = &\sum_{k=-2r}^{\lambda+2r-4|v(d)|} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_\gamma(k)}\right) (q^s)^k \\ &+ \sum_{k=-r-|v(d)|}^{\lambda+r-3|v(d)|} \mathbf{c}_\gamma(k)(-1)^k q^{r-|v(d)|}(q^s)^k. \end{aligned}$$

7.7 Derivatives of the orbital integral on $S_3(F)$

We now differentiate each of the three earlier theorems. Because of the similarity of the shapes of the orbitals, we prove the general results Lemma 7.7.1 and Lemma 7.7.3 and specialize them to the cases we need.

7.7.1 Analysis of the $\mathbf{n}_\gamma$ sum

Lemma 7.7.1 (Derivative of $\mathbf{n}_\gamma$ sum)**.** *Let C, W and H be integers with $W > 4H \geq 0$ and W odd. Consider sums of the form*

$$\Sigma_r(s) := \sum_{k=C-2r}^{C+2r+W} \left(1 + q + \cdots + q^{\operatorname{ARCH}_{[C-2r, C+2r+W]}(r,2H)(k)}\right) (-q^s)^k.$$

Then for any $r \geq 0$ we have

$$-\frac{1}{\log q} \frac{\partial}{\partial s} \Sigma_r(s) \bigg|_{s=0} = (-1)^{r+C} \sum_{j=r+1}^{r+H} \left(\frac{W+1}{2} + r - 2(j-r)\right) \cdot q^j$$

$$+\sum_{j=0}^{r}(-1)^{j+C}\left(\frac{W+1}{2}+2r-j\right)\cdot q^j.$$

Proof. Since $\Sigma_r(0)=0$, the effect of C is just multiplication by $(-1)^C$ and hence we assume without loss of generality that $C=0$. For each $0\le j\le r+H$ we consider the range of k such that $j\le \mathrm{ARCH}_{[-2r,2r+W]}(r,2H)(k)$. The derivative at $s=0$ will give the coefficient for q^j.

- For $0\le j\le r$, we have a contiguous range $-2r+j\le k\le 2r+W-j$. When we take the derivative of $\sum_k(-q^s)^k$ at $s=0$ we get we get $\log q\cdot\sum_k(-1)^k$ across this range. Since consecutive elements differ by 1, and W is odd, we get $(-1)^j$ times half the number of elements in the range, which is $\frac{(2r+W-j)-(-2r+j)+1}{2}=\frac{W+1}{2}+2r-j$.
- For $r+1\le j\le r+H$, we have a contiguous range $-2r+r+2(j-r)\le k\le 2r+W-(r+2(j-r))$ instead, and the same proof gives the coefficient of q^j. ☐

7.7.2 Analysis of the $\mathbf{c}_\gamma$ sum

We will use the following extremely easy lemma:

Lemma 7.7.2 (Extremely easy)**.** *If $a_0\le a_1$ are integers then*

$$\sum_{k=a_0}^{a_1}(k-a_0)\cdot k\cdot(-1)^k=(-1)^{a_1}\cdot\frac{a_1(a_1-a_0+1)}{2}-\frac{(-1)^{a_0}+(-1)^{a_1}}{4}\cdot a_0$$
$$\sum_{k=a_0}^{a_1}(a_1-k)\cdot k\cdot(-1)^k=(-1)^{a_0}\cdot\frac{a_0(a_1-a_0+1)}{2}-\frac{(-1)^{a_0}+(-1)^{a_1}}{4}\cdot a_1.$$

Proof. This follows trivially by induction on a_1. (Alternatively, use a symbolic engine like WolframAlpha; see here for the first sum and here for the second sum.) ☐

Lemma 7.7.3 (Derivative of $\mathbf{c}_\gamma$ sum)**.** *Let C, W, L be integers with $L\ge 1$ odd and $W\ge 0$.*

Consider sums of the form

$$\Sigma_r(s) \coloneqq \sum_{k=C-r}^{C+2W+L+r} \underset{[C-r,C+2W+L+r]}{\mathrm{ARCH}}(W, \min(2r, L))(k) \cdot (-q^s)^k.$$

Then for any $r \geq 0$ we have

$$\frac{(-1)^{r+W+C}}{\log q} \frac{\partial}{\partial s}\Sigma_r(s)\bigg|_{s=0} = \begin{cases} \frac{W}{2} - \frac{L-1}{2} \cdot r & \text{if } W \equiv 0 \pmod 2 \\ -\frac{W+L}{2} - \frac{L+1}{2} \cdot r & \text{if } W \equiv 1 \pmod 2. \end{cases}$$

Proof. Since $\Sigma_r(0) = 0$, the effect of C is just multiplication by $(-1)^C$ and hence we assume without loss of generality that $C = 0$. Also, note that change L to $L-1$ makes no difference since L is odd (in general the $\mathrm{ARCH}(w_0, w_1)$ only depends on $\lfloor w_1/2 \rfloor$). With this assumption, we could write explicitly

$$\Sigma_r(s) \coloneqq \sum_{k=-r}^{2W+L+r} \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, \min(2r, L-1))(k) \cdot (-q^s)^k.$$

Then for any $r \geq 0$ we have

$$\frac{1}{\log q} \frac{\partial}{\partial s}\Sigma_r(s)\bigg|_{s=0} = \sum_{k=-r}^{2W+L+r} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, \min(2r, L-1))(k).$$

Let $H \coloneqq \min(2r, L-1)$ for brevity, which is even. We split the sum now into five parts:

$$\begin{aligned} \frac{1}{\log q} \frac{\partial}{\partial s}\Sigma_r(s)\bigg|_{s=0} &= \sum_{k=-r}^{W-r-1} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, H)(k) \\ &\quad + \sum_{k=W-r}^{W-r+H-1} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, H)(k) \\ &\quad + \sum_{k=W-r+H}^{W+L+r-H} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, H)(k) \\ &\quad + \sum_{k=W+L+r-H+1}^{W+L+r} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W, H)(k) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=W+L+r+1}^{2W+L+r} (-1)^k \cdot k \cdot \underset{[-r,2W+L+r]}{\mathrm{ARCH}}(W,H)(k) \\
&= \sum_{k=-r}^{W-r-1} (-1)^k \cdot k \cdot (k+r) \\
&\quad + \sum_{k=W-r}^{W-r+H-1} (-1)^k \cdot k \cdot \left(W + \left\lfloor \frac{k-(W-r)}{2} \right\rfloor \right) \\
&\quad + \sum_{k=W-r+H}^{W+L+r-H} (-1)^k \cdot k \cdot \left(W + \frac{H}{2} \right) \\
&\quad + \sum_{k=W+L+r-H+1}^{W+L+r} (-1)^k \cdot k \cdot \left(W + \left\lfloor \frac{W+L+r-k}{2} \right\rfloor \right) \\
&\quad + \sum_{k=W+L+r+1}^{2W+L+r} (-1)^k \cdot k \cdot (2W+L+r-k).
\end{aligned}
$$

We can take out the contribution of r in the second, third, and fourth sum and write

$$
\sum_{k=W-r}^{W+L+r} (-1)^k \cdot k \cdot W = (-1)^{W-r+1} W \cdot \frac{L+2r+1}{2} \tag{7.6}
$$

The remaining part of the third sum is

$$
\sum_{k=W-r+H}^{W+L+r-H} (-1)^k \cdot k \cdot \frac{H}{2} = (-1)^{W-r+1} \frac{L+2r-2H+1}{2} \cdot \frac{H}{2} \tag{7.7}
$$

By Lemma 7.7.2, the first sum is

$$
\sum_{k=-r}^{W-r-1} (-1)^k \cdot k \cdot (k+r) = (-1)^{W-r-1} \cdot \frac{W(W-r-1)}{2} - \frac{(-1)^r + (-1)^{W+r-1}}{4} \cdot (-r)
$$

and the fifth sum is

$$
\begin{aligned}
\sum_{k=W+L+r+1}^{2W+L+r} (-1)^k \cdot k \cdot (2W+L+r-k) = (-1)^{W+L+r+1} \cdot \frac{W(W+L+r+1)}{2} \\
- \frac{(-1)^{r+1} + (-1)^{W+r}}{4} \cdot (2W+L+r).
\end{aligned}
$$

This sum equals

$$\begin{aligned}&\sum_{k=-r}^{W-r-1}(-1)^k\cdot k\cdot(k+r)+\sum_{k=W+L+r+1}^{2W+L+r}(-1)^k\cdot k\cdot(2W+L+r-k)\\&=(-1)^{W+r}\frac{W(L+2r+2)}{2}-(-1)^{W+r}(W\%2)\cdot\frac{2W+L+2r}{2}.\end{aligned}\tag{7.8}$$

Similarly, if we set aside the floors for the moment the second and fourth sum give

$$\begin{aligned}\sum_{k=W-r}^{W-r+H-1}(-1)^k\cdot k\cdot\frac{k-(W-r)}{2}&=\frac{(-1)^{W-r-1}}{4}\cdot H\cdot(W-r+H-1)\\\sum_{k=W+L+r-H+1}^{W+L+r}(-1)^k\cdot k\cdot\frac{W+L+r-k}{2}&=\frac{(-1)^{W+r}}{4}\cdot H\cdot(W+L+r-H+1)\end{aligned}$$

by Lemma 7.7.2. Their sum is

$$\begin{aligned}\sum_{k=W-r}^{W-r+H-1}(-1)^k\cdot k\cdot\frac{k-(W-r)}{2}&+\sum_{k=W+L+r-H+1}^{W+L+r}(-1)^k\cdot k\cdot\frac{W+L+r-k}{2}\\&=\frac{(-1)^{W+r}}{4}\cdot H\cdot(L+2r+2-2H).\end{aligned}\tag{7.9}$$

We now add in the correction terms of $-1/2$ from the floor:

$$\begin{aligned}\sum_{\substack{W-r\le k\le W-r+H-1\\k\equiv W+r+1 \bmod 2}}(-1)^k\cdot k\cdot\left(-\frac{1}{2}\right)&=\frac{(-1)^{W-r}}{2}\left((W-r)\cdot\frac{H}{2}+\left(\frac{H}{2}\right)^2\right)\\\sum_{\substack{W+L+r-H+1\le k\le W+L+r\\k\equiv W+L+r-1 \bmod 2}}(-1)^k\cdot k\cdot\left(-\frac{1}{2}\right)&=\frac{(-1)^{W+L+r}}{2}\left((W+L+r)\cdot\frac{H}{2}-\left(\frac{H}{2}\right)^2\right).\end{aligned}$$

As L is odd the sum of these is equal to

$$\begin{aligned}\sum_{\substack{W-r\le k\le W-r+H-1\\k\equiv W+r+1 \bmod 2}}(-1)^k\cdot k\cdot\left(-\frac{1}{2}\right)&+\sum_{\substack{W+L+r-H+1\le k\le W+L+r\\k\equiv W+L+r-1 \bmod 2}}(-1)^k\cdot k\cdot\left(-\frac{1}{2}\right)\\&=\frac{(-1)^{W+r}}{2}\cdot\left(\frac{H^2}{2}-(L+2r)\frac{H}{2}\right).\end{aligned}\tag{7.10}$$

Hence, if we sum all of (7.6), (7.7), (7.8), (7.9), (7.10) we obtain

$$\begin{aligned}
\frac{(-1)^{W+r}}{\log q}\frac{\partial}{\partial s}\Sigma_r(s)\bigg|_{s=0} &= -W\cdot\frac{L+2r+1}{2} - \frac{L+2r-2H+1}{2}\cdot\frac{H}{2} \\
&\quad + \frac{W(L+2r+2)}{2} - (W\%2)\cdot\frac{2W+L+2r}{2} \\
&\quad + \frac{1}{4}\cdot H\cdot(L+2r+2-2H) + \frac{1}{2}\left(\frac{H^2}{2} - (L+2r)\frac{H}{2}\right) \\
&= \frac{W}{2} - (W\%2)\cdot\frac{2W+L+2r}{2} - \frac{H}{4}\left((L-1)+2r-H\right).
\end{aligned}$$

Since $H = \min(L-1, 2r)$, it follows $H\cdot((L-1)+2r-H) = 2r(L-1)$, and finally the variable H is gone. Hence

$$\frac{(-1)^{W+r}}{\log q}\frac{\partial}{\partial s}\Sigma_r(s)\bigg|_{s=0} = \frac{W}{2} - (W\%2)\cdot\frac{2W+L+2r}{2} - \frac{r}{2}(L-1)$$

which equals the claimed expression. □

7.7.3 Proof of Theorem 5.6.1

All that remains is to apply the above lemmas with the correct inputs. To spell out the details:

- For ℓ odd, consult Theorem 5.5.2.
  - Apply Lemma 7.7.1 with $C = 0$, $W = \ell + 2\delta$ and $H = \frac{\ell-1}{2}$.
- For $\ell \geq 0$ even, consult Theorem 5.5.7.
  - Apply Lemma 7.7.1 with $C = 0$, $W = \lambda + 2\delta$ and $H = \frac{\ell}{2}$.
  - Apply Lemma 7.7.3 with $C = \ell$, $W = \delta - \frac{1}{2}\ell$ and $L = \lambda - \ell$.
- For $\ell < 0$ even, consult Theorem 5.5.7. If $r < |v(d)|$ then all the orbital integrals are zero anyway. Otherwise, we replace r by to $r - |v(d)| \geq 0$ and then apply the following lemmas:

– Apply Lemma 7.7.1 with $C = -2|v(d)|$, $W = \lambda$ and $H = 0$ and r replaced by $r - |v(d)|$.

– Apply Lemma 7.7.3 with $C = -2|v(d)|$, $W = 0$ and $L = \lambda$ and r replaced by $r - |v(d)|$.

Chapter 8

Synopsis of the weighted orbital integral $\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s)$ for $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))_{\mathrm{rs}}$ and $\phi \in \mathcal{H}(S_2(F))$

Throughout this section, $H = \mathrm{GL}_n(F)$ (rather than $H = \mathrm{GL}_{n-1}(F)$) and $K' = \mathrm{GL}_n(O_F)$. For the concrete calculation, we are mostly interested in the case $n = 2$. The goal of this chapter is to define the orbital integral in Theorem 1.3.1 and give a precise statement of the parameters used to state the formula, as well as the dependencies between the parameters that needs to hold in order for the matching to work.

8.1 Definition

For our conjecture, it will be enough to define the weighted orbital integral in the case where our function is of the form

$$\phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}$$

where $\phi \in \mathcal{H}(S_n(F))$ is the left component, and the right component is the indicator function defined in the obvious way:

$$\begin{aligned}
\mathbf{1}_{O_F^n \times (O_F^n)^\vee} \colon V_n'(F) &\to \{0,1\} \\
(\mathbf{u}, \mathbf{v}^\top) &\mapsto \begin{cases} 1 & \mathbf{u} \text{ and } \mathbf{v}^\top \text{ have } O_F\text{-entries} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Then, unsurprisingly from the definition of our action as

$$h \cdot (\gamma, \mathbf{u}, \mathbf{v}^\top) = (h\gamma h^{-1}, h\mathbf{u}, \mathbf{v}^\top h^{-1})$$

we analogously define the weighted orbital integral as follows.

Definition 8.1.1 ([Liu21, §1.3])**.** For brevity let $\eta(h) \coloneqq \eta(\det h)$ for $h \in H$. For $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in S_n(F) \times V_n'(F)$, $\phi \in \mathcal{H}(S_n(F))$, and $s \in \mathbb{C}$, we define the weighted orbital integral by

$$\begin{aligned}
&\operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s) \\
&\coloneqq \int_{h \in H} \phi(h^{-1}\gamma h) \mathbf{1}_{O_F^n \times (O_F^n)^\vee}(h\mathbf{u}, \mathbf{v}^\top h^{-1}) \eta(h) \left|\det(h)\right|_F^{-s} \, \mathrm{d}h.
\end{aligned}$$

Definition 8.1.2 (The abbreviation $\partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi)$)**.** Moving forward we abbreviate

$$\partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi) \coloneqq \left.\frac{\partial}{\partial s}\right|_{s=0} \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s).$$

As before it seems this weighted orbital integral should be related to an ordinary one. To define it, fix a self-dual lattice Λ_n in $\mathbb{V}_n^+$ of full rank. First, if $(g, u) \in \mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+$ and $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$, then we define an orbital integral for $\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+$ by

$$\operatorname{Orb}^{\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+}((g, u), f \otimes \mathbf{1}_{\Lambda_n}) \coloneqq \int_{\mathrm{U}(\mathbb{V}_n^+)} f(x^{-1}gx) \mathbf{1}_{\Lambda_n}(x^{-1}u) \, \mathrm{d}x. \tag{8.1}$$

Then in the spirit of [Liu21, Conjecture 1.9] and Theorem 5.1.3, we propose the following.

Conjecture 8.1.3 (Relative fundamental lemma in the semi-Lie case)**.** *Let $\phi \in \mathcal{H}(S_n(F))$ and $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$. Then*

$$\begin{aligned}
&\omega(\gamma, \mathbf{u}, \mathbf{v}^\top) \operatorname{Orb}(\phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, (\gamma, \mathbf{u}, \mathbf{v}^\top), 0) \\
&= \begin{cases} 0 & \text{if } (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}^- \\ \operatorname{Orb}^{\mathrm{U}(\mathbb{V}_n^+) \times \mathbb{V}_n^+}((g,u), \mathrm{BC}_{S_n}^{\eta^{n-1}}(\phi) \otimes \mathbf{1}_{\Lambda_n}) & \text{if } (\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}^+ \end{cases}
\end{aligned}$$

where the transfer factor ω is defined in Chapter 13.

Wei Zhang suggests that this conjecture can be proven by similar means to Theorem 5.1.3, but since it is not necessary for this paper we do not pursue this proof here.

8.2 Basis for the indicator functions in $\mathcal{H}(S_2(F))$

From now on assume $n = 2$. This section is almost an exact analog of Section 5.2, so we will be slightly terser. Again set

$$S_2(F) \coloneqq \{g \in \mathrm{GL}_2(E) \mid g\bar{g} = \mathrm{id}_2\}.$$

We again have a Cartan decomposition indexed by a single integer $r \geq 0$:

Lemma 8.2.1 (Cartan decomposition of $S_2(F)$)**.** *For each integer $r \geq 0$ let*

$$K'_{S,r} \coloneqq \mathrm{GL}_2(O_E) \cdot \begin{pmatrix} 0 & \varpi^r \\ \varpi^{-r} & 0 \end{pmatrix}$$

denote the orbit of $\begin{pmatrix} 0 & \varpi^r \\ \varpi^{-r} & 0 \end{pmatrix}$ under the left action of $\mathrm{GL}_2(O_E)$. Then we have a decompo-

sition

$$S_2(F) = \coprod_{r\geq 0} K'_{S,r}.$$

Proof. This is [LRZ24, Equation (7.1.7) and (7.1.8)]. □

Like last time, $K'_{S,r}$ is the part of $S_2(F)$ for which the most negative valuation among the nine entries is $-r$. And as before we abbreviate the $r = 0$ term specifically:

$$\begin{aligned} K'_S &\coloneqq K'_{S,0} \\ &= \mathrm{GL}_2(O_E) \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &= \mathrm{GL}_2(O_E) \cdot \mathrm{id}_2 = S_2(F) \cap \mathrm{GL}_2(O_E). \end{aligned}$$

Repeating the definition

$$K'_{S,\leq r} \coloneqq S_2(F) \cap \varpi^{-r}\,\mathrm{GL}_2(O_E) = K'_{S,0} \sqcup K'_{S,1} \sqcup \cdots \sqcup K'_{S,r}$$

we get a basis of indicator functions for the Hecke algebra $\mathcal{H}(S_2(F))$:

Corollary 8.2.2 (Basis of $\mathcal{H}(S_2(F))$)**.** *For $r \geq 0$, the indicator functions $\mathbf{1}_{K'_{S,\leq r}}$ form a basis of $\mathcal{H}(S_2(F))$.*

8.3 Parametrization of γ

From now on assume $n = 2$, and that $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V'_2)_{\mathrm{rs}}$ is regular.

8.3.1 Identifying an orbit representative

The weighted orbital integral depends only on the H-orbit of $(\gamma, \mathbf{u}, \mathbf{v}^\top)$. Consequently, we may assume without loss of generality (via multiplication by a suitable change-of-basis

$h \in H = \mathrm{GL}_2(F)$) that

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}^\top = \begin{pmatrix} 0 & e \end{pmatrix} \qquad e \in F.$$

(We know $\mathbf{u}$ is not the zero vector from the regular condition applied on $(\gamma, \mathbf{u}, \mathbf{v}^\top)$.)

Meanwhile, we will let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)$ for $a, b, c, d \in F$. Then, viewed as an element of $\mathrm{GL}_3(F)$ via the embedding we described earlier, we have

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 1 \\ 0 & e & 0 \end{pmatrix} \in \mathrm{Mat}_3(F).$$

Thus, our definition of regular requires that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is linearly independent from $\begin{pmatrix} b \\ d \end{pmatrix}$ and $\begin{pmatrix} 0 & e \end{pmatrix}$ is linearly independent from $\begin{pmatrix} c & d \end{pmatrix}$. This is just saying that b, c, e are all nonzero. We also know that $\gamma \in S_2(F)$, which gives us relations on a, b, c, d (the same as [LRZ24, Equation (7.3.2)]); we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \implies \begin{gathered} \bar{b}c = b\bar{c} = 1 - a\bar{a} = 1 - d\bar{d}, \\ d = -\bar{a}c/\bar{c} = -\bar{a}b/\bar{b}. \end{gathered}$$

8.3.2 Simplification due to the matching of non-split unitary group

Like before, we focus on the case where regular $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ matches an element in the non-split unitary group. As we described in Proposition 3.3.2, this is controlled by the parity of $v(\Delta)$, where

$$\Delta = \det\left(\left(\mathbf{v}^\top \gamma^{i+j} \mathbf{u} \right)_{0 \le i,j \le n-1} \right).$$

When $n = 2$, for the representatives we described before, we have

$$\left(\mathbf{v}^\top \gamma^{i+j} \mathbf{u}\right)_{0 \le i,j \le n-1} = \begin{pmatrix} e & de \\ de & bce + d^2 e \end{pmatrix}$$

so

$$\Delta = bce^2 = \frac{b}{\bar{b}}(1 - a\bar{a})e^2.$$

Hence, $v(\Delta)$ is odd if and only if $v(1 - a\bar{a})$ is odd. Thus, we restrict attention to the following situation:

Assumption 8.3.1. *We will assume that*

$$v(1 - a\bar{a}) \equiv 1 \pmod 2.$$

In particular, a must be a unit. And since $d = -\bar{a}c/\bar{c}$, it follows d is a unit. In other words, Assumption 8.3.1 gives the direct corollary

$$v(a) = v(d) = 0.$$

8.4 Parameters used in the calculation of the weighted orbital integral

The situation is simpler than Section 5.4 and we will state our derivative in terms of the five integers r, $v(b)$, $v(c)$, $v(e)$ and $v(d - a)$. From Assumption 8.3.1, we actually get that

Assumption 8.4.1. *We have that*

- $v(b) + v(c)$ *is an odd positive integer;*
- $v(d - a) \ge 0$.

These are the only constraints between these five numbers we will consider (together with $r \geq 0$). However, we mention that we will only be interested in the case when $v(e) \geq 0$ since in the case $v(e) < 0$ we will shortly see that $\operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s) = 0$ identically in s in that situation.

For convenience, we summarize all the assumptions on the shape of $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ in the following single lemma.

Lemma 8.4.2 (Parameters for $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))_{\mathrm{rs}}$)**.** *Suppose $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))_{\mathrm{rs}}$. Then one can choose a representative of the $\mathrm{GL}_2(F)$-orbit of $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ of the form*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix} \right)$$

where $a, b, c, d, e \in F$ satisfy $bce \neq 0$,

$$\bar{b}c = b\bar{c} = 1 - a\bar{a} = 1 - d\bar{d},$$

$$d = -\bar{a}c/\bar{c} = -\bar{a}b/\bar{b}.$$

Moreover, we always assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ matches an element of $(\mathrm{U}(\mathbb{V}_2^-) \times \mathbb{V}_2^-)_{\mathrm{rs}}$ rather than $(\mathrm{U}(\mathbb{V}_2^+) \times \mathbb{V}_2^+)_{\mathrm{rs}}$; this is equivalent to Assumption 8.3.1 which states that

$$v(1 - a\bar{a}) \equiv 1 \pmod 2.$$

In particular, we may assume $v(a) = v(d) = 0$ and $v(b) + v(c) \geq 1$ is odd (Assumption 8.4.1).

Chapter 9

Support of the weighted orbital integral for $S_2(F) \times V_2'(F)$

We assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is as in Lemma 8.4.2 throughout this chapter.

9.1 Iwasawa decomposition

The overall method is to take the Iwasawa decomposition in KAN form:

Lemma 9.1.1 (Iwasawa decomposition). *Every element in $h \in \mathrm{GL}_2(F)$ may be parametrized as*

$$h = k \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

where $k \in K' = \mathrm{GL}_2(O_F)$, $x_1, x_2 \in O_F^\times$ and $y \in O_F$.

Because the orbits are invariant under conjugation by K', the parameter k can be discarded. The Haar measure in these coordinates

$$\left| \frac{x_1}{x_2} \right| \mathrm{d}^\times x_1 \, \mathrm{d}^\times x_2 \, \mathrm{d}y$$

where we take multiplicative Haar measure on $F^\times$ (normalized so that $O_F^\times$ has volume 1) and additive Haar measure on F (so O_F has volume 1).

9.2 Action of upper triangular matrices on $(\gamma, \mathbf{u}, \mathbf{v}^\top)$

We now compute the action of an arbitrary

$$h = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

on $(\gamma, \mathbf{u}, \mathbf{v}^\top)$. The main term is given by

$$\begin{aligned} h\gamma h^{-1} &= \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} \\ &= \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} cy + a & -cy^2 + (d-a)y + b \\ c & -cy + d \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} \\ &= \begin{pmatrix} cy + a & \frac{x_1}{x_2} \cdot (-cy^2 + (d-a)y + b) \\ \frac{x_2}{x_1} \cdot c & -cy + d \end{pmatrix} \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} h\mathbf{u} &= \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 y \\ x_2 \end{pmatrix} \\ \mathbf{v}^\top h^{-1} &= \begin{pmatrix} 0 & e \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{e}{x_2} \end{pmatrix}. \end{aligned}$$

9.3 Description of support

From now on we fix the notation

$$n_1 \coloneqq v(x_1)$$
$$n_2 \coloneqq v(x_2).$$

Note that although $n_2 \geq 0$, the value of n_1 will often be non-positive. In fact n_1 is not particularly simple to work with and we will prefer to introduce the notation

$$m \coloneqq n_2 + v(c) + r - n_1 \tag{9.1}$$

instead to use as a summation variable. This is chosen so that $\frac{x_2}{x_1} \cdot c \in \varpi^{-r} O_F \iff m \geq 0$. Note that it follows we have

$$n_1 + n_2 = 2n_2 - m + v(c) + r \tag{9.2}$$

9.3.1 Collating the linear constraints

For a given $r \geq 0$, we find that h contributes to the integral exactly if $h\mathbf{u}$ and $\mathbf{v}^\top h^{-1}$ have O_F-entries, and all the entries of $h\gamma h^{-1}$ are in $\varpi^{-r} O_F$. The former condition is just saying that

$$v(y) \geq -n_1,$$
$$0 \leq n_2 \leq v(e).$$

Now we consider the entries of $h\gamma h^{-1}$. First, because a and d are units by Assumption 8.3.1, and $r \geq 0$, it follows that

$$\begin{aligned} cy + a, -cy + d \in \varpi^{-r} O_F &\iff cy \in \varpi^{-r} O_F \\ &\iff v(y) \geq -v(c) - r. \end{aligned}$$

Moreover,

$$\frac{x_2}{x_1} \cdot c \in \varpi^{-r} O_F \iff n_2 + v(c) - n_1 \geq -r \iff m \geq 0.$$

In summary, up until now we have the following requirements imposed:

$$\begin{aligned} 0 &\leq n_2 \leq v(e) \\ 0 &\leq m \\ v(y) &\geq \max(-n_1, -v(c) - r) \\ &= \max(m - n_2, 0) - v(c) - r. \end{aligned} \tag{9.3}$$

9.3.2 The quadratic constraint

As for the quadratic constraint, we seek y such that

$$\begin{aligned} &\frac{x_1}{x_2} \cdot (-cy^2 + (d-a)y + b) \in \varpi^{-r} O_F \\ \iff &v\left(-y^2 + \frac{d-a}{c} y + \frac{b}{c}\right) \geq n_2 - n_1 - v(c) - r \\ &\qquad = m - 2v(c) - 2r. \end{aligned}$$

As before, we complete the square:

$$-y^2 + \frac{d-a}{c} y + \frac{b}{c} = -\left(y - \frac{d-a}{2c}\right)^2 + \frac{b}{c} + \frac{(d-a)^2}{4c^2}.$$

Because $b\bar{c} = 1 - a\bar{a}$ has odd valuation, it follows that $\frac{b}{c} = \frac{1-a\bar{a}}{c\bar{c}}$ has odd valuation to. On the other hand, $\frac{(d-a)^2}{4c^2}$ has even valuation.

This motivates us to introduce the following parameter:

Definition 9.3.1 (θ)**.** We define

$$\theta \coloneqq \min\left(v(b) + v(c), 2v(d-a)\right) \geq 0.$$

Note that $v(b) + v(c)$ is odd, so θ takes the odd value if $v(b) + v(c) < 2v(d-a)$ and the even value otherwise. This definition ensures that

$$v\left(\frac{b}{c} + \frac{(d-a)^2}{4c^2}\right) = \theta - 2v(c).$$

9.4 Cases based on θ

Henceforth we consider two cases based on θ. We number these Case 5 and Case 6 to prevent confusion with the cases introduced in Chapter 6.

Case 5 Let's assume first that

$$\theta - 2v(c) \geq m - 2v(c) - 2r \iff m \leq \theta + 2r.$$

Then the only additional condition on y is that

$$v\left(y - \frac{d-a}{2c}\right) \geq \left\lceil \frac{m}{2} \right\rceil - v(c) - r.$$

We refer to this as **Case 5**.

Case 6$^+$ / **Case 6$^-$** Otherwise assume that

$$\theta - 2v(c) < m - 2v(c) - 2r \iff m > \theta + 2r.$$

Then in order for y to satisfy the constraint, we would need to be in a situation where $2v(y - \frac{d-a}{2c}) = \theta - 2v(c)$. So this case could only arise at all when θ is even, that is

$$0 \leq 2v(d-a) = \theta < v(b) + v(c)$$

(note that $v(d-a) \geq 0$ because a and d are units). As the quantity $\frac{b}{c} + \frac{(d-a)^2}{4c^2}$ must be a perfect square, we denote it by τ^2, with

$$v(\tau) = \frac{\theta}{2} - v(c).$$

This gives us the factorization

$$\frac{b}{c} = \tau^2 - \frac{(d-a)^2}{4c^2} = \left(\tau - \frac{d-a}{2c}\right)\left(\tau + \frac{d-a}{2c}\right).$$

The left-hand side has odd valuation $v(b) - v(c)$, so the two factors on the right have unequal valuations and hence exactly one of them has valuation the same as $v(\frac{d-a}{2c}) = v(\tau)$. Hence, we agree to fix the choice of the square root τ so that

$$\begin{aligned} v\left(\tau + \frac{d-a}{2c}\right) &= v(b) - v(c) - v(\tau) = v(b) - \frac{\theta}{2} \\ v\left(\tau - \frac{d-a}{2c}\right) &= v(\tau) = \frac{\theta}{2} - v(c) \end{aligned}$$

and in particular $v\left(\tau + \frac{d-a}{2c}\right) > v\left(\tau - \frac{d-a}{2c}\right)$.

In any case, the constraint on y is that

$$\begin{aligned} v\left(y - \left(\frac{d-a}{2c} \pm \tau\right)\right) &\geq (m - 2v(c) - 2r) - v(\tau) \\ &= (m - 2v(c) - 2r) - \left(\frac{\theta}{2} - v(c)\right) \\ &= m - \frac{\theta}{2} - v(c) - 2r \end{aligned}$$

$$v\left(y-\left(\frac{d-a}{2c}\mp\tau\right)\right)=v(\tau)=\frac{\theta}{2}-v(c).$$

By assumption, the second equation is true whenever the first inequality is and we may disregard it. **Case 6$^+$** refers to the situation where the $\pm$ sign is $+$ and **Case 6$^-$** refers to the situation where the $\mp$ sign is $-$. And these cases must be disjoint because the right-hand sides above are unequal.

9.4.1 Analysis of Case 5

The triple $(x_1, x_2, y) \in O_F^\times \times O_F^\times \times O_F$ contributes to the weighted orbital integral in Case 5 exactly if the following identities hold:

$$\begin{aligned} 0 &\le n_2 \le v(e) \\ 0 &\le m \le \theta + 2r \\ v(y) &\ge \max(m-n_2, 0) - v(c) - r \\ v\left(y-\frac{d-a}{2c}\right) &\ge \left\lceil\frac{m}{2}\right\rceil - v(c) - r. \end{aligned}$$

However, from the definitions we already know that

$$v\left(\frac{d-a}{2c}-0\right)\ge\frac{\theta-2v(c)}{2}\ge\frac{m}{2}-v(c)-r$$

so the disks in the last two conditions have nonempty intersection. Hence the earlier Lemma 2.2.2 applies to tell us that the locus of valid y is a single disk whose volume in O_F is given by

$$q^{-\max(m-n_2,\lceil m/2\rceil,0)+v(c)+r}=q^{-\max(m-n_2,\lceil m/2\rceil)+v(c)+r}.$$

The volume contribution for and $x_1 \in O_F^\times$ and $x_2 \in O_F^\times$ is also 1, because $v(x_1)$ and $v(x_2)$ are fixed. Hence the overall volume of the support in H for this pair (m, n_2) is given by

$$\begin{aligned}\left|\frac{x_1}{x_2}\right| q^{n_2-n_1} \operatorname{Vol}(\{y \mid \dots\}) &= q^{n_2-n_1-\max(m-n_2,\lceil m/2\rceil)+v(c)+r} \\ &= q^{m-\max(m-n_2,\lceil m/2\rceil)}.\end{aligned}$$

And again, this case is summed over

$$0 \leq n_2 \leq v(e), \qquad 0 \leq m \leq \theta + 2r.$$

9.4.2 Analysis of Case 6$^+$ and Case 6$^-$

Again, this case could only occur if θ is even. The triple $(x_1, x_2, y) \in O_F^\times \times O_F^\times \times O_F$ contributes to the weighted orbital integral in Case 6$^+$ and Case 6$^-$ exactly if the following identities hold:

$$\begin{aligned} 0 \leq n_2 &\leq v(e) \\ \theta + 2r &< m \\ v(y) &\geq \max(m - n_2, 0) - v(c) - r \\ v\left(y - \left(\frac{d-a}{2c} \pm \tau\right)\right) &\geq m - \frac{\theta}{2} - v(c) - 2r. \end{aligned}$$

The last two inequalities specify disks. So in each case, via Lemma 2.2.2 we get a nonzero contribution if and only if the distance between the centers 0 and $\frac{d-a}{2c} \pm \tau$ has valuation at least that of the smaller of the two right-hand sides, that is

$$\begin{aligned} v\left(\frac{d-a}{2c} \pm \tau\right) &\geq \min\left(\max(m-n_2, 0) - v(c) - r, m - \frac{\theta}{2} - v(c) - 2r\right) \\ &= \min\left(\max(m-n_2,0), m - \frac{\theta}{2} - r\right) - v(c) - r. \end{aligned}$$

Hence the upper bound on m is given by two different requirements, depending on which of the two values of $v\left(\frac{d-a}{2c} \pm \tau\right) + v(c) + r$ is given by the case:

- In Case 6^+, we need at least one of the inequalities

$$\begin{cases} \max(m - n_2, 0) \leq v(b) - \frac{\theta}{2} + v(c) + r, \\ m \leq v(b) + v(c) + 2r \end{cases}$$

  to hold. Now the inequality $0 \leq v(b) - \frac{\theta}{2} + v(c) + r$ is always true, as $\theta < v(b) + v(c)$, so we can disregard it. Therefore this can be rewritten as just

$$m \leq \max\left(r, n_2 - \frac{\theta}{2}\right) + v(b) + v(c) + r.$$

- In Case 6$^-$, we need at least one of the inequalities

$$\begin{cases} \max(m - n_2, 0) \leq \frac{\theta}{2} + r \\ m \leq \theta + 2r \end{cases}$$

  to hold. But $m \leq \theta + 2r$ is always false and $0 \leq \frac{\theta}{2} + r$ is always true, so this simplifies to

$$m \leq n_2 + \frac{\theta}{2} + r.$$

Assuming m lies in the valid range so that the locus of valid y is nonempty, it follows that the volume is given exactly by

$$q^{-\max(m-n_2, m-\frac{\theta}{2}-r, 0)-v(c)-r} = q^{-\max(m-n_2, m-\frac{\theta}{2}-r)-v(c)-r}.$$

Hence the overall volume of the support in H for this pair (m, n_2) is given by

$$\begin{aligned}
\left|\frac{x_1}{x_2}\right| q^{n_2-n_1} \operatorname{Vol}(\{y \mid \dots\}) &= q^{n_2-n_1-\max\left(m-n_2, m-\frac{\theta}{2}-r\right)+v(c)+r} \\
&= q^{m-\max\left(m-n_2, m-\frac{\theta}{2}-r\right)} \\
&= q^{\min\left(n_2, \frac{\theta}{2}+r\right)}.
\end{aligned}$$

And this sum is over two ranges of m (although the ranges obviously overlap, they set of y they cover is disjoint):

$$\begin{aligned}
&\theta + 2r < m \leq \max\left(r, n_2 - \frac{\theta}{2}\right) + v(b) + v(c) + r \\
&\theta + 2r < m \leq n_2 + \frac{\theta}{2} + r.
\end{aligned}$$

Note the second range could be empty if n_2 is small enough, but the first range is always nonempty.

Chapter 10

Evaluation of the weighted orbital integral for $S_2(F) \times V_2'(F)$

We continue to assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is as in Lemma 8.4.2 throughout this chapter.

We now aggregate the supports we found in the previous section together with the definition of the weighted orbital integral to extract the desired formulas.

Recall that the weighted orbital integral was defined as

$$\begin{aligned}
&\operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s) \\
&:= \int_{h \in H} \phi(h^{-1}\gamma h) \mathbf{1}_{O_F^n \times (O_F^n)^\vee}(h\mathbf{u}, \mathbf{v}^\top h^{-1}) \eta(h) \left|\det(h)\right|_F^{-s} \mathrm{d}h
\end{aligned}$$

and that after taking Iwasawa decomposition as

$$h = k \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

we broke the sum based on $n_1 = v(x_1)$ and $n_2 = v(x_2)$. For h as above, we know that

$$\eta(h) = (-1)^{n_1+n_2}$$

$$|\det(h)|_F^{-s} = (q^s)^{n_1+n_2}.$$

Applying (9.2) we find that

$$\eta(h)\,|\det(h)|_F^{-s} = (q^s)^{2n_2-m+v(c)+r}.$$

10.1 The contribution for Case 5

We assume $\theta+2r \geq 0$, because otherwise the entire sum is empty. Hence, the total contribution for **Case 5** is

$$\begin{aligned} I^5 &:= \sum_{n_2=0}^{v(e)} \sum_{m=0}^{\theta+2r} q^{m-\max(m-n_2,\lceil m/2\rceil)} (-q^s)^{2n_2-m+v(c)+r} \\ &:= \sum_{n_2=0}^{v(e)} \sum_{m=0}^{\theta+2r} q^{\min(n_2,\lfloor m/2\rfloor)} (-q^s)^{2n_2-m+v(c)+r}. \end{aligned}$$

We'll change the summation variable to

$$k := 2n_2 - m + v(c) + r \iff m = 2n_2 - k + v(c) + r.$$

Then

$$\begin{aligned} I^5 &:= \sum_{n_2=0}^{v(e)} \sum_{k=2n_2-\theta+v(c)-r}^{2n_2+v(c)+r} q^{\min\left(n_2,n_2+\left\lfloor \frac{v(c)+r-k}{2}\right\rfloor\right)} (-q^s)^k \\ &= \sum_{n_2=0}^{v(e)} \sum_{k=2n_2-\theta+v(c)-r}^{2n_2+v(c)+r} q^{n_2-\max\left(0,\left\lceil \frac{k-(v(c)+r)}{2}\right\rceil\right)} (-q^s)^k. \end{aligned}$$

We then interchange the order of summation so that k is outside. Then k runs from the lowest value of $k = -\theta + v(c) - r$ to the largest value $k = 2v(e) + v(c) + r$ over all choices of n_2. Since

$$2n_2 - \theta + v(c) - r \leq k \leq 2n_2 + v(c) + r$$

then in addition to $0 \leq n_2 \leq v(e)$ we also need

$$\frac{k - v(c) - r}{2} \leq n_2 \leq \frac{k + \theta - v(c) + r}{2}.$$

In other words, we obtain

$$\begin{aligned} I^5 &= \sum_{k=-\theta+v(c)-r}^{2v(e)+v(c)+r} (-1)^k (q^s)^k \sum_{n_2=\max\left(0,\left\lceil \frac{k-v(c)-r}{2}\right\rceil\right)}^{\min\left(v(e),\left\lfloor \frac{k+\theta-v(c)+r}{2}\right\rfloor\right)} q^{n_2-\max\left(0,\left\lceil \frac{k-(v(c)+r)}{2}\right\rceil\right)} \\ &= \sum_{k=-\theta+v(c)-r}^{2v(e)+v(c)+r} (-1)^k (q^s)^k \left(q^{\min\left(v(e),\left\lfloor \frac{k+\theta-v(c)+r}{2}\right\rfloor\right)-\max\left(0,\left\lceil \frac{k-v(c)-r}{2}\right\rceil\right)} + \cdots + q^0\right). \end{aligned}$$

Here, we retain the convention from Chapter 7 that ellipses of the form

$$q^i + \cdots + q^{i'}$$

will denote the expression $q^i + q^{i-1} + \cdots + q^{i'}$ (i.e. within any ellipses, the exponents are understood to decrease by 1, and the sums are always nonempty, meaning $i \geq i'$).

To simplify the exponent, write

$$\begin{aligned} &\min\left(v(e), \left\lfloor \tfrac{k+\theta-v(c)+r}{2}\right\rfloor\right) - \max\left(0, \left\lceil \tfrac{k-v(c)-r}{2}\right\rceil\right) \\ &= \min\left(v(e), \left\lfloor \tfrac{k+\theta-v(c)+r}{2}\right\rfloor\right) + \min\left(0, \left\lfloor \tfrac{v(c)+r-k}{2}\right\rfloor\right) \\ &= \min\left(\left\lfloor \tfrac{k+\theta-v(c)+r}{2}\right\rfloor, v(e) + \left\lfloor \tfrac{v(c)+r-k}{2}\right\rfloor, v(e), \left\lfloor \tfrac{k+\theta-v(c)+r}{2}\right\rfloor + \left\lfloor \tfrac{v(c)+r-k}{2}\right\rfloor\right). \end{aligned} \tag{10.1}$$

This already completes Theorem 1.3.1 in the situation when θ is odd since **Case 6**$^+$ and **Case 6**$^-$ do not appear at all. However, let's turn to the remaining cases first.

10.2 The contribution for Case 6^+ and Case 6^-

Herein we assume $\theta = 2v(d-a) > v(b) + v(c)$ is even, and in particular $\theta \geq 0$. We get a contribution of

$$I^{6+} := \sum_{n_2=0}^{v(e)} \sum_{m=\theta+2r+1}^{\max\left(r, n_2 - \frac{\theta}{2}\right) + v(b) + v(c) + r} q^{\min\left(n_2, \frac{\theta}{2}+r\right)} (-q^s)^{2n_2 - m + v(c) + r}$$

$$I^{6-} := \sum_{n_2=0}^{v(e)} \sum_{m=\theta+2r+1}^{n_2 + \frac{\theta}{2} + r} q^{\min\left(n_2, \frac{\theta}{2}+r\right)} (-q^s)^{2n_2 - m + v(c) + r}.$$

We will split I^{6+} into two parts:

$$\begin{aligned} I^{6+} &= \sum_{n_2=0}^{\frac{\theta}{2}+r} \sum_{m=\theta+2r+1}^{v(b)+v(c)+2r} q^{n_2} (-q^s)^{2n_2 - m + v(c) + r} \\ &\quad + q^{\frac{\theta}{2}+r} \sum_{n_2=\frac{\theta}{2}+r+1}^{v(e)} \sum_{m=\theta+2r+1}^{n_2 - \frac{\theta}{2} + v(b) + v(c) + r} (-q^s)^{2n_2 - m + v(c) + r}. \end{aligned}$$

Note that the second sum is nonempty only when $v(e) > \frac{\theta}{2} + r$. So we consider cases on this in what follows.

10.2.1 Sub-case where $v(e) \leq \frac{\theta}{2} + r$

First, suppose $v(e) \leq \frac{\theta}{2} + r$. Then the contribution of **Case 6⁻** is void, since the inner sum of I^{6-} contributes only when $n_2 > \frac{\theta}{2} + r$. We only need to consider

$$\begin{aligned} I^{6+} &= \sum_{n_2=0}^{v(e)} \sum_{m=\theta+2r+1}^{v(b)+v(c)+2r} q^{n_2} (-q^s)^{2n_2 - m + v(c) + r} \\ &= \sum_{n_2=0}^{v(e)} \sum_{k=2n_2 - v(b) - r}^{2n_2 - \theta + v(c) - r - 1} q^{n_2} (-q^s)^k. \end{aligned}$$

Swapping the summation order so that k is outside, the sum runs from the lowest value $k = -v(b) - r$ up to the highest value $k = 2v(e) - \theta + v(c) - r - 1$, subject to $0 \leq n_2 \leq v(e)$ and

$$\begin{aligned} & 2n_2 - v(b) - r \leq k \leq 2n_2 - \theta + v(c) - r - 1 \\ \iff & \left\lceil \frac{k + \theta - v(c) + r + 1}{2} \right\rceil \leq n_2 \leq \left\lfloor \frac{k + v(b) + r}{2} \right\rfloor . \end{aligned}$$

Thus,

$$I^{6+} = \sum_{k=-v(b)-r}^{2v(e)-\theta+v(c)-r-1} \sum_{n_2=\max(0,\left\lceil \frac{k+\theta-v(c)+r+1}{2} \right\rceil)}^{\min(v(e),\left\lfloor \frac{k+v(b)+r}{2} \right\rfloor)} q^{n_2}(-q^s)^k.$$

10.2.2 Sub-case where $v(e) > \frac{\theta}{2} + r$

We start on $I^{6\text{-}}$; note if $n_2 \leq \frac{\theta}{2} + r$ then the inner sum of $I^{6\text{-}}$ has empty range anyway. Consequently, we can simply write

$$I^{6\text{-}} = q^{\frac{\theta}{2}+r} \sum_{n_2=\frac{\theta}{2}+r+1}^{v(e)} \sum_{m=\theta+2r+1}^{n_2+\frac{\theta}{2}+r} (-q^s)^{2n_2-m+v(c)+r}$$

which in particular is nonempty. In that case, simplifying the inner sum gives

$$I^{6\text{-}} = q^{\frac{\theta}{2}+r} \sum_{n_2=\frac{\theta}{2}+r+1}^{v(e)} \left((-q^s)^{2n_2-\theta+v(c)-r-1} + \cdots + (-q^s)^{n_2-\frac{\theta}{2}+v(c)} \right).$$

We collect the coefficient of $(-q^s)^k$ for each k. The lowest value of k which appears is $k = v(c) + r + 1$; the highest one is $k = 2v(e) - \theta + v(c) - r - 1$. For these k, the coefficient is the number of integers n_2 such that

$$\frac{\theta}{2} + r + 1 \leq n_2 \leq v(e)$$

and

$$\begin{aligned}
&n_2 - \frac{\theta}{2} + v(c) \le k \le 2n_2 - \theta - r - 1 + v(c)\\
\iff &\frac{k+\theta - v(c) + r + 1}{2} \le n_2 \le k + \frac{\theta}{2} - v(c).
\end{aligned}$$

Note we already have $\frac{k+\theta-v(c)+r+1}{2} \ge \frac{\theta}{2} + r + 1$ for k in the desired range. Hence we have

$$\begin{aligned}
I^{6\text{-}} &= q^{\frac{\theta}{2}+r} \sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1} \Bigg(1 + \min\left(v(e), k + \frac{\theta}{2} - v(c)\right)\\
&\qquad - \max\left(\frac{\theta}{2} + r + 1, \left\lceil \frac{k+\theta-v(c)+r+1}{2} \right\rceil\right)\Bigg)(-q^s)^k\\
&= q^{\frac{\theta}{2}+r} \sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1} \left(1 + \min\left(v(e), k + \frac{\theta}{2} - v(c)\right) - \left\lceil \frac{k+\theta-v(c)+r+1}{2} \right\rceil\right)(-q^s)^k.
\end{aligned}$$

The second double sum of I^{6+} is again nonempty since $v(e) > \frac{\theta}{2} + r$. So we compute it in a similar way to $I^{6\text{-}}$ by putting

$$\begin{aligned}
&q^{\frac{\theta}{2}+r} \sum_{n_2=\frac{\theta}{2}+r+1}^{v(e)} \sum_{m=\theta+2r+1}^{n_2-\frac{\theta}{2}+v(b)+v(c)+r} (-q^s)^{2n_2-m+v(c)+r}\\
&= q^{\frac{\theta}{2}+r} \sum_{n_2=\frac{\theta}{2}+r+1}^{v(e)} \left((-q^s)^{2n_2-\theta+v(c)-r-1} + \cdots + (-q^s)^{n_2+\frac{\theta}{2}-v(b)}\right).
\end{aligned}$$

Again we calculate the coefficient of $(-q^s)^k$. The values of k run from the lowest value $k = \theta - v(b) + r + 1$ and end at the highest value $k = 2v(e) - \theta + v(c) - r - 1$. In this range we need $\frac{\theta}{2} + r + 1 \le n_2 \le v(e)$ and

$$\begin{aligned}
&n_2 + \frac{\theta}{2} - v(b) \le k \le 2n_2 - \theta + v(c) - r - 1\\
\iff &\frac{k+\theta-v(c)+r+1}{2} \le n_2 \le k - \frac{\theta}{2} + v(b).
\end{aligned}$$

The double sum therefore becomes

$$q^{\frac{\theta}{2}+r}\sum_{k=\theta-v(b)+r+1}^{2v(e)-\theta+v(c)-r-1}\Bigg(1+\min\left(v(e),k-\frac{\theta}{2}+v(b)\right)$$
$$-\max\left(\frac{\theta}{2}+r+1,\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)\Bigg)(-q^s)^k.$$

It is natural to split this sum into $k\le v(c)+r$ and $k>v(c)+r$. In the former case, we have both $k-\frac{\theta}{2}+v(b)\le v(e)$ and $\frac{\theta}{2}+r+1\ge\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil$; in the latter case we have just $\frac{\theta}{2}+r+1\le\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil$ instead. Hence, the double sum simplifies further to

$$\begin{aligned}
&q^{\frac{\theta}{2}+r}\sum_{k=\theta-v(b)+r+1}^{v(c)+r}\left(1+\left(k-\frac{\theta}{2}+v(b)\right)-\left(\frac{\theta}{2}+r+1\right)\right)(-q^s)^k\\
&+q^{\frac{\theta}{2}+r}\sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1}\left(1+\min\left(v(e),k-\frac{\theta}{2}+v(b)\right)-\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)(-q^s)^k\\
&=q^{\frac{\theta}{2}+r}\sum_{k=\theta-v(b)+r+1}^{v(c)+r}(k-\theta+v(b)-r)(-q^s)^k\\
&+q^{\frac{\theta}{2}+r}\sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1}\left(1+\min\left(v(e),k-\frac{\theta}{2}+v(b)\right)-\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)(-q^s)^k.
\end{aligned}$$

Meanwhile, the first sum within I^{6+} can be computed as

$$\begin{aligned}
\sum_{n_2=0}^{\frac{\theta}{2}+r}\sum_{m=\theta+2r+1}^{v(b)+v(c)+2r}q^{n_2}(-q^s)^{2n_2-m+v(c)+r}&=\sum_{n_2=0}^{\frac{\theta}{2}+r}q^{n_2}\sum_{m=\theta+2r+1}^{v(b)+v(c)+2r}(-q^s)^{2n_2-m+v(c)+r}\\
&=\sum_{n_2=0}^{\frac{\theta}{2}+r}q^{n_2}\sum_{k=2n_2-v(b)-r}^{2n_2-\theta+v(c)-r-1}(-q^s)^k.
\end{aligned}$$

We now interchange the summation so that k is outside, running from the lowest value $k=-v(b)-r$ to the highest value $k=v(c)+r-1$. From

$$2n_2-v(b)-r\le k\le 2n_2-\theta+v(c)-r-1$$

we require that $0 \leq n_2 \leq \frac{\theta}{2} + r$ and

$$\frac{k+\theta-v(c)+r+1}{2} \leq n_2 \leq \frac{k+r+v(b)}{2}.$$

In other words, we get

$$\sum_{k=-v(b)-r}^{v(c)+r-1} (-q^s)^k \sum_{n_2=\max\left(0,\left\lceil \frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)}^{\min\left(\frac{\theta}{2}+r,\left\lfloor \frac{v(b)+r+k}{2}\right\rfloor\right)} q^{n_2}.$$

Hence the total contribution from **Case 6** can be written as

$$\begin{aligned}
I^{6+} + I^{6\text{-}} = &\sum_{k=-v(b)-r}^{v(c)+r-1} (-q^s)^k \left(q^{\min\left(\frac{\theta}{2}+r,\left\lfloor \frac{v(b)+r+k}{2}\right\rfloor\right)} + \cdots + q^{\max\left(0,\left\lceil \frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)}\right) \\
&+ q^{\frac{\theta}{2}+r} \sum_{k=\theta-v(b)+r+1}^{v(c)+r} \left(k-\theta+v(b)-r\right)(-q^s)^k \\
&+ q^{\frac{\theta}{2}+r} \sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1} \left(2 + \min\left(v(e), k-\frac{\theta}{2}+v(b)\right) + \min\left(v(e), k+\frac{\theta}{2}-v(c)\right)\right. \\
&\qquad\qquad \left. - 2\left\lceil \frac{k+\theta-v(c)+r+1}{2}\right\rceil \right)(-q^s)^k.
\end{aligned}$$

We'd like to further simplify the coefficient of $q^{\frac{\theta}{2}+r}$ as follows. First, we may as well write

$$\begin{aligned}
2 - 2\left\lceil \frac{k+\theta-v(c)+r+1}{2}\right\rceil &= 2 - \left((k+\theta-v(c)+r+1) + \mathbf{1}_{k+\theta+v(c)+r\equiv 1 \bmod 2}\right) \\
&= \mathbf{1}_{k+\theta+v(c)+r\equiv 0 \bmod 2} + v(c) - \theta - k - r.
\end{aligned}$$

Set aside the indicator function $\mathbf{1}_{k+\theta+v(c)+r\equiv 0 \bmod 2}$ momentarily; we will merge it in a moment. To consolidate the minimum's in the third double sum, note that we have

$$v(c)+r+1 \leq v(e)+\frac{\theta}{2}-v(b) < v(e)+v(c)-\frac{\theta}{2} \leq 2v(e)-\theta+v(c)-r-1.$$

Hence, based on the value of k, we get the following coefficients:

- If $v(c)+r+1 \le k \le v(e)+\frac{\theta}{2}-v(b)$, we get

$$\begin{aligned}(v(c)-\theta-k-r)+\left(k-\frac{\theta}{2}+v(b)\right)+\left(k+\frac{\theta}{2}-v(c)\right)\\ =k-\theta+v(b)-r.\end{aligned}$$

- If $v(e)+\frac{\theta}{2}-v(b) \le k \le v(e)+v(c)-\frac{\theta}{2}$, we get

$$\begin{aligned}(v(c)-\theta-k-r)+v(e)+\left(k+\frac{\theta}{2}-v(c)\right)\\ =v(e)-\frac{\theta}{2}-r.\end{aligned}$$

- If $v(e)+v(c)-\frac{\theta}{2} \le k \le 2v(e)-\theta+v(c)-r$, we get

$$\begin{aligned}(v(c)-\theta-k-r)+v(e)+v(e)\\ =2v(e)+v(c)-\theta-r.\end{aligned}$$

Noting the expression in the first bullet also matches the coefficient of $(-q^s)^k$ for $\theta-v(b)+r+1 \le k \le v(c)+r$, we can now write

$$\begin{aligned}I^{6+}+I^{6\text{-}} &= \sum_{k=-v(b)-r}^{v(c)+r-1}(-q^s)^k\left(q^{\min\left(\frac{\theta}{2}+r,\left\lfloor\frac{v(b)+r+k}{2}\right\rfloor\right)}+\cdots+q^{\max\left(0,\left\lceil\frac{k+\theta-v(c)+r+1}{2}\right\rceil\right)}\right)\\ &+q^{\frac{\theta}{2}+r}\sum_{k=\theta-v(b)+r+1}^{2v(e)-\theta+v(c)-r-1}\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)(-q^s)^k\\ &+q^{\frac{\theta}{2}+r}\sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1}\mathbf{1}_{k+\theta+v(c)+r\equiv 0 \bmod 2}(-q^s)^k\end{aligned}$$

as the overall contribution from **Case 6**, where

$$\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) := \min\left(k-\theta+v(b)-r, v(e)-\frac{\theta}{2}-r, 2v(e)+v(c)-\theta-r\right).$$

10.3 Proof of Theorem 1.3.1

We now prove Theorem 1.3.1, which we restate here.

Theorem 1.3.1 (Explicit orbital integral on $S_2(F)\times V_2'(F)$)**.** *Let*

$$(\gamma,\mathbf{u},\mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix}\right) \in (S_2(F)\times V_2'(F))^-_{\mathrm{rs}}$$

satisfy the requirements in Lemma 8.4.2. Let $r \ge 0$.

If $v(e) < 0$ or $v(b)+v(c) < -2r$, then

$$\mathrm{Orb}((\gamma,\mathbf{u},\mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}} \otimes \mathbf{1}_{O_F^n\times(O_F^n)^\vee}, s) = 0$$

holds identically for all $s\in\mathbb{C}$.

Otherwise define

$$\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) := \min\left(\left\lfloor \tfrac{k+(v(b)+r)}{2}\right\rfloor, \left\lfloor \tfrac{(2v(e)+v(c)+r)-k}{2}\right\rfloor, N\right)$$

where

$$N := \min\left(v(e), \tfrac{v(b)+v(c)-1}{2}+r, v(d-a)+r\right).$$

Also, if $v(d-a) < v(e)-r$ and $v(b)+v(c) > 2v(d-a)$, then additionally define

$$\begin{aligned}\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = \min\big(&k-(2v(d-a)-v(b)+r),\\ &(2v(e)+v(c)-2v(d-a)-r)-k, v(e)-v(d-a)-r\big).\end{aligned}$$

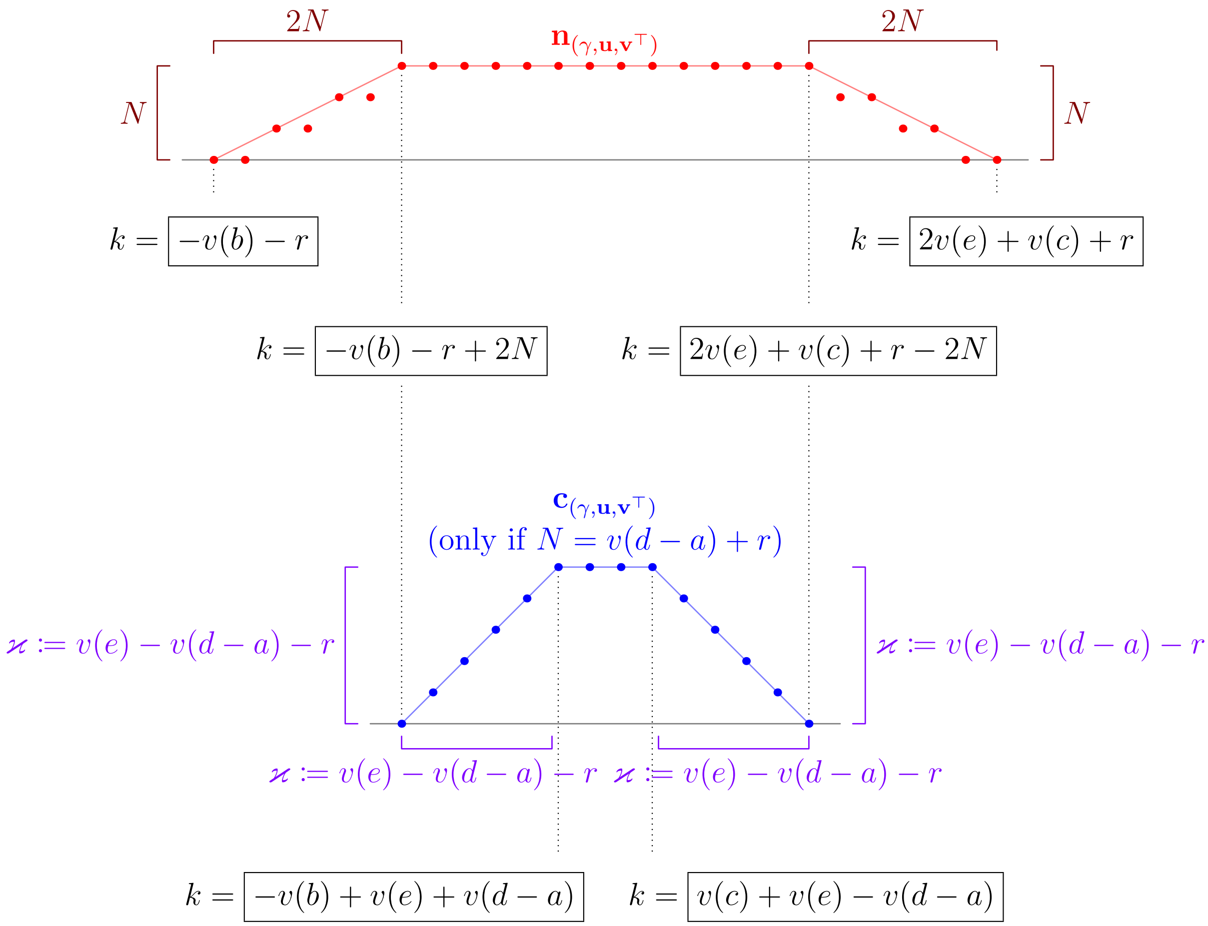

Figure 10.1: Sketch of the functions in Theorem 1.3.1. The boxed numbers indicate values of k.

Otherwise define $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = 0$. *Then we have*

$$\begin{aligned}\mathrm{Orb}((\gamma,\mathbf{u},\mathbf{v}^\top),\mathbf{1}_{K'_{S,\le r}},s) &= \sum_{k=-(v(b)+r)}^{2v(e)+v(c)+r} (-1)^k \left(1 + q + q^2 + \cdots + q^{\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)}\right)(q^s)^k \\ &+ \sum_{k=2v(d-a)-v(b)+r}^{2v(e)+v(c)-2v(d-a)-r} (-1)^k \mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) q^{v(d-a)+r} (q^s)^k.\end{aligned}$$

For reference, we provide Figure 10.1 sketching the shapes of $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ and $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$, which may be easier to think about.

Proof of Theorem 1.3.1. First, suppose $\theta = v(b) + v(c) < 2v(d-a)$ is odd. Then

$$\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) := \min\left(\left\lfloor \tfrac{k+(v(b)+r)}{2} \right\rfloor, \left\lfloor \tfrac{(2v(e)+v(c)+r)-k}{2} \right\rfloor, v(e), \tfrac{v(b)+v(c)-1}{2} + r\right)$$

and $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ terms do not appear. Now, in that case, the exponent (10.1) can be simplified, because

$$\begin{aligned}
\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor &= \left\lfloor \frac{k+v(b)+r}{2} \right\rfloor \\
v(e) + \left\lfloor \frac{v(c)+r-k}{2} \right\rfloor &= \left\lfloor \frac{(2v(e)+v(c)+r)-k}{2} \right\rfloor \\
\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor + \left\lfloor \frac{v(c)+r-k}{2} \right\rfloor &= \frac{k+\theta-v(c)+r}{2} + \frac{v(c)+r-k}{2} - \frac{1}{2} \\
&= \frac{\theta-1}{2} + r \\
&= \frac{v(b)+v(c)-1}{2} + r < v(d-a) + r.
\end{aligned}$$

Hence, $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ coincides with the exponent in (10.1). So the result is true in this case.

Now assume instead $\theta = 2v(d-a) < v(b) + v(c)$ is even. Notice that

$$\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor + \left\lfloor \frac{v(c)+r-k}{2} \right\rfloor = \frac{\theta}{2} + r - \mathbf{1}_{k+v(c)+r\equiv 0 \bmod 2}.$$

First assume that $v(e) \leq \frac{\theta}{2} + r$ and consider (10.1). For the range of values of k in I^{6+}, that is

$$-v(b) - r \leq k \leq 2v(e) - \theta + v(c) - r - 1$$

we have the first term of (10.1) is smallest, as

$$\begin{aligned}
\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor &< v(e) \\
&\leq v(e) + \frac{v(b)+v(c)-1}{2} \leq v(e) + \left\lfloor \frac{v(c)+r-k}{2} \right\rfloor \\
\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor &\leq v(e) - 1 \leq \frac{\theta}{2} + r - 1.
\end{aligned}$$

So the contributions from **Case 5** and **Case 6** fit together to give

$$\sum_{j=0}^{\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor} q^j + \left(q^{\min\left(v(e), \left\lfloor \frac{v(b)+r+k}{2} \right\rfloor\right)} + \cdots + q^{\max\left(0, \left\lceil \frac{k+\theta-v(c)+r+1}{2} \right\rceil\right)}\right)$$
$$= q^{\min\left(v(e), \left\lfloor \frac{v(b)+r+k}{2} \right\rfloor\right)} + \cdots + q^0$$

which thus matches the formula for $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)$.

Now suppose instead $v(e) > \frac{\theta}{2} + r$. First, a similar analysis gives that the first part of I^{6+} fits together with I^5 again. Indeed if

$$-v(b) - r \leq k \leq v(c) + r - 1$$

then in (10.1) we get the first exponent again, and hence we again get the fit

$$\sum_{j=0}^{\left\lfloor \frac{k+\theta-v(c)+r}{2} \right\rfloor} q^j + \left(q^{\min\left(\frac{\theta}{2}+r, \left\lfloor \frac{v(b)+r+k}{2} \right\rfloor\right)} + \cdots + q^{\max\left(0, \left\lceil \frac{k+\theta-v(c)+r+1}{2} \right\rceil\right)}\right)$$
$$= q^{\min\left(\frac{\theta}{2}+r, \left\lfloor \frac{v(b)+r+k}{2} \right\rfloor\right)} + \cdots + q^0$$

which matches the claimed formula for $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ in this range.

The remaining contribution from **Case 6$^+$ and Case 6$^-$** is

$$q^{\frac{\theta}{2}+r} \sum_{k=\theta-v(b)+r+1}^{2v(e)-\theta+v(c)-r-1} \mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)(-q^s)^k$$
$$+ q^{\frac{\theta}{2}+r} \sum_{k=v(c)+r+1}^{2v(e)-\theta+v(c)-r-1} \mathbf{1}_{k+\theta+v(c)+r \equiv 0 \bmod 2}(-q^s)^k.$$

The first sum matches the claimed coefficient $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ (except the summation in the theorem statement includes endpoints at $k = \theta - v(b) + r$ and $k = 2v(e) - \theta + v(c) - r$, but $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = 0$ at these two endpoints, so there is no change).

Meanwhile the second sum accounts for the discrepancy between the final term of (10.1) and the formula for $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$. That is, the range of k for which (10.1) achieves the last minimum is exactly

$$v(c)+r+1 \leq k \leq 2v(e)-\theta+v(c)-r-1$$

and only in those cases does (10.1) differs from $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ by exactly $\mathbf{1}_{k+\theta+v(c)+r\equiv 0 \bmod 2}$. This final step shows the claimed formulas coincide. □

Example 10.3.1 (The special case $v(e)=0$)**.** When $v(e)=0$ the expression is particularly simple. The assumption $v(d-a)\geq v(e)-r$ is automatically true, and $\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ is identically zero, so

$$\mathrm{Orb}((\gamma,\mathbf{u},\mathbf{v}^\top),\mathbf{1}_{K'_{S,\leq r}}\otimes\mathbf{1}_{O_F^n\times(O_F^n)^\vee},s)=\sum_{k=-(v(b)+r)}^{v(c)+r}(-q^s)^k.$$

Example 10.3.2. Suppose $r=14$, $v(b)=-5$, $v(c)=100$, $v(e)=3$. We have $v(d-a)\geq 0>-11=v(e)-r$. Hence the above formula reads

$$\begin{aligned}\mathrm{Orb}((\gamma,\mathbf{u},\mathbf{v}^\top),\mathbf{1}_{K'_{S,\leq 14}}\otimes\mathbf{1}_{O_F^n\times(O_F^n)^\vee},s)&=-q^{-9s}\\&+q^{-8s}\\&-(q+1)\cdot q^{-7s}\\&+(q+1)\cdot q^{-6s}\\&-(q^2+q+1)\cdot q^{-5s}\\&+(q^2+q+1)\cdot q^{-4s}\\&-(q^3+q^2+q+1)\cdot q^{-3s}\\&+(q^3+q^2+q+1)\cdot q^{-2s}\\&-(q^3+q^2+q+1)\cdot q^{-s}\\&+(q^3+q^2+q+1)\cdot q^{0}\\&-(q^3+q^2+q+1)\cdot q^{s}\end{aligned}$$

$$
\begin{aligned}
&+ (q^3+q^2+q+1)\cdot q^{2s}\\
&\vdots\\
&- (q^3+q^2+q+1)\cdot q^{111s}\\
&+ (q^3+q^2+q+1)\cdot q^{112s}\\
&- (q^3+q^2+q+1)\cdot q^{113s}\\
&+ (q^3+q^2+q+1)\cdot q^{114s}\\
&- (q^2+q+1)\cdot q^{115s}\\
&+ (q^2+q+1)\cdot q^{116s}\\
&- (q+1)\cdot q^{117s}\\
&+ (q+1)\cdot q^{118s}\\
&- q^{119s}\\
&+ q^{120s}.
\end{aligned}
$$

Example 10.3.3. Suppose $r = 2$, $v(b) = -5$, $v(c) = 100$, $v(e) = 20$, $v(d-a) = 1$. Then we have

$$
\begin{aligned}
\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq 2}} \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s) = &-q^{3s}\\
&+ q^{4s}\\
&- (q+1)\cdot q^{5s}\\
&+ (q+1)\cdot q^{6s}\\
&- (q^2+q+1)\cdot q^{7s}\\
&+ (q^2+q+1)\cdot q^{8s}\\
&- (q^3+q^2+q+1)\cdot q^{9s}\\
&+ (2q^3+q^2+q+1)\cdot q^{10s}\\
&- (3q^3+q^2+q+1)\cdot q^{9s}
\end{aligned}
$$

$$+ (4q^3 + q^2 + q + 1) \cdot q^{10s}$$

$$- (5q^3 + q^2 + q + 1) \cdot q^{9s}$$

$$+ (6q^3 + q^2 + q + 1) \cdot q^{10s}$$

$$\vdots$$

$$- (17q^3 + q^2 + q + 1) \cdot q^{25s}$$

$$+ (18q^3 + q^2 + q + 1) \cdot q^{26s}$$

$$- (18q^3 + q^2 + q + 1) \cdot q^{27s}$$

$$+ (18q^3 + q^2 + q + 1) \cdot q^{28s}$$

$$\vdots$$

$$- (18q^3 + q^2 + q + 1) \cdot q^{117s}$$

$$+ (18q^3 + q^2 + q + 1) \cdot q^{118s}$$

$$- (18q^3 + q^2 + q + 1) \cdot q^{119s}$$

$$+ (17q^3 + q^2 + q + 1) \cdot q^{120s}$$

$$- (16q^3 + q^2 + q + 1) \cdot q^{121s}$$

$$+ (15q^3 + q^2 + q + 1) \cdot q^{122s}$$

$$\vdots$$

$$+ (3q^3 + q^2 + q + 1) \cdot q^{134s}$$

$$- (2q^3 + q^2 + q + 1) \cdot q^{135s}$$

$$+ (q^3 + q^2 + q + 1) \cdot q^{136s}$$

$$- (q^2 + q + 1) \cdot q^{137s}$$

$$+ (q^2 + q + 1) \cdot q^{138s}$$

$$- (q + 1) \cdot q^{139s}$$

$$+ (q + 1) \cdot q^{140s}$$

$$- q^{141s}$$

$$+ q^{142s}.$$

10.4 Proof of Corollary 1.3.2

With Theorem 1.3.1 established, we aim to calculate the derivative of

$$\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K',S,\le r} \otimes \mathbf{1}_{O_F^n \times (O_F^n)^\vee}, s)$$

now at $s = 0$. Our goal is to prove Corollary 1.3.2:

Corollary 1.3.2 (Derivative at $s = 0$ for $S_2(F) \times V_2'(F)$)**.** *Retain the setting of Theorem 1.3.1. Also define $\varkappa := v(e) - (v(d-a) + r)$. If both $\varkappa \ge 0$ and $v(b) + v(c) > 2v(d-a)$, then we have the formula*

$$\begin{aligned}
&\frac{(-1)^{v(c)+r}}{\log q} \partial \,\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}}) \\
&\qquad = \sum_{j=0}^{N} \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j \\
&\qquad - q^{v(d-a)+r} \cdot \begin{cases} \frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r \right) - \frac{\varkappa}{2} & \text{if } \varkappa \equiv 1 \pmod 2. \end{cases}
\end{aligned}$$

Otherwise we instead have the formula

$$\frac{(-1)^{v(c)+r}}{\log q} \partial \,\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}}) = \sum_{j=0}^{N} \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j.$$

Proof of Corollary 1.3.2. For this calculation it will be more convenient to reformat Theorem 1.3.1 as a sum over q^j rather than $(-1)^k (q^s)^k$. To that, continuing to write

$$N := \min \left(v(e), \frac{v(b) + v(c) - 1}{2} + r, v(d-a) + r \right)$$

consider any index $0 \leq j \leq N$. Then

$$\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) \geq j \iff 2j - v(b) - r \leq k \leq 2v(e) + v(c) + r - 2j.$$

In other words, the first part of Corollary 1.3.3 can be rewritten as

$$\sum_{k=-(v(b)+r)}^{2v(e)+v(c)+r} (-1)^k \left(1 + \cdots + q^{\mathbf{n}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)}\right) (q^s)^k = q^j \sum_{j=0}^{N} \left(\sum_{k=2j-v(b)-r}^{2v(e)+v(c)+r-2j} (-q^s)^k \right).$$

If we take the derivative at $s = 0$ with respect to k, we get

$$\log q \sum_{j=0}^{N} \left(q^j \sum_{k=2j-v(b)-r}^{2v(e)+v(c)+r-2j} (-1)^k k \right).$$

The number of terms inside the summation is $2v(e) + v(b) + v(c) + 2r - 4j + 1$, an even number. Each consecutive pair differs by $(-1)^{v(c)+r}$. Hence we get

$$(-1)^{v(c)+r} \log q \sum_{j=0}^{N} \left(q^j \cdot \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \right).$$

Now in the case that $v(e) > v(d-a) + r$ and $2v(d-a) < v(b) + v(c)$, we have to handle the additional contribution obtained when we differentiate

$$q^{v(d-a)+r} \sum_{k=2v(d-a)-v(b)+r}^{2v(e)+v(c)-2v(d-a)-r} (-1)^k \mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k)(-q^s)^k \tag{10.2}$$

at $s = 0$. For brevity, we will define

$$\varkappa := v(e) - v(d-a) - r \geq 0.$$

(We allow the degenerate case $\kappa = 0$ for convenience, in which case $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}$ is still identically zero.)

We return to differentiating (10.2); to do so we reuse the (self-contained) extremely easy Lemma 7.7.2 from before. For the range of k given, we split it into three parts.

- For $-v(b) + r + 2v(d-a) \le k < -v(b) + v(e) + v(d-a)$, apply the first part of the lemma to conclude that the contribution to the derivative is

$$\log q \cdot \Big((-1)^{v(b)+v(e)+v(d-a)-1} \cdot \frac{(-v(b)+v(e)+v(d-a)-1)\cdot \varkappa}{2} \\ - \frac{(-1)^{v(b)+v(e)+v(d-a)+1} + (-1)^{v(b)+r}}{4} \cdot (-v(b)+r+2v(d-a)) \Big).$$

- For $v(c) + v(e) - v(d-a) < k \le 2v(e) + v(c) - 2v(d-a) - r$ apply the second part of the lemma to conclude that the contribution to the derivative is

$$\log q \cdot \Big((-1)^{v(c)+v(e)+v(d-a)+1} \cdot \frac{(v(c)+v(e)-v(d-a)+1)\cdot \varkappa}{2} \\ - \frac{(-1)^{v(c)+v(e)+v(d-a)+1} + (-1)^{v(c)+r}}{4} \cdot (2v(e)+v(c)-r-2v(d-a)) \Big).$$

- For the region $-v(b)+v(e)+v(d-a) \le k \le v(c)+v(e)-v(d-a)$, we have $\mathbf{c}_{(\gamma,\mathbf{u},\mathbf{v}^\top)}(k) = \varkappa$ and the values of k form $\left\lceil \frac{v(c)+v(b)-2v(d-a)}{2} \right\rceil$ consecutive pairs. So the contribution to the derivative here is exactly

$$\log q \cdot (-1)^{v(c)+v(e)+v(d-a)} \varkappa \cdot \frac{v(b)+v(c)-2v(d-a)+1}{2}.$$

If we sum all three, we get a contribution of $\varkappa \log q \cdot (-1)^{v(c)+v(e)+v(d-a)}$ times

$$\frac{v(b)+v(c)-2v(d-a)+1}{2} - \frac{v(c)+v(e)-v(d-a)+1}{2} \\ + \frac{-v(b)+v(e)+v(d-a)-1}{2} = -\frac{1}{2}$$

If $\varkappa$ is even, then that's all she wrote; we simply get

$$(-1)^{v(c)+r+1} \cdot \frac{\varkappa}{2} \cdot q^{v(d-a)+r} \log q.$$

On the other hand, if $\varkappa$ is odd we get instead

$$\begin{aligned} &(-1)^{v(c)+r} \frac{\varkappa}{2} q^{v(d-a)+r} \log q \\ -&\frac{1}{2} q^{v(d-a)+r} \log q \cdot (-1)^{v(c)+v(e)+v(d-a)} \cdot \Big((2v(e)+v(c)-r-2v(d-a)) \\ &\qquad\qquad\qquad - (-v(b)+r+2v(d-a)) \Big) \\ =& (-1)^{v(c)+r} q^{v(d-a)+r} \log q \cdot \left[- \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r \right) + \frac{\varkappa}{2} \right]. \end{aligned}$$

These formulas match Corollary 1.3.2, proving it. (Note that we changed $\varkappa > 0$ to $\varkappa \geq 0$, which makes no change since then the contribution is zero anyway.) $\square$

10.5 Proof of Corollary 1.3.3

From Corollary 1.3.2 we can now prove Corollary 1.3.3

Corollary 1.3.3 (The special case $\partial\,\mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq r}} + \mathbf{1}_{K'_{S,\leq (r-1)}})$). *Retain the setting of Theorem 1.3.1. Also define $\varkappa := v(e) - (v(d-a)+r)$. For $r \geq 1$ define*

$$C := \begin{cases} \frac{\varkappa-1}{2} & \text{if } \varkappa > 0 \text{ is odd and } v(b)+v(c) > 2v(d-a) \\ \frac{\varkappa+v(b)+v(c)-2v(d-a)-1}{2} & \text{if } \varkappa \geq 0 \text{ is even and } v(b)+v(c) > 2v(d-a) \\ v(e) - N & \text{if } v(e) \geq \frac{v(b)+v(c)-1}{2} + r \text{ and } 2v(d-a) > v(b)+v(c) \\ 0 & \text{otherwise} \end{cases}$$

$$C' := \begin{cases} C+1 & \text{if } \varkappa \geq 0 \text{ and } v(b)+v(c) > 2v(d-a) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\frac{(-1)^{v(c)+r}}{\log q}\partial\,\mathrm{Orb}((\gamma,\mathbf{u},\mathbf{v}^\top),\mathbf{1}_{K'_{S,\leq r}}+\mathbf{1}_{K'_{S,\leq(r-1)}})$$
$$=(q^N+q^{N-1}+\cdots+1)+Cq^N+C'q^{N-1}$$

Proof of Corollary 1.3.3. Observe that when we add the right-hand side of Corollary 1.3.2 to the same right-hand side with r replaced by $r-1$, almost all the terms cancel. Indeed, the main sum for $0\leq j\leq N-1$ line up:

$$(-1)^{r+v(c)}\sum_{j=0}^{N-1}q^j\cdot\left(\frac{2v(e)+v(b)+v(c)+1}{2}+r-2j\right)$$
$$(-1)^{(r-1)+v(c)}\sum_{j=0}^{N-1}q^j\cdot\left(\frac{2v(e)+v(b)+v(c)+1}{2}+(r-1)-2j\right)$$
$$=q^{N-1}+\cdots+q^0.$$

So we consider three cases:

- In the case where $v(d-a)+r>v(e)$ and $\frac{v(b)+v(c)-1}{2}+r>v(e)$ then the value of $N=v(e)$ does not change when we decrease r by one. Hence the term for $j=N$ cancels in both sums and we get exactly we get

$$(-1)^{r+v(c)}\log q(1+q+\cdots+q^N).$$

- Next suppose $v(e)\geq\frac{v(b)+v(c)-1}{2}+r$ and also $2v(d-a)>v(b)+v(c)$ (so the extra terms involving $\varkappa$ are absent). In that case $N=\frac{v(b)+v(c)-1}{2}+r$. Then we are left with $(-1)^{r+v(c)}\log q$ times

$$\sum_{j=0}^{N-1}(q^j)+\left(\frac{2v(e)+v(b)+v(c)+1}{2}+r-(v(b)+v(c)-1+2r)\right)q^N$$

$$= \sum_{j=0}^{N-1}(q^j) + \left(\frac{2v(e)-v(b)-v(c)-1}{2} - r\right) q^N$$
$$= \sum_{j=0}^{N}(q^j) + \left(v(e) - r - \frac{v(b)+v(c)-1}{2}\right) q^N.$$

This matches Theorem 1.3.1.

- Finally, suppose $\varkappa \geq 0$ and $v(b)+v(c) > 2v(d-a)$ so that $N = v(d-a)+r$. Then we are left with $(-1)^{r+v(c)}\log q$ times

$$\sum_{j=0}^{N-1}(q^j) + \left(\frac{2v(e)+v(b)+v(c)+1}{2} + r - 2(v(d-a)+r)\right) q^N$$
$$+ q^N \cdot \begin{cases} -\frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{\varkappa}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right) & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$
$$+ q^{N-1} \cdot \begin{cases} -\frac{\varkappa+1}{2} & \text{if } \varkappa+1 \equiv 0 \pmod 2 \\ \frac{\varkappa+1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - (r-1)\right) & \text{if } \varkappa+1 \equiv 1 \pmod 2 \end{cases}$$
$$= \sum_{j=0}^{N-1}(q^j) + q^N \cdot \begin{cases} \left(\frac{2v(e)+v(b)+v(c)+1}{2} + r - 2(v(d-a)+r)\right) - \frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{1}{2} + \frac{\varkappa}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$
$$- q^{N-1} \cdot \begin{cases} \frac{\varkappa-1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - (r-1)\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ -\frac{\varkappa+1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$
$$= \sum_{j=0}^{N}(q^j) + q^N \cdot \begin{cases} \left(\frac{2v(e)+v(b)+v(c)-1}{2} + r - 2(v(d-a)+r)\right) - \frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{\varkappa-1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$
$$- q^{N-1} \cdot \begin{cases} \frac{\varkappa-1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - (r-1)\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ -\frac{\varkappa+1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$

$$= \sum_{j=0}^{N}(q^j) + q^N \cdot \begin{cases} \left(\frac{v(e)+v(b)+v(c)-1}{2} - \frac{3}{2}v(d-a) - \frac{1}{2}r\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{\varkappa+1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$

$$- q^{N-1} \cdot \begin{cases} -\frac{1}{2} - \left(\frac{1}{2}v(e) + \frac{v(b)+v(c)}{2} - \frac{3}{2}v(d-a) - \frac{1}{2}r\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ -\frac{\varkappa+1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$

$$= \sum_{j=0}^{N}(q^j) + q^N \cdot \begin{cases} \left(\frac{\varkappa+v(b)+v(c)-1-2v(d-a)}{2}\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{\varkappa-1}{2} & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases}$$

$$+ q^{N-1} \cdot \begin{cases} \frac{\kappa+v(b)+v(c)+1-2v(d-a)}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ -\frac{\varkappa+1}{2} & \text{if } \varkappa \equiv 1 \pmod 2). \end{cases}$$

This matches Theorem 1.3.1, and the proof is complete. □

Chapter 11

Large kernel and large image

In this chapter we use Corollary 1.3.3 to prove both Theorem 1.3.5 and Theorem 1.3.6 for the orbital integral on $S_2(F) \times V_2'(F)$, We also comment on an analogous result for $S_3(F)$, although we do not provide all the details.

We assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is as in Lemma 8.4.2 throughout this chapter.

11.1 In the semi-Lie case, the kernel is trivial if we allow $v(e)$ to vary, for every fixed choice of γ

We prove Theorem 1.3.5 in this section. We treat γ as fixed, satisfying the requirements of Lemma 8.4.2, and we let

$$\theta \coloneqq \min\left(v(b) + v(c), 2v(d-a)\right) \geq 0$$

as we did in Chapter 9.

Lemma 11.1.1 (The matrix of $\partial \operatorname{Orb}$'s has full rank)**.** *Fix γ. Let $N \geq 0$ be a nonnegative integer. We define an $(N + \lfloor \frac{\theta}{2} \rfloor + 2) \times (N+1)$ matrix M as follows: for $0 \leq i \leq N + \lfloor \frac{\theta}{2} \rfloor + 1$*

and $0 \le r \le N$, *the* i^{th} *row and* r^{th} *column takes the value*

$$M_{i,r} := \frac{(-1)^r}{\log q} \partial \operatorname{Orb}\left(\left(\gamma, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & \varpi^i \end{pmatrix}\right), \mathbf{1}_{K', \le r}\right).$$

Then M *has full rank.*

The basic strategy of the proof will be to perform some sequences of row operations. Specifically, we introduce the following definition.

- For each $i = N + \left\lfloor \frac{\theta}{2} \right\rfloor, \ldots, 0$ in that order, subtract the i^{th} row of M from the $(i+1)^{\text{th}}$ row of M. Denote the new matrix as M'.
- For each $i = N + \left\lfloor \frac{\theta}{2} \right\rfloor - 1, \ldots, 0$ in that order, subtract the i^{th} row of M' from the $(i+2)^{\text{nd}}$ row of M'. Denote the new matrix as M''.

Then the basic premise is to show that M'' has an upper triangular submatrix. This is easier to see with some illustrations, which we give below.

Example 11.1.2. For example, for $N = 4$ and $v(b) + v(c) = 1$, $v(d-a) = 0$ (hence $\theta = 0$), we have

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & q+3 & 2q+4 & 3q+5 & 4q+6 \\ 2 & q+4 & q^2+3q+5 & 2q^2+4q+6 & 3q^2+5q+7 \\ 2 & 2q+5 & q^2+4q+6 & q^3+3q^2+5q+7 & 2q^3+4q^2+6q+8 \\ 3 & 2q+6 & 2q^2+5q+7 & q^3+4q^2+6q+8 & q^4+3q^3+5q^2+7q+9 \\ 3 & 3q+7 & 2q^2+6q+8 & 2q^3+5q^2+7q+9 & q^4+4q^3+6q^2+8q+10 \end{pmatrix}$$

hence

$$M' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & q+1 & 2q+1 & 3q+1 & 4q+1 \\ 1 & 1 & q^2+q+1 & 2q^2+q+1 & 3q^2+q+1 \\ 0 & q+1 & q+1 & q^3+q^2+q+1 & 2q^3+q^2+q+1 \\ 1 & 1 & q^2+q+1 & q^2+q+1 & q^4+q^3+q^2+q+1 \\ 0 & q+1 & q+1 & q^3+q^2+q+1 & q^3+q^2+q+1 \end{pmatrix}$$

and finally

$$M'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & q+1 & 2q+1 & 3q+1 & 4q+1 \\ 0 & -1 & q^2+q-2 & 2q^2+q-3 & 3q^2+q-4 \\ 0 & 0 & -q & q^3+q^2-2q & 2q^3+q^2-3q \\ 0 & 0 & 0 & -q^2 & q^4+q^3-2q^2 \\ 0 & 0 & 0 & 0 & -q^3 \end{pmatrix}.$$

Example 11.1.3. For example, for $N = 4$, $v(b) + v(c) = 17$, $v(d - a) = 2$ (hence $\theta = 4$), we have

$$M = \begin{pmatrix} 9 & 10 & 11 & \dots \\ 8q+10 & 9q+11 & 10q+12 & \dots \\ 7q^2+9q+11 & 8q^2+10q+12 & 9q^2+11q+13 & \dots \\ q^2+10q+12 & 7q^3+9q^2+11q+13 & 8q^3+10q^2+12q+14 & \dots \\ 8q^2+11q+13 & q^3+10q^2+12q+14 & 7q^4+9q^3+11q^2+13q+15 & \dots \\ 2q^2+12q+14 & 8q^3+11q^2+13q+15 & q^4+10q^3+12q^2+14q+16 & \dots \\ 9q^2+13q+15 & 2q^3+12q^2+14q+16 & 8q^4+11q^3+13q^2+15q+17 & \dots \\ 3q^2+14q+16 & 9q^3+13q^2+15q+17 & 2q^4+12q^3+14q^2+16q+18 & \dots \end{pmatrix}$$

hence

$$M' = \begin{pmatrix} 9 & 10 & 11 & 12 & \dots \\ 8q+1 & 9q+1 & 10q+1 & 11q+1 & \dots \\ 7q^2+q+1 & 8q^2+q+1 & 9q^2+q+1 & 10q^2+q+1 & \dots \\ -6q^2+q+1 & 7q^3+q^2+q+1 & 8q^3+q^2+q+1 & 9q^3+q^2+q+1 & \dots \\ 7q^2+q+1 & -6q^3+q^2+q+1 & 7q^4+q^3+q^2+q+1 & 8q^4+\cdots+1 & \dots \\ -6q^2+q+1 & 7q^3+q^2+q+1 & -6q^4+q^3+q^2+q+1 & 7q^5+\cdots+1 & \dots \\ 7q^2+q+1 & -6q^3+q^2+q+1 & 7q^4+q^3+q^2+q+1 & -6q^5+\cdots+1 & \dots \\ -6q^2+q+1 & 7q^3+q^2+q+1 & -6q^4+q^3+q^2+q+1 & 7q^5+\cdots+1 & \dots \end{pmatrix}$$

and finally

$$M'' = \begin{pmatrix} 9 & 10 & 11 & 12 & 13 \\ 8q+1 & 9q+1 & 10q+1 & 11q+1 & 12q+1 \\ 7q^2+q-8 & 8q^2+q-9 & 9q^2+q-10 & 10q^2+q-11 & 11q^2+q-12 \\ -6q^2-7q & 7q^3+q^2-8q & 8q^3+q^2-9q & 9q^3+q^2-10q & 10q^3+q^2-11q \\ 0 & -6q^3-7q^2 & 7q^4+q^3-8q^2 & 8q^4+q^3-9q^2 & 9q^4+q^3-10q^2 \\ 0 & 0 & -6q^4-7q^3 & 7q^5+q^4-8q^3 & 8q^5+q^4-9q^3 \\ 0 & 0 & 0 & -6q^5-7q^4 & 7q^6+q^5-8q^4 \\ 0 & 0 & 0 & 0 & -6q^6-7q^5 \end{pmatrix}.$$

Example 11.1.4. For example, for $N = 4$, $v(b) + v(c) = 5$, $v(d - a) = 8$ (hence $\theta = 5$), we have

$$M = \begin{pmatrix} 3 & 4 & 5 & \dots \\ 2q+4 & 3q+5 & 4q+6 & \dots \\ q^2+3q+5 & 2q^2+4q+6 & 3q^2+5q+7 & \dots \\ 2q^2+4q+6 & q^3+3q^2+5q+7 & 2q^3+4q^2+6q+8 & \dots \\ 3q^2+5q+7 & 2q^3+4q^2+6q+8 & q^4+3q^3+5q^2+7q+9 & \dots \\ 4q^2+6q+8 & 3q^3+5q^2+7q+9 & 2q^4+4q^3+6q^2+8q+10 & \dots \\ 5q^2+7q+9 & 4q^3+6q^2+8q+10 & 3q^4+5q^3+7q^2+9q+11 & \dots \\ 6q^2+8q+10 & 5q^3+7q^2+9q+11 & 4q^4+6q^3+8q^2+10q+12 & \dots \end{pmatrix}$$

hence

$$M' = \begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ 2q+1 & 3q+1 & 4q+1 & 5q+1 & 6q+1 \\ q^2+q+1 & 2q^2+q+1 & 3q^2+q+1 & 4q^2+q+1 & 5q^2+q+1 \\ q^2+q+1 & q^3+q^2+q+1 & 2q^3+q^2+q+1 & 3q^3+q^2+q+1 & 4q^3+q^2+q+1 \\ q^2+q+1 & q^3+q^2+q+1 & q^4+\cdots+1 & 2q^4+\cdots+1 & 3q^4+\cdots+1 \\ q^2+q+1 & q^3+q^2+q+1 & q^4+\cdots+1 & q^5+\cdots+1 & 2q^5+\cdots+1 \\ q^2+q+1 & q^3+q^2+q+1 & q^4+\cdots+1 & q^5+\cdots+1 & q^6+\cdots+1 \\ q^2+q+1 & q^3+q^2+q+1 & q^4+\cdots+1 & q^5+\cdots+1 & q^6+\cdots+1 \end{pmatrix}$$

and finally

$$M'' = \begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ 2q+1 & 3q+1 & 4q+1 & 5q+1 & 6q+1 \\ q^2+q-2 & 2q^2+q-3 & 3q^2+q-4 & 4q^2+q-5 & 5q^2+q-6 \\ q^2-q & q^3+q^2-2q & 2q^3+q^2-3q & 3q^3+q^2-4q & 4q^3+q^2-5q \\ 0 & q^3-q^2 & q^4+q^3-2q^2 & 2q^4+q^3-3q^2 & 3q^4+q^3-4q^2 \\ 0 & 0 & q^4-q^3 & q^5+q^4-2q^3 & 2q^5+q^4-3q^3 \\ 0 & 0 & 0 & q^5-q^4 & q^6+q^5-2q^4 \\ 0 & 0 & 0 & 0 & q^6-q^5 \end{pmatrix}.$$

Proof of Lemma 11.1.1. In order to prove M has full rank, it suffices to prove M'' has full rank. We now confirm the patterns shown by the example above.

By quoting Corollary 1.3.2 we will write

$$\begin{aligned} M_{i,r} = &\sum_{j=0}^{\min(i, r+\lfloor \frac{\theta}{2} \rfloor)} \left(i + \frac{v(b)+v(c)+1}{2} + r - 2j \right) q^j \\ &- \mathbf{1}_{\substack{\theta \equiv 0 \bmod 2 \\ i \geq r+\theta/2}} \cdot q^{v(d-a)+r} \cdot \left(\frac{i-r}{2} + t_i \right) \end{aligned}$$

where

$$t_i = \begin{cases} -\frac{v(d-a)}{2} & \text{if } i+r \equiv v(d-a) \pmod 2 \\ \frac{v(b)+v(c)-3v(d-a)}{2} & \text{if } i+r \not\equiv v(d-a) \pmod 2 \end{cases}$$

depends only on the parity of i. Hence for $i \geq 1$ we always have

$$\begin{aligned} M'_{i,r} &= M_{i,r} - M_{i-1,r} \\ &= -\mathbf{1}_{\substack{\theta \equiv 0 \bmod 2 \\ i \geq r+\theta/2}} \cdot q^{v(d-a)+r} \cdot \left(\frac{i-r}{2} + t_i \right) \\ &\quad + \mathbf{1}_{\substack{\theta \equiv 0 \bmod 2 \\ i-1 \geq r+\theta/2}} \cdot q^{v(d-a)+r} \cdot \left(\frac{i-1-r}{2} + t_{i-1} \right) \end{aligned}$$

$$+ \mathbf{1}_{i \le r + \lfloor \frac{\theta}{2} \rfloor} \cdot \left(\frac{v(b)+v(c)+1}{2} + r - i \right) q^i$$
$$+ \sum_{j=0}^{\min(i-1, r+\lfloor \frac{\theta}{2} \rfloor)} q^j.$$

From this we can make the following deductions on

$$M''_{i,r} = M'_{i,r} - M'_{i-2,r}$$

by cancelling most of the terms.

- If $i \ge r + \left\lfloor \frac{\theta}{2} \right\rfloor + 3$ then $M''_{i,r} = 0$ is clear.
- If $i = r + \left\lfloor \frac{\theta}{2} \right\rfloor + 2$ we contend that $M''_{i,r} = 0$ too.
  - When $\theta = v(b) + v(c)$ is odd, the surviving terms are

    $$-\left(\frac{v(b)+v(c)+1}{2} + r - (i-2) \right) q^{i-2} + q^{r+\lfloor \frac{\theta}{2} \rfloor}$$

    Substituting in $i = r + \frac{v(b)+v(c)-1}{2} + 2$ gives zero, as needed.
  - When $\theta = 2v(d-a)$ is even, the surviving terms are

$$\begin{aligned}
&- q^{v(d-a)+r} \cdot \left(\frac{i-r}{2} + t_i \right) + q^{v(d-a)+r} \cdot \left(\frac{(i-2)-r}{2} + t_{i-2} \right) \\
&\quad + q^{v(d-a)+r} \cdot \left(\frac{i-1-r}{2} + t_{i-1} \right) \\
&\quad - \left(\frac{v(b)+v(c)+1}{2} + r - (i-2) \right) q^{i-2} \\
&\quad + q^{r+\lfloor \frac{\theta}{2} \rfloor} \\
&= q^{v(d-a)+r} \cdot \left(-1 + \frac{i-1-r}{2} + t_{i-1} - \frac{v(b)+v(c)+1}{2} - r + i - 2 + 1 \right) \\
&= q^{v(d-a)+r} \cdot \left(-\frac{3}{2} r + t_{i-1} - \frac{v(b)+v(c)}{2} + \frac{3}{2} i \right)
\end{aligned}$$

Substituting in $i = r + v(d-a) + 2$ (and since $t_i = t_{i-2}$), we also get exactly 0, since $t_{i-1} = \frac{v(b)+v(c)-3v(d-a)}{2}$.

- If $i = r + \left\lfloor \frac{\theta}{2} \right\rfloor + 1$, we consider again cases on the parity of θ.
  - When $\theta = v(b) + v(c)$ is odd, the surviving terms are

$$-\left(\frac{v(b)+v(c)+1}{2} + r - (i-2)\right) q^{i-2} + q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} + q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1} = q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} - q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1}.$$

  - When $\theta = 2v(d-a)$ is even, the surviving terms are

$$\begin{aligned}
&- q^{v(d-a)+r} \cdot \left(\frac{i-r}{2} + t_i\right) \\
&\quad + q^{v(d-a)+r} \cdot \left(\frac{i-1-r}{2} + t_{i-1}\right) \\
&\quad - \left(\frac{v(b)+v(c)+1}{2} + r - (i-2)\right) q^{i-2} \\
&\quad + q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} + q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1} \\
&= \left(t_{i-1} - t_i + \frac{1}{2}\right) q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} \\
&\quad - \left(\frac{v(b)+v(c)+1}{2} + r - (i-2) - 1\right) q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1} \\
&= \left(-\frac{v(d-a)}{2} - \frac{v(b)+v(c)-3v(d-a)}{2} + \frac{1}{2}\right) q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} \\
&\quad - \left(\frac{v(b)+v(c)+1}{2} + r - ((r+v(d-a)+1)-2) - 1\right) q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1} \\
&= -\frac{v(b)+v(c)-1-2v(d-a)}{2} q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} \\
&\quad - \frac{v(b)+v(c)+1-2v(d-a)}{2} q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1}.
\end{aligned}$$

    In the edge case where $r = 0$ and $\theta \leq 1$, it can be checked the same formula still holds with the last term omitted.

Hence we have found a diagonal of M'' below with all entries are zero, and on which all entries are nonzero except possibly $M''_{1,0} = 0$ in the case where $r = 0$ and $\theta \leq 1$.

Assuming $r+\lfloor\frac{\theta}{2}\rfloor \geq 2$, suppose we take the rows from $i = r+\lfloor\frac{\theta}{2}\rfloor+1$ up to $i = r+\lfloor\frac{\theta}{2}\rfloor+N+1$. Then the resulting matrix is upper triangular. The determinant is the product of the diagonal entries; up to multiplication by sign and a power of q, it equals and the determinant is equal to

$$\begin{cases} (q-1)^{N+1} & \text{if } \theta \equiv 1 \pmod 2 \\ \left(\frac{v(b)+v(c)-1-2v(d-a)}{2}q + \frac{v(b)+v(c)+1-2v(d-a)}{2}\right)^{N+1} & \text{if } \theta \equiv 0 \pmod 2 \end{cases}$$

which is manifestly nonzero for any odd prime power q.

In the situation where $r + \lfloor\frac{\theta}{2}\rfloor = 1$, we use the same rows except that we replace the row for $i = 1$ with the row for $i = 0$, which has leftmost entry $M_{0,0} = \frac{v(b)+v(c)+1}{2} > 0$. Hence the same proof still shows that the determinant is nonzero. $\square$

11.1.1 Proof of Theorem 1.3.5

We can now deduce:

Theorem 1.3.5 ($\partial\operatorname{Orb}$ is injective even for fixed $\gamma \in S_2(F)$)**.** *Fix any $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$. Then there doesn't exist any nonzero function $\phi \in \mathcal{H}(S_2(F))$ such that*

$$\partial\operatorname{Orb}\left((\gamma, \mathbf{u}, \varpi^i\mathbf{v}^\top), \phi\right) = 0$$

holds for every integer i. Thus Conjecture 1.2.4 holds for $n = 2$.

Proof of Theorem 1.3.5. Suppose we are given some function

$$\phi = \sum_{r=0}^{N} (-1)^r c_r \mathbf{1}_{K'_{S,\leq r}} \in \mathcal{H}(S_2(F)).$$

Letting M be the matrix given in Lemma 11.1.1, we are supposed to have

$$M \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix} = \mathbf{0}.$$

Since M has full rank, it follows that $c_0 = \cdots = c_N = 0$. $\square$

11.2 In the semi-Lie case, the kernel has finite codimension for fixed $v(e)$

We prove Theorem 1.3.6 in this section. We start with the following lemma.

Lemma 11.2.1 (A combination vanishing for large r)**.** *Let* $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$. *If* $r \geq v(e) + 2$, *we have*

$$\partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\leq r}} + 2\mathbf{1}_{K'_{S,\leq (r-1)}} + \mathbf{1}_{K'_{S,\leq (r-2)}}\right) = 0.$$

Proof. This follows directly from Corollary 1.3.3 which gives

$$\begin{aligned}\sum_{j=0}^{v(e)} q^j &= \frac{(-1)^r}{\log q} \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \left(\mathbf{1}_{K'_{S,\leq r}} + \mathbf{1}_{K'_{S,\leq (r-1)}}\right)\right) \\ &= \frac{(-1)^{r-1}}{\log q} \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \left(\mathbf{1}_{K'_{S,\leq (r-1)}} + \mathbf{1}_{K'_{S,\leq (r-2)}}\right)\right).\end{aligned}$$

$\square$

We need one more lemma:

Lemma 11.2.2. *There is no* $Y \in \mathbb{C}^\times$ *such that*

$$q^3(Y^r + Y^{-r}) - 3q^2(Y^{r-1} + Y^{-(r-1)}) + 3q(Y^{r-2} + Y^{-(r-2)}) - (Y^{r-3} + Y^{-(r-3)}) = 0$$

holds for all sufficiently large integers r.

Proof. Assume for contradiction such a $Y \in \mathbb{C}^\times$ existed. Let $Y^{\pm k} := Y^k + Y^{-k}$ for brevity for every integer $k \geq 1$. By writing the recursion relations

$$\begin{aligned} Y^{\pm(r-1)} &= Y^{\pm 1} \cdot Y^{\pm(r-2)} - Y^{\pm(r-3)} \\ Y^{\pm r} &= Y^{\pm 1} \cdot Y^{\pm(r-1)} - Y^{\pm(r-2)} \\ &= ((Y^{\pm 1})^2 - 1) \cdot Y^{\pm(r-2)} - Y^{\pm 1} \cdot Y^{\pm(r-3)} \end{aligned}$$

we can deduce that

$$\begin{aligned} 0 &= q^3 \cdot Y^{\pm r} - 3q^2 \cdot Y^{\pm(r-1)} + 3q \cdot Y^{\pm(r-2)} - Y^{\pm(r-3)} \\ &= \left[q^3 \cdot \left((Y^{\pm 1})^2 - 1\right) - 3q^2 \cdot Y^{\pm 1} + 3q\right] \cdot Y^{\pm(r-2)} - \left[q^3 \cdot Y^{\pm 1} - 3q^2 + 1\right] \cdot Y^{\pm(r-3)}. \end{aligned}$$

Now, in general there is no complex number $Y \in \mathbb{C}^\times$ such that $Y^r + Y^{-r} = 0$ for two consecutive values of r. Hence, if either bracketed coefficient is zero, then so must be the other one. However, in that case, we would conclude that $Y^{\pm 1} = \frac{3q^2-1}{q^3}$ from the second bracketed coefficient, meaning

$$0 = q^3 \cdot \left(\left(\frac{3q^2-1}{q^3}\right)^2 - 1\right) - 3q^2 \cdot \frac{3q^2-1}{q^3} + 3q = \frac{(q^2-1)^3}{q^3}$$

which is a contradiction, because $q > 1$.

Hence neither bracketed coefficient can be zero, from which we conclude that there is some nonzero constant c such that

$$Y^{\pm(r-2)} = c \cdot Y^{\pm(r-3)} \neq 0$$

holds for all large r. But then

$$c \cdot Y^{\pm(r-3)} = Y^{\pm 1} \cdot Y^{\pm(r-3)} - Y^{\pm(r-4)} = Y^{\pm 1} \cdot Y^{\pm(r-3)} - \frac{Y^{\pm(r-3)}}{c}$$

and hence $c = Y^{\pm 1} - \frac{1}{c}$. So either $c = Y$ or $c = \frac{1}{Y}$. Then from $Y^{\pm(r-2)} = c \cdot Y^{\pm(r-3)}$ we derive that $Y = \pm 1$.

But substituting $Y = 1$ in the original equation would imply $(q-1)^3 = 0$ while $Y = -1$ would imply $(q+1)^3 = 0$ neither of which is possible. This contradiction completes the proof of the lemma. $\square$

11.2.1 Proof of Theorem 1.3.6

We now prove:

Theorem 1.3.6 (The kernel of $\partial\,\mathrm{Orb}$ is large for fixed $(\mathbf{u}, \mathbf{v}^\top) \in V_2'(F)$)**.** *Let $N \geq 0$ be an integer. Consider all $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$ for which $v(\mathbf{u}\mathbf{v}^\top) \leq N$. Then the space of $\phi \in \mathcal{H}(S_2(F))$ for which*

$$\partial\,\mathrm{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi\right) = 0$$

holds for all such $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is a $\mathbb{Q}$-vector subspace of $\mathcal{H}(S_2(F))$ whose codimension is at most $N + 2$.

Moreover, this subspace of $\mathcal{H}(S_2(F))$ is not contained in any maximal ideal of $\mathcal{H}(S_2(F))$ when $\mathcal{H}(S_2(F))$ is viewed as a ring under the isomorphism of Chapter 4.

Proof of Theorem 1.3.6. The first part of Theorem 1.3.6 follows directly from Lemma 11.2.1.

It remains to show the kernel is not contained in any maximal ideal. Consider the composed isomorphism from Chapter 4 given by

$$\mathcal{H}(S_2(F)) \xrightarrow{\mathrm{BC}_S^\eta} \mathcal{H}(\mathrm{U}(\mathbb{V}_2^+)) \xrightarrow{\mathrm{Sat}} \mathbb{Q}[Y + Y^{-1}].$$

By combining [LRZ24, Equation (7.1.9)] (which is also Lemma 15.3.1 later) and [LRZ24, Equation (7.1.4)] we find that

$$\begin{aligned}\operatorname{Sat}\left(\mathrm{BC}_S^\eta\left(\mathbf{1}_{K',\le r}+\mathbf{1}_{K',\le(r-1)}\right)\right)&=(-1)^r\operatorname{Sat}(\mathbf{1}_{\varpi^{-r}\operatorname{Mat}_2(O_E)\cap\mathbb{V}_n^+})\\&=(-1)^r\left(q^r\sum_{j=-r}^{r}Y^j-q^{r-1}\sum_{j=-(r-1)}^{r-1}Y^j\right).\end{aligned}$$

Hence, if we define the polynomial

$$\begin{aligned}P_r(Y)&\coloneqq(-1)^r\operatorname{Sat}\left(\mathrm{BC}_S^\eta\left(\mathbf{1}_{K',\le r}+2\mathbf{1}_{K',\le(r-1)}+\mathbf{1}_{K',\le(r-2)}\right)\right)\\&=\left(q^r\sum_{j=-r}^{r}Y^j-q^{r-1}\sum_{j=-(r-1)}^{r-1}Y^j\right)-\left(q^{r-1}\sum_{j=-(r-1)}^{r-1}Y^j-q^{r-2}\sum_{j=-(r-2)}^{r-2}Y^j\right)\\&=q^r\sum_{j=-r}^{r}Y^j-2q^{r-1}\sum_{j=-(r-1)}^{r-1}Y^j+q^{r-2}\sum_{j=-(r-2)}^{r-2}Y^j\end{aligned}$$

then for any $r\ge N+2$, all the polynomials $P_r(Y)-P_{N+2}(Y)$ lie in the kernel.

We now prove there is no choice of a single $Y\in\mathbb{C}^\times$ for which $P_r(Y)$ is eventually constant, which would complete the proof. Indeed, if we write

$$P_r(Y)-qP_{r-1}(Y)=q^r(Y^r+Y^{-r})-2q^{r-1}(Y^{r-1}+Y^{-(r-1)})+q^{r-2}(Y^{r-2}+Y^{-(r-2)})$$

then

$$\begin{aligned}(P_r(Y)-qP_{r-1}(Y))-(P_{r-1}(Y)-qP_{r-2}(Y))&=q^3(Y^r+Y^{-r})-3q^2(Y^{r-1}+Y^{-(r-1)})\\&\quad+3q(Y^{r-2}+Y^{-(r-2)})-(Y^{r-3}+Y^{-(r-3)}).\end{aligned}$$

So we are done by Lemma 11.2.2. $\square$

11.3 A sequence of test functions almost lying in the kernel in the semi-Lie case

We make one additional remark on the kernel that unifies both of the preceding two sections. For this section, we define the following indicator function for $r \geq 3$:

$$\phi_r \coloneqq \mathbf{1}_{K',\leq r} + \mathbf{1}_{K',\leq(r-1)} - q^2(\mathbf{1}_{K',\leq(r-2)} + \mathbf{1}_{K',\leq(r-3)}).$$

We give the following theorem which can be thought of as a simultaneously refined version of both Lemma 11.1.1 and Lemma 11.2.1. Roughly, it says that we can define a sequence of test functions

$$\phi_r + (q+1)\phi_{r-1} + q\phi_{r-2} \qquad r \geq 5$$

such that for any fixed $(\gamma, \mathbf{u}, \mathbf{v}^\top)$, there are at most three values of r for which the orbital integral does not vanish.

Theorem 11.3.1 (A sequence in $\mathcal{H}(S_2(F))$)**.** *Suppose $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}$ is as in Lemma 8.4.2. Then*

$$\partial \operatorname{Orb}\big((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi_r + (q+1)\phi_{r-1} + q\phi_{r-2}\big) = 0$$

holds for all $r \geq 5$ with at most three exceptions, namely those r with

$$v(e) - \min\left(\frac{v(b)+v(c)-1}{2}, v(d-a)\right) + 2 \leq r \leq v(e) - \min\left(\frac{v(b)+v(c)-1}{2}, v(d-a)\right) + 4.$$

Proof. As always $\theta \coloneqq \min\left(v(b)+v(c), 2v(d-a)\right)$ as in Chapter 9. Consider Corollary 1.3.3, and let N, C, C' be as in the statement. Let $N^{\flat\flat}$, $C^{\flat\flat}$, $(C')^{\flat\flat}$ be the changes to those constants when one replaces r by $r-2$. Then one can record the changes to these parameters explicitly, see Table 11.1.

Assumptions	Parameters for r	Parameters for $r-2$
$r \geq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 3$	$N = v(e)$ $C = C' = 0$	$N^{\flat\flat} = v(e)$ $C^{\flat\flat} = (C')^{\flat\flat} = 0$
$r = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 2$ $\theta = 2v(d-a)$ (exceptional)	$N = v(e)$ $C = C' = 0$	$N^{\flat\flat} = (r-2) + \left\lfloor \frac{\theta}{2} \right\rfloor = v(e)$ $C^{\flat\flat} = \frac{-1+(v(b)+v(c)-2v(d-a))}{2}$ $(C')^{\flat\flat} = C^{\flat\flat} + 1$
$r = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 1$ $\theta = 2v(d-a)$	$N = v(e)$ $C = C' = 0$	$N^{\flat\flat} = (r-2) + \left\lfloor \frac{\theta}{2} \right\rfloor = v(e) - 1$ $C^{\flat\flat} = 0$ $(C')^{\flat\flat} = 1$
$r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor$ $\theta = 2v(d-a)$ $\varkappa \equiv 0 \pmod 2$	$N = r + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C = \frac{\varkappa-1+(v(b)+v(c)-2v(d-a))}{2}$ $C' = C + 1$	$N^{\flat\flat} = (r-2) + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C^{\flat\flat} = \frac{(\varkappa-2)-1+(v(b)+v(c)-2v(d-a))}{2}$ $(C')^{\flat\flat} = C^{\flat\flat} + 1$
$r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor$ $\theta = 2v(d-a)$ $\varkappa \equiv 1 \pmod 2$	$N = r + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C = \frac{\varkappa-1}{2}$ $C' = C + 1$	$N^{\flat\flat} = (r-2) + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C^{\flat\flat} = \frac{(\varkappa-2)-1}{2}$ $(C')^{\flat\flat} = C^{\flat\flat} + 1$
$r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor$ $\theta = v(b) + v(c)$	$N = r + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + r$ $C' = 0$	$N^{\flat\flat} = (r-2) + \left\lfloor \frac{\theta}{2} \right\rfloor$ $C^{\flat\flat} = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + (r-2)$ $(C')^{\flat\flat} = 0$

Table 11.1: Comparison of N to $N^{\flat\flat}$, etc., needed to carry out the proof of Theorem 11.3.1. Note the exceptional case $r = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 2$ differs from all the others because $C - C^{\flat\flat}$ can be large.

Note in particular except for the single exceptional value $r = v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 2$ we should always have $(N^{\flat\flat} + 2) - N \in \{0, 1, 2\}$, $C - C^{\flat\flat} \in \{0, 1, 2\}$, $(C') - (C')^{\flat\flat} \in \{0, 1\}$. More explicitly, we have the following result from Table 11.1 for every $r \geq 3$:

$$\frac{(-1)^{r+v(c)}}{\log q} \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi_r\right)$$
$$= \begin{cases} -q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} - q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor - 1} + q + 1 & \text{if } r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 1 \text{ and } \theta = v(b) + v(c) \\ -2q^{r+\left\lfloor \frac{\theta}{2} \right\rfloor} + q + 1 & \text{if } r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 1 \text{ and } \theta = 2v(d-a) \\ -q^{v(e)+2} - q^{v(e)+1} + q + 1 & \text{if } r \geq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 3. \end{cases}$$

It follows that for $r \geq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 4$ we have

$$\frac{(-1)^{r+v(c)}}{\log q} \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi_r + \phi_{r-1}\right) = 0 \tag{11.1}$$

while when $r \leq v(e) - \left\lfloor \frac{\theta}{2} \right\rfloor + 1$ we have

$$\frac{(-1)^{r+v(c)}}{\log q} \partial \operatorname{Orb}\left((\gamma, \mathbf{u}, \mathbf{v}^\top), \phi_r + q\phi_{r-1}\right) = 1 - q^2. \tag{11.2}$$

Then Theorem 11.3.1 follows directly from (11.1) and (11.2). □

11.4 Proof of Theorem 1.3.11 for the group AFL

We give the short proof of the following:

Theorem 1.3.11 ($\partial \operatorname{Orb} \colon \mathcal{H}(S_3(F)) \to \mathcal{C}^\infty(S_3(F)^-_{\mathrm{rs}})$ has large image)**.** *There is no nontrivial $\phi \in \mathcal{H}(S_3(F))$ such that*

$$\partial \operatorname{Orb}(\gamma, \phi) = 0$$

holds for every $\gamma \in S_3(F)^-_{\mathrm{rs}}$*. In other words, the map*

$$\begin{aligned}\partial \operatorname{Orb}\colon \mathcal{H}(S_3(F)) &\to \mathcal{C}^\infty(S_3(F)^-_{\mathrm{rs}}) \\ \phi &\mapsto (\gamma \mapsto \partial \operatorname{Orb}(\gamma, \phi))\end{aligned}$$

is injective, i.e., has image as large as possible, for $n = 3$*.*

Proof of Theorem 1.3.11. Suppose that

$$\phi = \sum_{r=0}^{N} a_j \mathbf{1}_{K'_{S,\le r}}$$

for some $N > 0$. We pick our γ according to Lemma 5.3.3 subject to

$$v(b) = v(d) = -N.$$

In that case, we have

$$\operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\le r}}, s) = 0 \text{ for all } r = 0, 1, \dots, N-1$$

and so there is certainly no contribution to its derivative. However, for a_N according to Example 5.5.11 we have

$$\partial \operatorname{Orb}(\gamma, \mathbf{1}_{K'_{S,\le N}}) = \frac{\partial}{\partial s}\bigg|_{s=0} \left(\sum_{k=-r}^{\lambda-2r} (-1)^k (q^s)^k \right) = (-1)^{r+1} \cdot \frac{\lambda+1}{2} \neq 0.$$

Hence $N > 0$ cannot hold.

For $N = 0$, we instead take, say, $\ell = \delta = 1$ in Theorem 5.5.2 in which case $\operatorname{Orb}(\gamma, \mathbf{1}_{K'_S}, s) = -q^{3s} + q^{2s} - q^s + 1$ whose derivative is -2. The proof is complete. □

11.5 A sequence of test functions almost lying in the kernel in the group AFL case

We mention the following $S_3(F)$ analog of Theorem 11.3.1.

Theorem 11.5.1 (A sequence in $\mathcal{H}(S_3(F))$). *Choose $\gamma \in S_3(F)^-_{\mathrm{rs}}$ and let ℓ be as in Lemma 5.3.3. If $\ell \geq 0$, then for any $r \geq 3$,*

$$\partial \operatorname{Orb}\left(\gamma, \mathbf{1}_{K'_{S,r}} + 2q\mathbf{1}_{K'_{S,r-1}} + q^2\mathbf{1}_{K'_{S,r-2}}\right) = -(2q+2)\log q.$$

The result also holds for $\ell < 0$ if $r \geq -\frac{1}{2}\ell + 3$; while for $r < -\frac{1}{2}\ell$ the left-hand is 0 instead.

To prove it, we first state the following corollaries of Lemma 7.7.1 and Lemma 7.7.3, respectively.

Corollary 11.5.2. *Retain the setting of Lemma 7.7.1. Also let $\Sigma'_r(s) = \Sigma_r(s) - \Sigma_{r-1}(s)$ for all $r \geq 1$. Then for all $r \geq 3$,*

$$\frac{\partial}{\partial s}\left(\Sigma'_r(s) + 2q\Sigma'_{r-1}(s) + q^2\Sigma'_{r-2}(s)\right)\bigg|_{s=0} = (-1)^{C+1}(2q+2)\log q.$$

Proof. From Lemma 7.7.1 it follows that for any $r \geq 1$ we have

$$-\frac{\partial}{\partial s}\left(\Sigma_r(s) + q\Sigma_{r-1}(s)\right)\bigg|_{s=0} = (-1)^r \sum_{j=r+1}^{r+H} q^j + \sum_{j=1}^{r}(-1)^j q^j + \frac{W+1}{2} + 2r.$$

Hence it follows that for any $r \geq 2$,

$$-\frac{\partial}{\partial s}\left(\Sigma_r(s) + q\Sigma_{r-1}(s) + q\left(\Sigma_r(s) + q\Sigma_{r-1}(s)\right)\right)\bigg|_{s=0} = \left(\frac{W-3}{2} + 2r\right)q + \frac{W+1}{2} + 2r.$$

Subtracting this from the same expression for $r-1$ yields the corollary. $\square$

Corollary 11.5.3. *Retain the setting of Lemma 7.7.3. Also let* $\Sigma'_r(s) = \Sigma_r(s) - \Sigma_{r-1}(s)$ *for all* $r \geq 1$*. Then for all* $r \geq 2$*,*

$$\frac{\partial}{\partial s}\left(\Sigma'_r(s) + \Sigma'_{r-1}(s)\right)\bigg|_{s=0} = 0.$$

Proof. Since Lemma 7.7.3 implies $\Sigma'_r(s) = -\frac{L-(-1)^W}{2}(-1)^{r+W+C}$ this is immediate. □

Proof of Theorem 11.5.1. Assume the notation in Lemma 5.3.3 and Lemma 5.4.1. We apply Corollary 11.5.2 and Corollary 11.5.3 directly using the same inputs as the proof of Theorem 5.6.1.

For $\ell \geq 0$, the $\mathbf{c}_\gamma$ sum (when ℓ is even) has no effect, while by Corollary 11.5.2 we always get $-(2q+2)\log q$ (as we only use C even in Corollary 11.5.2).

When $\ell < 0$, if $r \geq -\frac{1}{2}\ell + 3$ then the replacement of r by $r - |v(d)| \geq 3$ allows the analysis to carry through. And where $r < -\frac{1}{2}\ell$, the entire orbital integral is identically zero anyways, as needed. □

11.5.1 Generation of $\mathcal{H}(S_3(F))$ as an ideal

In the spirit of the conjecture proposed in [LRZ24, Conjecture 1.0.2], we mention the following result as well.

Proposition 11.5.4. *The subspace of* $\mathcal{H}(S_3(F))$ *spanned by*

$$\mathbf{1}_{K'_{S,r}} + 2q\mathbf{1}_{K'_{S,r-1}} + \mathbf{1}_{K'_{S,r-2}}$$

for $r \geq 3$ *is (in addition to being of codimension at most* 3*) not contained in any maximal ideal of* $\mathcal{H}(S_3(F))$*.*

Proof. This requires us to invoke the explicit results from Chapter 4. Consider the composed isomorphisms

$$\mathcal{H}(S_3(F)) \xrightarrow{\mathrm{BC}_S} \mathcal{H}(\mathrm{U}(\mathbb{V}_3^+)) \xrightarrow{\mathrm{Sat}} \mathbb{Q}[Y + Y^{-1}].$$

Let $K = \mathrm{GL}_n(O_E) \cap \mathrm{U}(\mathbb{V}_3^+)$ and note for any $r \geq 2$ we have

$$\begin{aligned}
\mathbf{1}_{K\varpi^{(r,0,-r)}K} &\xmapsto{\mathrm{BC}_S^{-1}} \mathbf{1}_{K'_{S,r}} + 2q\mathbf{1}_{K'_{S,r-1}} + 2q^2\mathbf{1}_{K'_{S,r-2}} + 2q^3\mathbf{1}_{K'_{S,r-3}} + \dots \\
\implies \mathbf{1}_{K\varpi^{(r,0,-r)}K} - q^2\mathbf{1}_{K\varpi^{(r-2,0,-(r-2))}K} &\xmapsto{\mathrm{BC}_S^{-1}} \mathbf{1}_{K'_{S,r}} + 2q\mathbf{1}_{K'_{S,r-1}} + q^2\mathbf{1}_{K'_{S,r-2}}.
\end{aligned}$$

Hence for $r \geq 3$ if we define

$$\begin{aligned}
P_r(Y) &:= \mathrm{Sat}\left(\mathrm{BC}_S(\mathbf{1}_{K'_{S,r}} + 2q\mathbf{1}_{K'_{S,r-1}} + q^2\mathbf{1}_{K'_{S,r-2}})\right) \\
&= \mathrm{Sat}\left(\mathbf{1}_{K\varpi^{(r,0,-r)}K} - q^2\mathbf{1}_{K\varpi^{(r-2,0,-(r-2))}K}\right) \\
&= \mathrm{Sat}\Big(\mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_3(O_E)\cap\mathrm{U}(\mathbb{V}_3^+)} - \mathbf{1}_{\varpi^{-(r-1)}\,\mathrm{Mat}_3(O_E)\cap\mathrm{U}(\mathbb{V}_3^+)} \\
&\qquad - q^2\mathbf{1}_{\varpi^{-(r-2)}\,\mathrm{Mat}_3(O_E)\cap\mathrm{U}(\mathbb{V}_3^+)} + q^2\mathbf{1}_{\varpi^{-(r-3)}\,\mathrm{Mat}_3(O_E)\cap\mathrm{U}(\mathbb{V}_3^+)}\Big) \\
&= \left(q^{2r}\sum_{j=-r}^{r} Y^j + q^{2r-1}\sum_{j=-(r-1)}^{r-1} Y^j\right) - \left(q^{2r-2}\sum_{j=-(r-1)}^{r-1} Y^j + q^{2r-3}\sum_{j=-(r-2)}^{r-2} Y^j\right) \\
&\qquad - \left(q^{2r-2}\sum_{j=-(r-2)}^{r-2} Y^j + q^{2r-3}\sum_{j=-(r-3)}^{r-3} Y^j\right) + \left(q^{2r-4}\sum_{j=-(r-3)}^{r-3} Y^j + q^{2r-5}\sum_{j=-(r-4)}^{r-4} Y^j\right)
\end{aligned}$$

then it follows that $P_r(Y) - P_3(Y)$ is contained in the kernel for any $r \geq 3$.

To show this kernel generates the entire ring, it would be sufficient to prove there is no $Y \in \mathbb{C}^\times$ such that $P_3(Y) = P_4(Y) = P_5(Y) = P_6(Y) = \cdots$. However, using the explicit formula for $P_r(Y)$ above, a direct calculation gives the following two identities:

$$\begin{aligned}
P_5(Y) - \frac{q^2}{Y}P_4(Y) - q^2YP_4(Y) - \frac{q^2}{Y}P_3(Y) = -q^5 \\
P_6(Y) - \frac{q^2}{Y}P_5(Y) - q^2YP_5(Y) - \frac{q^2}{Y}P_4(Y) = 0.
\end{aligned}$$

So such a common root Y cannot exist. $\square$

Chapter 12

Transfer factors

In this short chapter we document the definitions of the transfer factors appearing in Conjecture 1.2.1 and Conjecture 1.2.2.

12.1 Transfer factor for the inhomogeneous group AFL

The definition of the transfer factor in Conjecture 1.2.1 is given below:

Definition 12.1.1 ([Zha24c, Equation 2.7])**.** Choose any $\gamma \in S_n(F)_{\mathrm{rs}}$. Let $\mathbf{e} = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^\top \in F^n$ be a column vector. Then the transfer factor $\omega(\gamma)$ is defined by

$$\omega(\gamma) := \eta\left(\det\left(\left(\gamma^i \mathbf{e}\right)_{i=0}^{n-1}\right)\right).$$

For $n = 3$, because we gave our answer in terms of a representative of an H'-orbit, it is not trivial to state the transfer factor in terms of the a, b, d in Lemma 5.3.3.

However, we don't need this transfer factor anyway in this paper, so we do not evaluate it here.

12.2 Transfer factor for the semi-Lie AFL

Definition 12.2.1 ([Zha24c, Equation 2.2])**.** Choose any $(\gamma, \mathbf{u}, \mathbf{v}^\top) \in (S_n(F) \times V_n'(F))_{\mathrm{rs}}$. Then the transfer factor $\omega(\gamma)$ is defined by

$$\omega(\gamma, \mathbf{u}, \mathbf{v}^\top) \coloneqq \eta\left(\det\left(\left(\gamma^i \mathbf{u}\right)_{i=0}^{n-1}\right)\right).$$

For the purposes of our $n = 2$ calculation, we compute the transfer factor when

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix}\right) \in (S_2(F) \times V_2'(F))^-_{\mathrm{rs}}.$$

is as described in Lemma 8.4.2. Applying the definition above, we find that

$$\omega(\gamma, \mathbf{u}, \mathbf{v}^\top) = \eta\left(\det\left(\gamma^0 \mathbf{u}, \gamma^1 \mathbf{u}\right)\right) = \eta\left(\det\begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}\right) = (-1)^{v(b)} = (-1)^{v(c)+1}. \tag{12.1}$$

Chapter 13

The geometric side

13.1 Rapoport-Zink spaces

We briefly recall the theory of Rapoport-Zink spaces. This follows the exposition in [Zha24a, §4.1].

Let $\breve{F}$ denote the completion of a maximal unramified extension of F, and let $\mathbb{F}$ denote the residue field of $O_{\breve{F}}$. Suppose S is a $\operatorname{Spf} O_{\breve{F}}$-scheme. Then we can consider triples (X, ι, λ) consisting of the following data.

- X is a formal ϖ-divisible n-dimensional O_F-module over S whose relative height is $2n$.
- $\iota \colon O_E \to \operatorname{End}(X)$ is an action of O_E such that the induced action of O_F on $\operatorname{Lie} X$ is via the structure morphism $O_F \to \mathcal{O}_S$.

  We require that ι satisfies the Kottwitz condition of signature $(n-1, 1)$, meaning that for all $a \in O_E$, the characteristic polynomial of $\iota(a)$ on $\operatorname{Lie} X$ is exactly

$$(T - a)^{n-1}(T - \bar{a}) \in \mathcal{O}_S[T].$$

- $\lambda \colon X \to X^\vee$ is a principal O_F-relative polarization.

We require that the Rosati involution of λ induces the map $a \mapsto \bar{a}$ on O_F (i.e. the nontrivial automorphism of $\mathrm{Gal}(E/F)$).

The triple is called supersingular if X is a supersingular strict O_F-module.

For each $n \geq 1$, over $\mathbb{F}$ we choose a supersingular triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$; it's unique up to O_E-linear quasi-isogeny compatible with the polarization, and refer to it as the *framing object*. We can now define the Rapoport-Zink space:

Definition 13.1.1 (Rapoport-Zink space; [LRZ24, §5.1])**.** For each $n \geq 1$, we let $\mathcal{N}_n$ denote the functor over $\operatorname{Spf} O_{\breve{F}}$ defined as follows. Let S be an $\operatorname{Spf} O_{\breve{F}}$ scheme, and let $\overline{S} \coloneqq S \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \mathbb{F}$. For every $\operatorname{Spf} O_{\breve{F}}$ scheme, we let $\mathcal{N}_n(S)$ be the set of isomorphism classes of quadruples

$$(X, \iota, \lambda, \rho)$$

where (X, ι, λ) is one of the triples as we described, and

$$\rho\colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\operatorname{Spec} \mathbb{F}} \overline{S}$$

is a *framing*, meaning it is a height zero O_F-linear quasi-isogeny and satisfies

$$\rho^*((\lambda_{\mathbb{X}_n})_{\overline{S}}) = \lambda_{\overline{S}}.$$

Then $\mathcal{N}_n$ is formally smooth over $O_{\breve{F}}$ of relative dimension $n-1$.

Henceforth, we also make the following abbreviation.

Definition 13.1.2 ($\mathcal{N}_{m,n}$)**.** For integers m and n,

$$\mathcal{N}_{m,n} \coloneqq \mathcal{N}_m \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n.$$

13.2 A realization of the non-split Hermitian space $\mathbb{V}_n^-$ of dimension n

For the following definition (and later on), we need a variation of $\mathcal{N}_1$:

Definition 13.2.1 ($\mathbb{E}$)**.** Let $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ be the unique triple over $\mathbb{F}$ whose Rosati involution has signature $(1,0)$ (note this is different from $\mathcal{N}_1$ where the signature is $(0,1)$ instead).

At the same time, we can define the following Hermitian space.

Definition 13.2.2 ([LRZ24, §5.2])**.** For each $n \geq 1$, let

$$\mathbb{V}_n^- \coloneqq \operatorname{Hom}^\circ_{O_E}(\mathbb{E}, \mathbb{X}_n)$$

which we call the space of special homomorphisms. When endowed with the form

$$\langle x, y \rangle = \lambda_{\mathbb{E}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}_n} \circ x \in \operatorname{End}^\circ_E(\mathbb{E}) \simeq E$$

it becomes an n-dimensional E/F-Hermitian space.

Proposition 13.2.3 (Realization of $\mathbb{V}_n^-$)**.** *Up to isomorphism, $\mathbb{V}_n^-$ is the unique n-dimensional nondegenerate non-split E/F-Hermitian space.*

Proof. See the comment in [LRZ24, §5.2] or the comment after [Zha24a, Equation (4.2)]. □

As described in [Zha24a, Equation (4.3)], there is an action of $\mathrm{U}(\mathbb{V}_n^-)$ on $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ and hence each $g \in \mathrm{U}(\mathbb{V}_n^-)$ acts on $\mathcal{N}_n$ by

$$g \cdot (X, i, \lambda, \rho) \coloneqq (X, i, \lambda, g \circ \rho).$$

13.3 Intersection numbers for the group version of AFL for the full spherical Hecke algebra

Here we reproduce the definition of the intersection number used in Conjecture 1.2.1.

Compared to the formulation of the group version and semi-Lie version of the AFL, the intersection number requires the introduction of a *Hecke operator* $\mathbb{T}_\varphi$ for an element

$$\varphi \in \mathcal{H}(G^\flat \times G, K^\flat \times K)$$

as introduced in [LRZ24]. This definition is too involved to reproduce here in its entirety, we give a summary for this special cases in which we need.

First consider the given $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$. The main work of the construction is to define another derived formal scheme $\mathcal{T}_{\mathbf{1}_{K^\flat}\otimes f}$ (see [LRZ24, §6.1]) together with two projection maps

$$\begin{tikzcd} & \mathcal{T}_{\mathbf{1}_{K^\flat}\otimes f} \arrow[dl] \arrow[dr] & \\ \mathcal{N}_{n-1,n} & & \mathcal{N}_{n-1,n}. \end{tikzcd}$$

This definition is carried out in [LRZ24, §5], by defining it first for so-called *atomic elements* of the spherical Hecke algebra, which form basis elements of a certain presentation of this Hecke algebra as the unitary group for a polynomial algebra; we refer the reader to *loc. cit.* for the full details.

Now, take the natural closed embedding

$$\mathcal{N}_{n-1} \to \mathcal{N}_n$$

and let

$$\Delta\colon \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_{n-1,n}$$

be the associated graph morphism; let $\Delta_{\mathcal{N}_{n-1}}$ denote the image, with an inclusion $\iota\colon \Delta_{\mathcal{N}_{n-1}} \hookrightarrow$

$\mathcal{N}_{n-1,n}$. Once this is done, consider then the diagram

$$\begin{tikzcd}
 & \pi_1^*(\Delta_{\mathcal{N}_{n-1}}) \arrow[dl] \arrow[dr] \arrow[rr] & & \mathcal{T}_{\mathbf{1}_{K^\flat}\otimes f} \arrow[dll, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\Delta_{\mathcal{N}_{n-1}} \arrow[r, hook, "\iota"'] & \mathcal{N}_{n-1,n} & (\pi_2)_*(\pi_1^*(\Delta_{\mathcal{N}_{n-1,n}})) \arrow[rr] & & \mathcal{N}_{n-1,n}.
\end{tikzcd}$$

That is, one takes the pullback of $\Delta_{\mathcal{N}_{n-1}} \xhookrightarrow{\iota} \mathcal{N}_{n-1,n}$ along the projection

$$\mathcal{N}_{n-1,n} \xleftarrow{\pi_1} \mathcal{T}_{\mathbf{1}_{K^\flat}\otimes f}$$

and then takes the pushforward along the other projection

$$\mathcal{T}_{\mathbf{1}_{K^\flat}\otimes f} \xrightarrow{\pi_2} \mathcal{N}_{n-1,n}.$$

Definition 13.3.1 (Hecke operator)**.** Set

$$\mathbb{T}_{\mathbf{1}_{K^\flat}\otimes f}(\Delta_{\mathcal{N}_{n-1}}) \coloneqq (\pi_2)_*(\pi_1^*(\Delta_{\mathcal{N}_{n-1}})).$$

This is the part of the intersection number depending on f (or rather $\mathbb{T}_{\mathbf{1}_{K^\flat}\otimes f}$). As for our $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$, consider the translation $(1,g)\cdot\Delta_{\mathcal{N}_{n-1}}$. The intersection number is then defined as by taking the intersection of these two objects using the derived tensor product $\otimes^{\mathbf{L}}$ of the structure sheaves.

Definition 13.3.2 ($\mathrm{Int}((1,g),\mathbf{1}_{K^\flat}\otimes f)$; [LRZ24, Equation (6.1.1)] or [Zha24a, Equation (4.4)])**.** We define the intersection number in Conjecture 1.2.1 by

$$\begin{aligned}
\mathrm{Int}((1,g),\mathbf{1}_{K^\flat}\otimes f) &\coloneqq \left\langle \mathbb{T}_{\mathbf{1}_{K^\flat}\otimes f}\Delta_{\mathcal{N}_{n-1}}, (1,g)\cdot\Delta_{\mathcal{N}_{n-1}} \right\rangle_{\mathcal{N}n-1,n} \\
&\coloneqq \chi_{\mathcal{N}_{n-1,n}}\left(\mathcal{O}_{\mathbb{T}_{\mathbf{1}_{K^\flat}\otimes f}(\Delta_{\mathcal{N}_{n-1}})} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n-1,n}}}{\otimes}} \mathcal{O}_{(1,g)\cdot\Delta_{\mathcal{N}_{n-1}}}\right).
\end{aligned}$$

Here χ denotes the Euler-Poincaré characteristic, meaning that if X is a formal scheme over $\operatorname{Spf} O_{\breve{F}}$ then given a finite complex $\mathcal{F}$ of $\mathcal{O}_X$-modules we set

$$\chi_X(\mathcal{F}) = \sum_i \sum_j (-1)^{i+j} \operatorname*{len}_{O_{\breve{F}}} H^j(X, H_i(\mathcal{F}))$$

provided all the lengths are finite.

Remark 13.3.3. In general we could adapt this definition so $(1, g)$ is replaced by by an element of $(\mathrm{U}(\mathbb{V}_{n-1}^-) \times \mathrm{U}(\mathbb{V}_n^-))_{\mathrm{rs}}$ if we wish to work with the full group version of the AFL rather than just the inhomogeneous version. However, this simpler definition will be sufficient for our purposes.

13.4 Intersection numbers for the semi-Lie version of AFL for the full spherical Hecke algebra

Now we continue to define an intersection number needed for the proposed Conjecture 1.2.2 from earlier. The definition mirrors the one given in the last section. Here we reproduce the definition of the intersection number used in Conjecture 1.2.1.

We work here with $\mathcal{N}_{n,n}$ rather than $\mathcal{N}_{n-1,n}$. The change is that we need to incorporate the new $u \in \mathbb{V}_n^-$ that was not present before. In order to do this one considers a certain relative Cartier divisor $\mathcal{Z}(u)$ on $\mathcal{N}_n$ for each nonzero $u \in \mathbb{V}_n^-$. This divisor was defined by Kudala and Rapport in [KR11] and accordingly we call it a *KR-divisor* following [Zha24a, §4.3]. The definition is given as follows.

Definition 13.4.1 ($\mathcal{Z}(u)$; [KR11, Definition 3.2])**.** Recall that $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ is the unique triple over $\mathbb{F}$ whose Rosati involution has signature $(1, 0)$. Hence the formal O_F-module has a unique lifting called its *canonical lifting*, which we denote by the triple $(\mathcal{E}, \iota_{\mathcal{E}}, \lambda_{\mathcal{E}})$.

Then the KR-divisor $\mathcal{Z}(u)$ is the locus where the quasi-homomorphism

$$u\colon \mathbb{E} \to \mathbb{X}_n$$

lifts to a homomorphism from $\mathcal{E}$ to the universal object over $\mathcal{N}_n$. More explicitly, it consists of those $X \in \mathcal{N}_n$ such that there exists a map $\varphi : \mathcal{E} \to X$ with the following property. Let S be an $\operatorname{Spf} O_{\breve{F}}$ scheme and consider the map on special fiber

$$\mathcal{E} = \mathbb{E} \times_S \overline{S} \xrightarrow{\varphi \times_S \overline{S}} X' \times_S \overline{S}.$$

Since $X \in \mathcal{N}_n$ we also have $\rho\colon X \times_S \overline{S} \to \mathbb{X}_n \times_{\operatorname{Spec}\mathbb{F}} \overline{S}$. Moreover, u gives a map

$$\mathbb{E} \times_{\operatorname{Spec}\mathbb{F}} \overline{S} \xrightarrow{u \times_{\operatorname{Spec}\mathbb{F}} \overline{S}} \mathbb{X}_n \times_{\operatorname{Spec}\mathbb{F}} \overline{S}.$$

Then we require the following diagram to commute:

$$\begin{tikzcd}
\mathbb{E} \times_S \overline{S} \arrow[r, "\varphi \times_S \overline{S}"] \arrow[d, "\rho_{\mathcal{E}}"] & X \times_S \overline{S} \arrow[d, "\rho"] \\
\mathbb{E} \times_{\operatorname{Spec}\mathbb{F}} \overline{S} \arrow[r, "u \times_{\operatorname{Spec}\mathbb{F}} \overline{S}"] & \mathbb{X}_n \times_{\operatorname{Spec}\mathbb{F}} \overline{S}.
\end{tikzcd}$$

That is, $\mathcal{Z}(u)$ is the locus where u lifts to a homomorphism $\mathcal{E} \to \mathcal{X}_n$. Note also by the definition that $g\mathcal{Z}(u) = \mathcal{Z}(gu)$. See [KR11] for a full definition.

The main change is then that we can consider $\Delta_{\mathcal{Z}(u)}$ as the image of

$$\mathcal{Z}(u) \hookrightarrow \mathcal{N}_n \xrightarrow{\Delta} \mathcal{N}_{n,n}$$

where $\Delta\colon \mathcal{N}_n \to \mathcal{N}_{n,n}$ now denotes the diagonal map. If one defines an appropriate space $\mathcal{T}_{\mathbf{1}_K \otimes f}$ for $f \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ again following [LRZ24], together with

$$\begin{tikzcd}
& \mathcal{T}_{\mathbf{1}_K \otimes f} \arrow[dl] \arrow[dr] & \\
\mathcal{N}_{n,n} & & \mathcal{N}_{n,n}
\end{tikzcd}$$

then one can then repeat the diagram from before:

$$
\begin{tikzcd}
& \pi_1^*(\Delta_{\mathcal{Z}(u)}) \arrow[rr] \arrow[dl] \arrow[drr] & & \mathcal{T}_{\mathbf{1}_K \otimes f} \arrow[dll, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\Delta_{\mathcal{Z}(u)} \arrow[r, hook, "\iota"'] & \mathcal{N}_{n,n} & & (\pi_2)_*(\pi_1^*(\Delta_{\mathcal{Z}(u)})) \arrow[r] & \mathcal{N}_{n,n}.
\end{tikzcd}
$$

In other words, we again take a pullback followed by a pushforward but this time of $\Delta_{\mathcal{Z}(u)} \hookrightarrow \mathcal{N}_{n,n}$. This lets us write an analogous definition:

Definition 13.4.2 (Hecke operator)**.** Set

$$\mathbb{T}_{\mathbf{1}_K \otimes f}(\Delta_{\mathcal{Z}(u)}) \coloneqq (\pi_2)_*(\pi_1^*(\Delta_{\mathcal{Z}(u)})).$$

Meanwhile to replace $(1, g)\Delta_{\mathcal{N}_{n,n-1}}$, we let

$$\Gamma_g \subseteq \mathcal{N}_{n,n}$$

denote the graph of the automorphism of $\mathcal{N}_n$ induced by g. This finally allows us to write a definition of the intersection number in the semi-Lie case:

Definition 13.4.3 ($\mathrm{Int}((g, u), f)$)**.** In analog to Definition 13.3.2 for the group version of the AFL, we now define the intersection number in Conjecture 1.2.2 as

$$
\begin{aligned}
\mathrm{Int}((g, u), f) &\coloneqq \left\langle \mathbb{T}_{\mathbf{1}_K \otimes f} \Delta_{\mathcal{Z}(u)}, \Gamma_g \right\rangle_{\mathcal{N}_{n,n}} \\
&\coloneqq \chi_{\mathcal{N}_{n,n}} \left(\mathcal{O}_{\mathbb{T}_{\mathbf{1}_K \otimes f}(\Delta_{\mathcal{Z}(u)})} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n,n}}}{\otimes}} \mathcal{O}_{\Gamma_g} \right).
\end{aligned}
$$

In the situation where $f = \mathbf{1}_K$, this coincides with the existing definition in [Zha24a, Equation (4.9)] and [Zha21a, Remark 3.1].

13.5 An analogy between the geometric and analytic sides

With the intersection number now defined for Conjecture 1.2.2, we provide some intuitive discussion about the connection. All of this is for philosophical cheerleading only, and is not meant to formally assert any definitions or results. But it may help in motivating the formulation of the conjecture.

For this section write $G \coloneqq \mathrm{U}(\mathbb{V}_n^+)$ and $K \coloneqq G \cap \mathrm{GL}_n(O_E)$ the hyperspecial maximal compact subgroup of G. For simplicity we only focus on the semi-Lie AFL originally proposed by Liu to start; which is the special case of Conjecture 1.2.2 when $f = \mathbf{1}_K$ and $\phi = \mathbf{1}_{K'}$.

The geometric side On the geometric side, $\mathcal{N}_n$ is the RZ-space acted on by $\mathrm{U}(\mathbb{V}_n^-)$, and hence $\mathrm{U}(\mathbb{V}_n^-) \times \mathrm{U}(\mathbb{V}_n^-)$ acts on $\mathcal{N}_{n,n}$. Roughly speaking, we are considering the two morphisms

$$\mathcal{N}_n \xrightarrow{\Delta} \mathcal{N}_{n,n} \xleftarrow{\Gamma_g} \mathcal{N}_n$$

with Δ being thought of as the diagonal morphism and Γ_g as the graph under multiplication by $g \in \mathrm{U}(\mathbb{V}_n^-)_{\mathrm{rs}}$.

Hence loosely speaking, the intersection $\mathrm{Int}\left((g,u), \mathbf{1}_K\right)$ can be thought of as the intersection of three images in $\mathcal{N}_{n,n}$:

- A "diagonal" object Δ;
- A "graph" object Γ;
- A third object $\mathcal{Z}(u)$, the KR-divisor, parametrized by diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{X}_n \\ \Big| & & \Big| \\ \mathbb{E} & \xrightarrow{u} & \mathbb{X}_n. \end{array}$$

The derived tensor product $\otimes^{\mathbf{L}}$ is used together with some formalism to make this intersection idea precise. The intersection of the "diagonal" and "graph" is the *fixed point locus*, and in fact could be formally defined as the intersection

$$\Gamma_g \cap \Delta_{\mathcal{N}_n}$$

viewed as a closed formal subscheme of $\mathcal{N}_n$ (or $\mathcal{N}_{n,n}$); see [Zha24a, Equation (4.6)].

The analytic side On the other hand, consider the analytic side. We will try to explain how the weighted orbital integral in Definition 8.1.1 can be thought of as some weighted intersection of analogous objects.

Note the quotient G/K can be identified as

$$G/K \simeq \left\{\Lambda \subseteq \mathbb{V}_n^+ \mid \Lambda^\vee = \Lambda\right\}$$

that is, the set of self-dual lattices Λ of full rank, which thus has a natural action of G. Henceforth we denote elements of G/K by h, and fix one particular such lattice Λ_0, acted on by O_E. Hence G/K can be thought of as

$$G/K \simeq \{h\Lambda_0 \mid h \in G/K\}.$$

Recall from (8.1) that we have an orbital integral on the unitary side of the shape

$$\int_{h \in G/K} \mathbf{1}_K(h^{-1}gh)\mathbf{1}_{\Lambda_0}(h \cdot u)\,\mathrm{d}h$$

where $u \in \mathbb{V}_n^+$, and Λ_0 is a fixed particular lattice in G/K. See for example the "relative fundamental lemma" stated as [Liu21, Conjecture 1.9].

Like before, we can consider two maps

$$G/K \xrightarrow{\Delta} G/K \times G/K \xleftarrow{\Gamma_g} G/K$$

which are the diagonal morphism and the graph of the action of g. Hence the intersection are those cosets hK for which

$$hK = ghK \iff h^{-1}gh \in K.$$

Hence the indicator function $\mathbf{1}_K(h^{-1}gh)$ plays the analog of the fixed point locus in the geometric side.

Meanwhile, the term $h \cdot u$ plays a role analogous to the KR-divisor on the geometric side, giving the third intersection object. We have

$$h \cdot u \in \Lambda_0 \iff u \in h^{-1}\Lambda_0$$

and so the object corresponding to the KR-divisor $\mathcal{Z}(u)$ is the subset in G/K of those lattices containing u, that is

$$\{\Lambda' \mid \Lambda' \ni u\}.$$

The analog to the earlier diagram that we described for $\mathcal{Z}(u)$ is then

$$\begin{aligned} O_E &\to \Lambda' \\ 1 &\mapsto u. \end{aligned}$$

Up until now this whole section is written for $f = \mathbf{1}_K$ and $\phi = \mathbf{1}_{K'}$. In the general situation, if one replaces $\mathbf{1}_K$ in the above integral by a general f, then this corresponds to changing the analog of the fixed point locus; the idea of [LRZ24] is that this should correspond to replacing $\Delta_{\mathcal{Z}(u)}$ with $\mathbb{T}_f(\Delta_{\mathcal{Z}(u)})$ on the geometric side.

Chapter 14

Intersection numbers for $\mathrm{Int}((g, u), f)$ for $n = 2$

This chapter is dedicated to computing intersection numbers for the semi-Lie version of AFL in the special case $n = 2$.

14.1 Background on quaternion division algebra

Through this section we let $\mathcal{D}$ be a quaternion division algebra over F, with a fixed maximal order $O_{\mathcal{D}}$. We will make $\mathcal{D}$ explicit in the following way for our calculations to follow.

14.1.1 Structure as a noncommutative algebra

As F-vector spaces we will write

$$\mathcal{D} = E \oplus E\Pi$$

where Π is selected so that $\Pi^2 = \varpi$. We endow $\mathcal{D}$ with a noncommutative multiplication according to

$$\Pi t = \bar{t}\Pi \qquad \text{for all } t \in E$$

where $\bar{t}$ is the image of $t \in E$ under the nontrivial element of $\mathrm{Gal}(E/F)$.

14.1.2 Conjugation of elements of $\mathcal{D}$

In general, suppose $x \in \mathcal{D}$ is any element decomposed as $x = a + b\Pi$ for $a, b \in E$. Then we denote by $\bar{x} \in \mathcal{D}$ the conjugate in $\mathcal{D}$ defined by

$$\bar{x} := \bar{a} - b\Pi$$

where, again, $\bar{a}$ is the image of $a \in E$ under the nontrivial element of $\mathrm{Gal}(E/F)$. It is an anti-involution, meaning that $\bar{\bar{x}} = x$ and $\overline{xy} = \bar{y}\bar{x}$.

(Notice that we have a slight abuse of notation here in that we have used the same notation to denote both conjugation under the Galois action of $\mathrm{Gal}(E/F)$ as well as the conjugation in $\mathcal{D}$. However, there is no ambiguity resulting because when E is viewed as a subset of $\mathcal{D}$, the two symbols denote the same element of E: that is we have

$$\overline{a + 0\Pi} = \bar{a} + 0\Pi$$

in any event. In other words, the restriction of the quaternion conjugation to E coincides with the nontrivial element of $\mathrm{Gal}(E/F)$, so we do not need to introduce a separate notation for it.)

This allows us to define the reduced norm and trace in $\mathcal{D}$. The reduced trace is given by

$$\operatorname{tr} x := x + \bar{x} = \operatorname{Tr} E/F(a) = 2x_0 \in F.$$

We may thus define

$$\mathcal{D}^{\mathrm{tr}=0} := \{u \in \mathcal{D} \mid \operatorname{tr} u = 0\}$$

which has codimension 1 inside $\mathcal{D}$ (i.e. is three-dimensional as an F-vector space). Since

$\operatorname{tr}(a+b\Pi) = \operatorname{Tr}_{E/F}(a)$, we could also write

$$\mathcal{D}^{\mathrm{tr}=0} = \{a + b\Pi \mid a, b \in E \text{ and } \operatorname{Tr}_{E/F}(a) = 0\}.$$

The reduced norm is similarly defined by

$$\begin{aligned}\operatorname{Nm} x = x\bar{x} &= (a + b\Pi)(\bar{a} - b\Pi) \\ &= a\bar{a} + b\Pi\bar{a} - ab\Pi - b\Pi b\Pi \\ &= a\bar{a} - b\bar{b}\varpi \\ &= \mathrm{N}_{E/F}(a) - \mathrm{N}_{E/F}(b)\varpi \in F.\end{aligned}$$

As an F-vector space, $\mathcal{D}$ has a basis given by $\{1, \sqrt{\varepsilon}, \Pi, \sqrt{\varepsilon}\Pi\}$, that is

$$\mathcal{D} = F \oplus F\sqrt{\varepsilon} \oplus F\Pi \oplus F\sqrt{\varepsilon}\Pi.$$

It will be convenient to introduce the following notation:

Definition 14.1.1 (x_0 and x_-)**.** For $x \in \mathcal{D}$, we introduce the notation x_0 and x_- to mean

- x_0 is the projection into the first component F; and
- $x_- = x - x_0$ is the projection into $\mathcal{D}^{\mathrm{tr}=0} = F\sqrt{\varepsilon} \oplus F\Pi \oplus F\sqrt{\varepsilon}\Pi$.

In particular, the formula for conjugation then reads as the simpler

$$\bar{x} = x_0 - x_-.$$

14.1.3 Hermitian structure

We view $\mathcal{D}$ as an E/F-Hermitian space under left multiplication by E; that is, for $a, b, t \in E$ we consider

$$t \cdot (a + b\Pi) = at + bt\Pi.$$

as the action of E on $\mathcal{D}$. Then we equip $\mathcal{D}$ with a E/F-Hermitian form $\langle \bullet, \bullet \rangle : \mathcal{D} \times \mathcal{D} \to E$ defined by

$$\langle x, y \rangle = \frac{1}{2} \operatorname{Tr}_{\mathcal{D}/E}(x\bar{y})$$

i.e. the projection of $x\bar{y} \in \mathcal{D} = E \oplus E\Pi$ onto the first component. In particular, note that

$$\langle x, x \rangle = x\bar{x} = \operatorname{Nm} x$$

or equivalently

$$\langle a + b\Pi, a + b\Pi \rangle = a\bar{a} - b\bar{b}\varpi.$$

14.1.4 Identification of $\mathbb{V}_n^-$ with $\mathcal{D}$

We continue using the notation $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ as the triple over $\mathbb{F}$ whose Rosati involution has signature $(1, 0)$. Moreover, we will take the identification

$$\operatorname{End}(\mathbb{E}) \simeq O_{\mathcal{D}}$$

see [KR11, Remark 2.5], and hence the corresponding identification

$$\mathbb{V}_n^- \simeq \mathcal{D}.$$

14.2 The invariants for the orbit of (g, u)

We specialize to the situation where $u \in O_{\mathcal{D}}$ and $g \in \mathrm{U}(\mathbb{V}_n^-)$.

14.2.1 Coordinates for g

To impose coordinates on g, we appeal to the following fact.

Lemma 14.2.1 (Description of $\mathrm{U}(\mathbb{V}_2^-)$). *Every unitary map in* $\mathrm{U}(\mathbb{V}_2^-)$ *can be described in the form*

$$x \mapsto \lambda^{-1} x(\alpha + \beta\Pi)$$

for some quaternion $\alpha + \beta\Pi \in \mathcal{D}^\times$ *and an element* $\lambda \in E^\times$ *such that*

$$\mathrm{Nm}(\alpha + \beta\Pi) = \mathrm{N}_{E/F}(\lambda).$$

Moreover, such a description is unique up to multiplication by elements of F. *In other words,*

$$\mathrm{U}(\mathbb{V}_2^-) \simeq (E^\times \times \mathcal{D}^\times)^\circ / \Delta(F^\times)$$

where $(\mathcal{D}^\times \times E^\times)^\circ$ *denotes those pairs* $(\lambda, \alpha + \beta\Pi)$ *with* $\mathrm{N}_{E/F}(\lambda) = \mathrm{Nm}(\alpha + \beta\Pi)$, *and* $\Delta(F^\times)$ *is the diagonal embedding of* $F^\times$.

Remark 14.2.2. In this paper we will not have a need to compose multiple such unitary maps. However, if we did, then our notation above swaps the multiplication order. That is, if we have $g_1, g_2 \in \mathrm{U}(\mathbb{V}_2^-)$ represented by pairs $g_1 \leftrightarrow (\lambda_1, \alpha_1 + \beta\Pi_1)$ and $g_2 \leftrightarrow (\lambda_2, \alpha_2 + \beta\Pi_2)$ under the isomorphism above, then

$$g_1 \circ g_2 \leftrightarrow (\lambda_2\lambda_1, (\alpha_2 + \beta_2\Pi)(\alpha_1 + \beta\Pi)).$$

Note that in the definition for g, if $v(\lambda) \neq 0$ then we can factor out powers of $\varpi = \bar{\varpi}$ from λ and put them into α and β instead. Hence, by relabeling α and β, we may assume without loss of generality that:

Assumption 14.2.3. *We assume WLOG that* $v(\lambda) = 0$ *(and hence* $v(\alpha) = 0$, $v(\beta) \geq 0$*).*

This frees us from having to deal with $v(\lambda)$ offsets in subsequent calculation.

We encode g as a matrix on $\mathcal{D}$ now (with the obvious E-basis 1, Π, again viewing $\mathcal{D}$ as a left-E module) so we can compute its determinant and trace. We have

$$g(1) = \lambda^{-1} \cdot 1 \cdot (\alpha + \beta\Pi) = \lambda^{-1}\alpha + \lambda^{-1}\beta\Pi$$
$$g(\Pi) = \lambda^{-1} \cdot \Pi(\alpha + \beta\Pi) = \lambda^{-1}\bar{\beta}\varpi + \lambda^{-1}\bar{\alpha}\Pi.$$

Hence, written as a matrix with respect to the obvious basis $\{1, \Pi\}$ we have

$$g = \lambda^{-1}\begin{pmatrix} \alpha & \bar{\beta}\varpi \\ \beta & \bar{\alpha} \end{pmatrix}.$$

14.2.2 Coordinates for u

We also impose coordinates for u according to

$$u = s + t\Pi \in O_{\mathcal{D}} \qquad s, t \in E.$$

To make the calculation that follows less complicated, we are going to make the following assumption on u.

Assumption 14.2.4. *We assume WLOG that either $u \in E$ or $u \in E\Pi$. That is, either $s = 0$ or $t = 0$.*

This assumption can be made without loss of generality because the invariants and the intersection only depend on the SU(2)-orbit of the pair (g, u), and any element $u \in \mathcal{D}^{\times}$ can be mapped under an element of SU(2) into a pair for which $u \in E$ or $u \in E\Pi$.

In order for (g, u) to be regular semisimple we require that

$$\begin{aligned} u &= s + t\Pi \\ g(u) &= \lambda^{-1}(s + t\Pi)(\alpha + \beta\Pi) \\ &= \lambda^{-1}\left((s\alpha + t\bar{\beta}\varpi) + (s\beta + t\bar{\alpha})\Pi\right) \end{aligned}$$

are linearly independent, meaning

$$
\begin{aligned}
0 \neq \det \begin{pmatrix} s & s\alpha + t\bar{\beta}\varpi \\ t & s\beta + t\bar{\alpha} \end{pmatrix} &= st(\bar{\alpha} - \alpha) + \beta s^2 - \bar{\beta} t^2 \varpi \\
&= \beta s^2 - \bar{\beta} t^2 \varpi \\
&= \begin{cases} -\bar{\beta} t^2 \varpi & \text{if } s = 0 \\ \beta s^2 & \text{if } t = 0. \end{cases}
\end{aligned}
$$

Hence, we have a requirement that $\beta \neq 0$ and s and t are not both zero (we require $st = 0$ from Assumption 14.2.4).

Remark 14.2.5 ($\alpha \neq 0$)**.** Note that necessarily α is nonzero as well. This follows from the requirement that $\alpha\bar{\alpha} - \beta\bar{\beta}\varpi = \lambda\bar{\lambda}$; if $\alpha = 0$ we would get a left-hand side with odd valuation but a right-hand side with even valuation.

14.2.3 The invariants of the matching

At this point we can state:

Lemma 14.2.6. *Under the coordinates we just described, the four corresponding invariants of Definition 3.3.1 are:*

$$
\begin{aligned}
\operatorname{Tr} g &= \lambda^{-1}(\alpha + \bar{\alpha}) \\
\det g &= \lambda^{-2}(\alpha\bar{\alpha} - \beta\bar{\beta}\varpi) \\
\langle u, u \rangle &= s\bar{s} - t\bar{t}\varpi \\
\langle g(u), u \rangle &= \begin{cases} \lambda^{-1}\bar{\alpha} \operatorname{Nm} u & \textit{if } s = 0 \\ \lambda^{-1}\alpha \operatorname{Nm} u & \textit{if } t = 0. \end{cases}
\end{aligned}
$$

Proof. The first three are immediate. The last one follows by computing

$$\begin{aligned}\langle g(u), u\rangle &= \left\langle \lambda^{-1}(s+t\Pi)(\alpha+\beta\Pi), s+t\Pi\right\rangle \\ &= \lambda^{-1}\left\langle (s\alpha+t\bar{\beta}\varpi)+(t\bar{\alpha}+s\beta)\Pi, s+t\Pi\right\rangle \\ &= \lambda^{-1}\cdot\frac{1}{2}\operatorname{Tr}_{\mathcal{D}/E}\left[((s\alpha+t\bar{\beta}\varpi)+(t\bar{\alpha}+s\beta)\Pi)(\bar{s}-t\Pi)\right] \\ &= \lambda^{-1}\left(s\bar{s}\alpha-t\bar{t}\bar{\alpha}\varpi\right)\end{aligned}$$

and recalling that $st = 0$. □

14.3 A basis for $\mathcal{H}(\mathrm{U}(\mathbb{V}_2^+))$

As before $K = \mathrm{GL}_2(O_E) \cap \mathrm{U}(\mathbb{V}_2^+)$denotes the maximal hyperspecial compact subgroup of $\mathrm{U}(\mathbb{V}_2^+)$. For each $r \geq 0$, we define

$$\mathbf{1}_{K,\leq r} \coloneqq \mathbf{1}_{\varpi^{-r}\operatorname{Mat}_2(O_E)\cap\mathrm{U}(\mathbb{V}_2^+)} \in \mathcal{H}(\mathrm{U}(\mathbb{V}_2^+)).$$

For convenience $\mathbf{1}_{K,\leq r} = 0$ for $r < 0$. We also set

$$\mathbf{1}_{K,r} \coloneqq \mathbf{1}_{K,\leq r} - \mathbf{1}_{K,\leq(r-1)}$$

which is the indicator function for the coset

$$K\begin{pmatrix} 0 & \varpi^r \\ \varpi^{-r} & 0 \end{pmatrix}K.$$

Note when $r = 0$, $\mathbf{1}_{K,0} = \mathbf{1}_K = \mathbf{1}_{K,\leq 0}$.

Analogous to Section 8.2 we then have the following result.

Proposition 14.3.1 ($\mathbf{1}_{K,\leq r}$ basis)**.** *The functions $\mathbf{1}_{K,\leq r}$ (for $r \geq 0$) form a basis of $\mathcal{H}(\mathrm{U}(\mathbb{V}_2^+))$. (Similarly, so do $\mathbf{1}_{K,r}$ for $r \geq 0$.)*

Proof. This follows from the fact that

$$\mathrm{U}(\mathbb{V}_2^+) = \coprod_{r\geq 0} K \begin{pmatrix} 0 & \varpi^r \\ \varpi^{-r} & 0 \end{pmatrix} K.$$

See the comment in [LRZ24, Equation (7.1.5)]. □

The base change for this basis is given later in Lemma 15.3.1.

14.4 Background on special divisors for $n = 2$

14.4.1 The Rapoport-Zink space $\mathcal{N}_2$ and $\mathcal{T}_{\mathbf{1}_{K,\leq r}}$

Recall $\mathcal{N}_2$ from Chapter 13. With the Hecke operator $\mathbb{T}$ from [LRZ24] (see Chapter 13 for discussion) we introduce $\mathcal{T}_{\mathbf{1}_{K,\leq r}} = \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,\leq r}}(\Delta_{\mathcal{N}_n})$ so that we have the diagram

$$\begin{array}{ccccc} & & \mathcal{T}_{\mathbf{1}_{K,\leq r}} & & \\ & \swarrow & & \searrow & \\ \mathcal{N}_2 & & & & \mathcal{N}_2 \end{array}$$

14.4.2 The Lubin-Tate space $\mathcal{M}_2$

We introduce the notation $\mathcal{M}_2$ for the *Lubin-Tate space* for $n = 2$. It is defined almost in the same way as $\mathcal{N}_2$ except that we replace $\mathbb{X}_2$ with $\mathbb{E}$ now.

Definition 14.4.1 (Lubin-Tate space)**.** We let $\mathcal{M}_2$ denote the functor over $\operatorname{Spf} O_{\breve{F}}$ defined as follows. Let S be an $\operatorname{Spf} O_{\breve{F}}$ scheme, and let $\overline{S} \coloneqq S \times_{\operatorname{Spf} O_{\breve{F}}} \operatorname{Spec} \mathbb{F}$. For every $\operatorname{Spf} O_{\breve{F}}$ scheme, we let $\mathcal{N}_n(S)$ be the set of isomorphism classes of quadruples

$$(Y, \iota, \lambda, \rho)$$

where (Y, ι, λ) is one of the triples as we described, and

$$\rho\colon Y \times_S \overline{S} \to \mathbb{E} \times_{\operatorname{Spec}\mathbb{F}} \overline{S}$$

is a *framing*, meaning it is a height zero O_F-linear quasi-isogeny and satisfies

$$\rho^*((\lambda_{\mathbb{E}})_{\overline{S}}) = \lambda_{\overline{S}}.$$

Proposition 14.4.2 ([LRZ24, Example 5.5.6]). *The Serre tensor construction produces an identification*

$$\mathrm{ST}\colon \mathcal{N}_2 \xrightarrow{\sim} \mathcal{M}_2.$$

By abuse of notation we will also use the same symbol for the map

$$\mathrm{ST}\colon \mathcal{N}_{2,2} \xrightarrow{\sim} \mathcal{M}_2 \times \mathcal{M}_2.$$

Recall we have an action of $\mathrm{U}(\mathbb{V}_2^-)$ (actually $\mathrm{PU}(\mathbb{V}_2^-)$) on $\mathcal{N}_2$. We describe now the corresponding action on $\mathcal{M}_2$. We have an isomorphism of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & O_E^\times/O_F \arrow[r] \arrow[d, equal] & (O_{\mathcal{D}}^\times \times O_E^\times)^\circ/\Delta(O_F^\times) \arrow[r] \arrow[d, equal] & O_{\mathcal{D}}^\times/O_F^\times \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \mathrm{U}(\mathbb{V}_2^-) \arrow[r] & \mathrm{PU}(\mathbb{V}_2^-) \arrow[r] & 1.
\end{tikzcd}$$

The image of $\alpha + \beta\Pi \in O_{\mathcal{D}}^\times/O_F^\times$ is then $(\lambda, \alpha + \beta\Pi) \in \mathrm{PU}(2)$ for any choice of λ with $\lambda\bar{\lambda} = \mathrm{Nm}(\alpha + \beta\Pi)$.

14.4.3 The divisor $\mathcal{Z}^\dagger_{\mathrm{SO}(4)}$ on $\mathcal{M}_2 \times \mathcal{M}_2$

Now we define the special orthogonal divisor $\mathcal{Z}^\dagger_{\mathrm{SO}(4)}(u)$ on $\mathcal{M}_2 \times \mathcal{M}_2$ as follows.

Definition 14.4.3 ($\mathcal{Z}^\dagger_{\mathrm{SO}(4)}(u)$). Let $u \in O_{\mathcal{D}}$. Then we define the divisor $\mathcal{Z}^\dagger_{\mathrm{SO}(4)}(u)$ to be the pairs $(Y, Y') \in \mathcal{M}_2 \times \mathcal{M}_2$ such that there exists $\varphi\colon Y \to Y'$ with the following property. Let

S be an $\operatorname{Spf} O_{\breve{F}}$ scheme and consider the map on special fiber

$$Y \times_S \overline{S} \xrightarrow{\varphi \times_S \overline{S}} Y' \times_S \overline{S}.$$

Also, from $Y \in \mathcal{M}_2$ and $Y' \in \mathcal{M}_2$ we have the data of framings $\rho\colon Y \times_S \overline{S} \to \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}$ and $\rho'\colon Y' \times_S \overline{S} \to \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}$. Moreover, u gives a map

$$\mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S} \xrightarrow{u \times_{\operatorname{Spec} \mathbb{F}} \overline{S}} \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}.$$

Then we require the following diagram to commute:

$$\begin{tikzcd}
Y \times_S \overline{S} \arrow[r, "\varphi \times_S \overline{S}"] \arrow[d, "\rho"] & Y' \times_S \overline{S} \arrow[d, "\rho'"] \\
\mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S} \arrow[r, "u \times_{\operatorname{Spec} \mathbb{F}} \overline{S}"] & \mathbb{E} \times_{\operatorname{Spec} \mathbb{F}} \overline{S}.
\end{tikzcd}$$

We propose that the Serre tensor construction identifies the space $\mathcal{T}_{\mathbf{1}_{K,\leq r}}$ we previously described with the following $\mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}$ divisor:

Conjecture 14.4.4 ($\mathcal{T}_{\mathbf{1}_{K,\leq r}} \simeq \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\varpi^r)$)**.** *The Serre tensor construction gives an isomorphism*

$$\mathcal{T}_{\mathbf{1}_{K,\leq r}} \simeq \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\varpi^r)$$

such that we get an analogous diagram

$$\begin{tikzcd}
 & \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\varpi^r) \arrow[dl] \arrow[dr] & \\
\mathcal{M}_2 & & \mathcal{M}_2.
\end{tikzcd}$$

We assume this conjecture henceforth. (In fact, for $n = 2$ we could even go as far as to take this as a definition of $\mathcal{T}_{\mathbf{1}_{K,\leq r}}$, but then the generalization Conjecture 1.2.2 would be less obvious, since a definition like this would not easily extend to $n > 2$.)

14.4.4 The divisor $\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}$ on $\mathcal{M}_2$

Turning to $\mathcal{M}_2 \times \mathcal{M}_2$, we will henceforth always identify $\mathcal{M}_2$ with its image under the diagonal map

$$\mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1) \xrightarrow{\sim} \mathcal{M}_2 \xhookrightarrow{\Delta_{\mathcal{M}_2}} \mathcal{M}_2 \times \mathcal{M}_2.$$

Definition 14.4.5 ($\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(u)$)**.** Suppose now $u \in \mathcal{D}^{\mathrm{tr}=0}$. Then we define the divisor $\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(u)$ to be those $X \in \mathcal{M}_2$ for which we have a diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, no head] & X \arrow[d, no head] \\ \mathbb{E} \arrow[r, "u"] & \mathbb{E}. \end{tikzcd}$$

Note that basically by definition, for $u \in O_{\mathcal{D}}$ and $\operatorname{tr} u = 0$ we have

$$\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(u) \simeq \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(u) \cap \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1)$$

when we identify $\mathcal{M}_2$ with its image in $\mathcal{M}_2 \times \mathcal{M}_2$.

14.5 Comparison of the unitary and orthogonal special divisors

We now relate $\mathcal{Z}(u)$ to $\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(u)$ through our isomorphism $\mathcal{N}_{2,2} \xrightarrow{\sim} \mathcal{M}_2 \times \mathcal{M}_2$. Recall that we have the notation

$$\mathcal{Z}(u)^{\circ} \coloneqq \mathcal{Z}(u) - \mathcal{Z}\left(\frac{u}{\varpi}\right).$$

Define $\mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(u)^{\circ}$ and $\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(u)^{\circ}$ similarly.

Lemma 14.5.1 ($\mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(\bar{u}\sqrt{\varepsilon}u)^{\circ}$, [CLZ])**.** *Let $u \in \mathbb{V}_n^-$, and consider it as an element $u \in \mathcal{D}$. Then pullback along the Serre tensor construction identifies*

$$\mathrm{ST}^* \mathcal{Z}(u)^{\circ} \simeq \mathcal{Z}^{\dagger}_{\mathrm{SO}(3)}(\bar{u}\sqrt{\varepsilon}u)^{\circ}.$$

Proof. This is shown in the in-preparation [CLZ] by Ryan C. Chen, Weixiao Lu, and Wei Zhang. (Although [CLZ] is technically written for $F = \mathbb{Q}_p$, that restriction is for other parts of the paper that don't affect this lemma.) $\square$

14.6 The intersection number as a triple product

We return to the intersection number

$$\operatorname{Int}((g,u), \mathbf{1}_{K,\le r}) = \chi_{\mathcal{N}_{n,n}} \left(\mathcal{O}_{\mathbb{T}_{\mathbf{1} \otimes \mathbf{1}_{K,\le r}}(\Delta_{\mathcal{Z}(u)})} \overset{\mathbf{L}}{\underset{\mathcal{O}_{\mathcal{N}_{n,n}}}{\otimes}} \mathcal{O}_{\Gamma_g} \right)$$

which we will rewrite more succinctly using angle brackets as

$$\operatorname{Int}((g,u), \mathbf{1}_{K,\le r}) = \left\langle \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,\le r}} \Delta_{\mathcal{Z}(u)}, \Gamma_g \right\rangle_{\mathcal{N}_{2,2}}$$

in analogy to [LRZ24, §6.1]. (Note that Δ here is the diagonal map $\mathcal{N}_2 \to \mathcal{N}_{2,2}$.) For this calculation, it would be sufficient to split

$$\mathcal{Z}(u) = \sum_{i \ge 0} \mathcal{Z}(u/\varpi^i)^\circ.$$

Accordingly, let us introduce the notation

$$\begin{aligned} \operatorname{Int}^\circ((g,u), \mathbf{1}_{K,\le r}) &\coloneqq \operatorname{Int}((g,u), \mathbf{1}_{K,\le r}) - \operatorname{Int}\left(\left(g, \frac{u}{\varpi}\right), \mathbf{1}_{K,\le r}\right) \\ &= \left\langle \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,\le r}} \Delta_{\mathcal{Z}(u)^\circ}, \Gamma_g \right\rangle_{\mathcal{N}_{2,2}}. \end{aligned}$$

From Lemma 14.5.1 we get

$$\begin{aligned} \operatorname{Int}^\circ((g,u), \mathbf{1}_{K,\le r}) &= \left\langle \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(g \cdot \varpi^r),\ \Delta(\mathrm{ST}^*(\mathcal{Z}(u))^\circ) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} \\ &= \left\langle \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(g \cdot \varpi^r),\ \mathcal{Z}^\dagger_{\mathrm{SO}(3)}(\bar{u}\sqrt{\varepsilon}u)^\circ \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(g \cdot \varpi^r),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1)^{\circ},\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\bar{u}\sqrt{\varepsilon}u)^{\circ} \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} \\
&= \left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(g \cdot \varpi^r),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\bar{u}\sqrt{\varepsilon}u) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} \\
&\quad - \left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(g \cdot \varpi^r),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}\left(\frac{\bar{u}\sqrt{\varepsilon}u}{\varpi}\right) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2}.
\end{aligned}
$$

In that case we have

$$
\mathrm{Int}((g,u), \mathbf{1}_{K,\leq r}) = \sum_{i \geq 0} \mathrm{Int}^{\circ}\left(\left(g, \frac{u}{\varpi^i}\right), \mathbf{1}_{K,\leq r}\right). \tag{14.1}
$$

14.7 The formula of Gross-Keating

In what follows, we let

$$
\langle x, y \rangle_0 = \frac{\langle x, y \rangle + \overline{\langle x, y \rangle}}{2}
$$

denote half the E/F-trace of $\langle x, y \rangle \in E$. Let $O_{\mathcal{D}}^{\mathrm{tr}=0} \coloneqq O_{\mathcal{D}} \cap \mathcal{D}^{\mathrm{tr}=0}$.

Proposition 14.7.1 (Gross-Keating)**.** *Let $x, y \in O_{\mathcal{D}}^{\mathrm{tr}=0}$ and let*

$$
\begin{aligned}
n_1 &= \min\left(v(\langle x, x \rangle_0), v(\langle x, y \rangle_0), v(\langle y, y \rangle_0)\right) \\
n_2 &= v\left(\langle x, x \rangle_0 \langle y, y \rangle_0 - \langle x, y \rangle_0^2\right) - n_1
\end{aligned}
$$

so that $0 \leq n_1 \leq n_2$. Then if n_1 is odd, we have

$$
\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(x),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(y) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} = \sum_{j=0}^{\frac{n_1-1}{2}} (n_1 + n_2 - 4j) q^j
$$

while if n_1 is even we instead have

$$
\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(x),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(y) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} = \frac{n_2 - n_1 + 1}{2} q^{n_1/2} + \sum_{j=0}^{n_1/2-1} (n_1 + n_2 - 4j) q^j.
$$

Proof. This is a rewriting of [Kud97, Proposition 14.6] which is itself a special case of [GK93, Proposition 5.4]. □

We now compute all the quantities needed to invoke the Gross-Keating formula. We start by writing

$$\begin{aligned}\bar{u}\sqrt{\varepsilon}u &= (\bar{s} - t\Pi)\sqrt{\varepsilon}(s + t\Pi) \\ &= (\bar{s} - t\Pi)(s\sqrt{\varepsilon} + t\sqrt{\varepsilon}\Pi) \\ &= s\bar{s}\sqrt{\varepsilon} - t\overline{s\sqrt{\varepsilon}}\Pi + \bar{s}t\sqrt{\varepsilon}\Pi - t\overline{t\sqrt{\varepsilon}}\varpi \\ &= (s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon} + 2\bar{s}t\sqrt{\varepsilon}\Pi.\end{aligned}$$

We now invoke Assumption 14.2.4 to simplify this to just

$$\bar{u}\sqrt{\varepsilon}u = (s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon}.$$

This assumption will also let us write

$$(s\bar{s} + t\bar{t}\varpi)^2 = (s\bar{s} - t\bar{t}\varpi)^2 = (\operatorname{Nm} u)^2.$$

Next we consider

$$g \cdot \varpi^r = \varpi^r(\alpha + \beta\Pi).$$

(Here the action of g is the one on $\mathcal{M}_2$, which is why we write $g \cdot \varpi^r$ rather than $g(\varpi^r)$.) From now on, let's write

$$\alpha = \alpha_0 + \alpha_1\sqrt{\varepsilon}$$

for $\alpha_0, \alpha_1 \in F$. Then we use the notation

$$\begin{aligned}x &:= \bar{u}\sqrt{\varepsilon}u = (s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon} \in O_{\mathcal{D}}^{\mathrm{tr}=0} \\ y &:= (g \cdot \varpi^r)_- = \varpi^r(\alpha_1\sqrt{\varepsilon} + \beta\Pi) \in O_{\mathcal{D}}^{\mathrm{tr}=0}.\end{aligned}$$

Then we can compute

$$\begin{aligned}
\langle x, x\rangle_0 &= \operatorname{Nm} x \\
&= \mathrm{N}_{E/F}((s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon}) \\
&= -\varepsilon(s\bar{s} + t\bar{t}\varpi)^2 = -\varepsilon(\operatorname{Nm} u)^2 \\
\langle y, y\rangle_0 &= \operatorname{Nm}\left(\varpi^r(\alpha_1\sqrt{\varepsilon} + \beta\Pi)\right) \\
&= \varpi^{2r}(-\alpha_1^2\varepsilon - \beta\bar{\beta}\varpi) \\
\langle x, y\rangle_0 &= (\bar{x}y)_0 \\
&= \left[-(s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon}\cdot\varpi^r(\alpha_1\sqrt{\varepsilon} + \beta\Pi)\right]_0 \\
&= -\varpi^r\alpha_1\varepsilon(s\bar{s} + t\bar{t}\varpi).
\end{aligned}$$

This lets us compute the determinant

$$\begin{aligned}
\langle x, x\rangle_0\langle y, y\rangle_0 - \langle x, y\rangle_0^2 &= -\varepsilon(\operatorname{Nm} u)^2\cdot\varpi^{2r}(-\alpha_1^2\varepsilon - \beta\bar{\beta}\varpi) - (\varpi^r\alpha_1\varepsilon(s\bar{s} + t\bar{t}\varpi))^2 \\
&= \varepsilon(\operatorname{Nm} u)^2\cdot\varpi^{2r}\cdot(\varpi\beta\bar{\beta}).
\end{aligned}$$

Hence we arrive at an exact formula for

$$\left\langle \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(x),\ \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(y)\right\rangle_{\mathcal{M}_2\times\mathcal{M}_2}$$

in terms of the valuations of the above formulas, which we will explicate in the next section after matching (g, u) to the corresponding element in $(S_2(F)\times V_2'(F))_{\mathrm{rs}}$.

Chapter 15

Proof of Theorem 1.3.8

We now put together all the results from the previous chapters to prove Theorem 1.3.8.

On the orbital side, we assume $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ is as in Lemma 8.4.2 throughout this chapter. On the geometric side, we assume (g, u) are as described in Section 14.2.

15.1 Matching $(\gamma, \mathbf{u}, \mathbf{v}^\top)$ and (g, u), and the invariants for the matching

15.1.1 The invariants for the orbit of $(\gamma, \mathbf{u}, \mathbf{v}^\top)$

Recall the relations in Lemma 8.4.2. The invariants in this case as described in Definition 3.3.1 are:

- $\operatorname{Tr} \gamma = a + d$
- $\det \gamma = ad - bc$
- $\mathbf{v}^\top \mathbf{u} = e$
- $\mathbf{v}^\top \gamma \mathbf{u} = \begin{pmatrix} 0 & e \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = de.$

Note that the parameters b and c are absent; but we have

$$v(b) + v(c) = v(\det\gamma - ad).$$

15.1.2 Matching

We now take these results and line them up with Lemma 14.2.6 to deduce the following lemma.

Lemma 15.1.1 (Explicit matching of invariants of $\mathrm{U}(\mathbb{V}_2^-)$ and $(S_2(F) \times V_2'(F))_{\mathrm{rs}}^-$)**.** *Let*

$$g = \lambda^{-1}\begin{pmatrix} \alpha & \bar{\beta}\varpi \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathrm{U}(\mathbb{V}_2^-)$$

and $u = s + t\Pi$ with $st = 0$ and $v(\lambda) = 0$. Suppose (g, u) matches with

$$(\gamma, \mathbf{u}, \mathbf{v}^\top) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & e \end{pmatrix} \right) \in (S_2(F) \times V_2'(F))_{\mathrm{rs}}.$$

Then we have

$$a = \begin{cases} \lambda^{-1}\bar{\alpha} & \text{if } s = 0 \\ \lambda^{-1}\alpha & \text{if } t = 0 \end{cases}$$

$$d = \begin{cases} \lambda^{-1}\alpha & \text{if } s = 0 \\ \lambda^{-1}\bar{\alpha} & \text{if } t = 0 \end{cases}$$

$$bc = \lambda^{-2}\beta\bar{\beta}\varpi$$

$$e = \operatorname{Nm} u.$$

Thus we also have the identity

$$v(d-a) = v(\alpha_1).$$

Proof. Setting the invariants from the previous subsection equal to the ones determined in Lemma 14.2.6 gives

$$\begin{aligned} a+d &= \lambda^{-1}(\alpha+\bar{\alpha}) \\ \det\gamma = ad-bc = \det g &= \lambda^{-2}(\alpha\bar{\alpha}-\beta\bar{\beta}\varpi) \\ e &= \operatorname{Nm} u \\ de &= \begin{cases} \lambda^{-1}\bar{\alpha}\operatorname{Nm} u & \text{if } s=0 \\ \lambda^{-1}\alpha\operatorname{Nm} u & \text{if } t=0. \end{cases} \end{aligned}$$

So the equations for e, d and a are immediate. In both cases we get $ad = \lambda^{-2}\alpha\bar{\alpha}$ and hence

$$\lambda^{-2}\beta\bar{\beta}\varpi = -(\det g - \lambda^{-2}\alpha\bar{\alpha}) = -(\det\gamma - ad) = bc. \qquad \square$$

Remark 15.1.2 (Deriving Lemma 8.4.2 from Lemma 15.1.1)**.** Note that many of the assumptions in Lemma 8.4.2 can also be extracted from Lemma 15.1.1. For example, taking the valuation of

$$\lambda\bar{\lambda} = \alpha\bar{\alpha} - \beta\bar{\beta}\varpi$$

implies that (since the left-hand side is a unit)

$$v(a) = v(\alpha) = 0 < 2v(\beta)+1 = v(b)+v(c).$$

So indeed $v(1-a\bar{a}) = v(\bar{b}c)$ must be odd.

15.2 Translation of the Gross-Keating data to the orbital side

We combine the results of Section 14.7 with Lemma 15.1.1. Retaining the notation

$$x \coloneqq \bar{u}\sqrt{\varepsilon}u = (s\bar{s} + t\bar{t}\varpi)\sqrt{\varepsilon} \in O_{\mathcal{D}}^{\mathrm{tr}=0}$$
$$y \coloneqq (g \cdot \varpi^r)_- = (\alpha_1\sqrt{\varepsilon} + \beta\Pi) \in O_{\mathcal{D}}^{\mathrm{tr}=0}$$

from Section 14.7, we obtain the following:

$$\begin{aligned}
v(\langle x, x\rangle_0) &= 2v(\mathrm{Nm}\, u) \\
&= 2v(e) \\
v\left(\langle x, y\rangle_0\right) &= r + v(\alpha_1) + v(\mathrm{Nm}\, u) \\
&= r + v(d-a) + v(e) \\
v(\langle x, x\rangle_0 \langle y, y\rangle_0 - \langle x, y\rangle_0^2) &= 2r + 2v(\mathrm{Nm}\, u) + v(\beta\bar{\beta}\varpi) \\
&= 2r + 2v(e) + v(b) + v(c).
\end{aligned}$$

Notice that the last valuation is odd. Therefore we can always extract $v(\langle y, y\rangle_0)$ by writing

$$\begin{aligned}
v\left(\langle x, x\rangle_0\right) + v\left(\langle y, y\rangle_0\right) &= \min\left(v\left(\langle x, x\rangle_0 \langle y, y\rangle_0 - \langle x, y\rangle_0^2\right), v(\langle x, y\rangle_0^2)\right) \\
\implies v\left(\langle y, y\rangle_0\right) &= \min(2r + 2v(e) + v(b) + v(c), 2r + 2v(d-a) + 2v(e)) - 2v(e) \\
&= 2r + \min(v(b) + v(c), 2v(d-a)).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\min\left(v(\langle x, x\rangle_0), v(\langle x, y\rangle_0), v(\langle y, y\rangle_0),\right) \\
&= \min\left(2v(e), r + v(d-a) + v(e), 2r + \min(v(b) + v(c), 2v(\varsigma))\right)
\end{aligned}$$

$$= \min\left(2v(e), r + v(d-a) + v(e), 2r + v(b) + v(c), 2r + 2v(\varsigma)\right)$$
$$= \min\left(2v(e), v(b) + v(c) + 2r, 2v(d-a) + 2r\right)$$

where we can drop $r + v(d-a) + v(e)$ from the minimum because it equals $\frac{2v(e)+(2v(d-a)+2r)}{2}$.

Now recall the right-hand side of Proposition 14.7.1, that is

$$\begin{cases} \sum_{j=0}^{\frac{n_1-1}{2}} (n_1 + n_2 - 4j) \cdot q^j & \text{if } n_1 \equiv 1 \pmod 2 \\ \frac{n_2-n_1+1}{2} q^{n_1/2} + \sum_{j=0}^{n_1/2-1} (n_1 + n_2 - 4j) \cdot q^j & \text{if } n_1 \equiv 0 \pmod 2. \end{cases}$$

for $0 \le n_1 \le n_2$. Then apply Proposition 14.7.1 to obtain that

$$\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\bar{u}\sqrt{\varepsilon}u),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}((g \cdot \varpi^r)_-) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2}$$

is equal to the above formula applied at

$$n_1 \coloneqq \min(2v(e), v(b) + v(c) + 2r, 2v(d-a) + 2r)$$
$$n_2 \coloneqq 2v(e) + v(b) + v(c) + 2r - n_1.$$

Note that

$$n_1 + n_2 = 2v(e) + v(b) + v(c) + 2r.$$

For brevity, we henceforth introduce the symbol GK for the sum above; hence we have

$$\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(\bar{u}\sqrt{\varepsilon}u),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}((g \cdot \varpi^r)_-) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} = \mathrm{GK}(r, v(b), v(c), v(e), v(d-a)). \tag{15.1}$$

In that case we also have

$$\left\langle \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}\left(\frac{\bar{u}\sqrt{\varepsilon}u}{\varpi}\right),\ \mathcal{Z}^{\dagger}_{\mathrm{SO}(4)}((g \cdot \varpi^r)_-) \right\rangle_{\mathcal{M}_2 \times \mathcal{M}_2} = \mathrm{GK}(r, v(b), v(c), v(e)-1, v(d-a)).$$

Subtracting the two gives

$$\begin{aligned}\operatorname{Int}^\circ((g,u),\mathbf{1}_{K,\le r}) &= \left\langle \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(1),\ \mathcal{Z}^\dagger_{\mathrm{SO}(4)}(\bar{u}\sqrt{\varepsilon}u)^\circ,\ \mathcal{Z}^\dagger_{\mathrm{SO}(4)}((g\cdot\varpi^r)_-)\right\rangle_{\mathcal{M}_2\times\mathcal{M}_2} \\ &= \mathrm{GK}(r,v(b),v(c),v(e),v(d-a)) \\ &\qquad - \mathrm{GK}(r,v(b),v(c),v(e)-1,v(d-a)).\end{aligned} \tag{15.2}$$

15.3 Base change

The base change for $n=2$ was already calculated in [LRZ24] and we simply recall the result here.

As in Chapter 14, we define

$$\begin{aligned}\mathbf{1}_{K,\le r} &\coloneqq \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_2(O_E)\cap \mathrm{U}(\mathbb{V}_n^+)} \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+)) \\ \mathbf{1}_{K,r} &\coloneqq \mathbf{1}_{K,\le r} - \mathbf{1}_{K,\le(r-1)} \\ &= \mathbf{1}_{\varpi^{-r}\,\mathrm{Mat}_2(O_E)\cap \mathrm{U}(\mathbb{V}_n^+)} \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))\end{aligned}$$

Lemma 15.3.1 ([LRZ24, Lemma 7.1.1])**.** *For $n=2$ and $r\ge 1$ we have*

$$\begin{aligned}\mathrm{BC}^\eta_S(\mathbf{1}_{K'_{S,\le r}} + \mathbf{1}_{K'_{S,\le(r-1)}}) &= (-1)^r\mathbf{1}_{K,r} \\ &= (-1)^r(\mathbf{1}_{K,\le r} - \mathbf{1}_{K,\le(r-1)}).\end{aligned}$$

Proof. This follows directly from [LRZ24, Equation (7.1.9)]. ∎

15.4 Transfer factor

As stated in (12.1), the transfer factor is

$$\omega(\gamma, \mathbf{u}, \mathbf{v}^\top) = (-1)^{v(c)+1}.$$

15.5 Comparison to orbital formula

In what follows, we introduce the shorthand

$$\partial \operatorname{Orb}(r, v(b), v(c), v(e), v(d-a)) \coloneqq \partial \operatorname{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}}).$$

The main claim is that the following formula holds:

Theorem 15.5.1 (GK is a difference of two orbitals)**.** *We have*

$$\begin{aligned}&\frac{(-1)^{r+v(c)}}{\log q}\Big(\partial \operatorname{Orb}(r, v(b), v(c), v(e), v(d-a)) + \partial \operatorname{Orb}(r, v(b), v(c), v(e)-1, v(d-a))\Big) \\ &= \operatorname{GK}(r, v(b), v(c), v(e), v(d-a)).\end{aligned}$$

We continue to use the notation N and $\varkappa$ from Chapter 10 defined by

$$\begin{aligned} N &\coloneqq \min\left(v(e), \tfrac{v(b)+v(c)-1}{2} + r, v(d-a) + r\right) \\ \varkappa &\coloneqq v(e) - v(d-a) - r \ge 0 \end{aligned}$$

and prove Theorem 15.5.1 by exhausting the cases based on which value of N is smallest.

15.5.1 Proof of Theorem 15.5.1 when $v(e) \leq \frac{v(b)+v(c)-1}{2} + r$ and $v(e) \leq v(d-a) + r$

In the Gross-Keating formula we have simply

$$n_1 = 2v(e)$$
$$n_2 = v(b) + v(c) + 2r.$$

Hence, we have

$$\begin{aligned}\mathrm{GK}(r, v(b), v(c), v(e), v(d-a)) = &\frac{-2v(e) + v(b) + v(c) + 2r + 1}{2} q^{v(e)} \\ &+ \sum_{j=0}^{v(e)-1} (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j.\end{aligned}$$

On the orbital side, we refer to Corollary 1.3.2 and compare it to the single instance of Gross-Keating above. The exponent of j runs up to $v(e)$ in one case and $v(e) - 1$ in the second; that is we need

$$\begin{aligned}&\sum_{j=0}^{v(e)} \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j \\ &\quad + \sum_{j=0}^{v(e)-1} \left(\frac{2(v(e) - 1) + v(b) + v(c) + 1}{2} + r - 2j \right) \cdot q^j \\ &= \frac{-2v(e) + v(b) + v(c) + 2r + 1}{2} q^{v(e)} + \sum_{j=0}^{v(e)-1} (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j\end{aligned}$$

which is obvious.

15.5.2 Proof of Theorem 15.5.1 when $\frac{v(b)+v(c)-1}{2}+r<v(e)$ and $v(b)+v(c)<2v(d-a)$

Set $N=\frac{v(b)+v(c)-1}{2}+r$. In the Gross-Keating formula we have simply

$$n_1 = 2N+1$$
$$n_2 = 2v(e).$$

Hence

$$\mathrm{GK}(r,v(b),v(c),v(e),v(d-a)) = \sum_{j=0}^{N}(2v(e)+v(b)+v(c)+2r-4j)\cdot q^j.$$

We compare this to Corollary 1.3.2; we check

$$\begin{aligned}
&\sum_{j=0}^{N}\left(\frac{2v(e)+v(b)+v(c)+1}{2}+r-2j\right)\cdot q^j \\
&\quad+\sum_{j=0}^{N}\left(\frac{2(v(e)-1)+v(b)+v(c)+1}{2}+r-2j\right)\cdot q^j \\
&=\sum_{j=0}^{N}(2v(e)+v(b)+v(c)+2r-4j)\cdot q^j
\end{aligned}$$

which is clear.

15.5.3 Proof of Theorem 15.5.1 when $v(d-a)+r<v(e)$ and $2v(d-a)<v(b)+v(c)$

Hence $N=v(d-a)+r$ and $\varkappa:=v(e)-(v(d-a)+r)>0$. In the Gross-Keating side formula, we now have

$$n_1 := 2v(d-a)+2r = 2N$$

$$n_2 = 2v(e) + v(b) + v(c) - 2v(d-a).$$

Hence

$$\begin{aligned}\mathrm{GK}(r, v(b), v(c), v(e), v(d-a)) = {}& \frac{2v(e) + v(b) + v(c) - 4v(d-a) - 2r + 1}{2} q^N \\ &+ \sum_{j=0}^{N-1} (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j.\end{aligned}$$

This time, the relevant combination of Corollary 1.3.2 is

$$\begin{aligned}
&\sum_{j=0}^{N} q^j \cdot \left(\frac{2v(e) + v(b) + v(c) + 1}{2} + r - 2j\right) \\
&\quad + q^N \cdot \begin{cases} -\frac{\varkappa}{2} & \text{if } \varkappa \equiv 0 \pmod 2 \\ \frac{\varkappa}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right) & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases} \\
&\quad + \sum_{j=0}^{N} q^j \cdot \left(\frac{2(v(e)-1) + v(b) + v(c) + 1}{2} + r - 2j\right) \\
&\quad + q^N \cdot \begin{cases} -\frac{\varkappa-1}{2} & \text{if } \varkappa - 1 \equiv 0 \pmod 2 \\ \frac{\varkappa-1}{2} - \left((v(e)-1) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right) & \text{if } \varkappa - 1 \equiv 1 \pmod 2 \end{cases} \\
&= \sum_{j=0}^{N} q^j \cdot (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j \\
&\quad + q^N \cdot \begin{cases} -\frac{\varkappa}{2} + \frac{\varkappa-1}{2} - \left((v(e)-1) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right) & \text{if } \varkappa \equiv 0 \pmod 2 \\ -\frac{\varkappa-1}{2} + \frac{\varkappa}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right) & \text{if } \varkappa \equiv 1 \pmod 2 \end{cases} \\
&= \sum_{j=0}^{N} q^j \cdot (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j \\
&\quad + q^N \cdot \left(\frac{1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r\right)\right) \\
&= \sum_{j=0}^{N-1} q^j \cdot (2v(e) + v(b) + v(c) + 2r - 4j) \cdot q^j
\end{aligned}$$

$$+ q^N \cdot \left((2v(e) + v(b) + v(c) + 2r - 4N) + \frac{1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r \right) \right).$$

The coefficient of q^N is given by

$$\begin{aligned}
&(2v(e) + v(b) + v(c) + 2r - 4(v(d-a) - r)) + \frac{1}{2} - \left(v(e) + \frac{v(b)+v(c)}{2} - 2v(d-a) - r \right) \\
&= \frac{2v(e) + v(b) + v(c) - 4v(d-a) - 2r + 1}{2}
\end{aligned}$$

which matches the one from Gross-Keating. Hence Theorem 15.5.1 is completely proved.

15.6 Conclusion (proof of Theorem 1.3.8)

From Theorem 15.5.1 we have

$$\begin{aligned}
\mathrm{GK}(r, v(b), v(c), v(e), v(d-a)) &= \frac{(-1)^{r+v(c)}}{\log q} \Big(\partial \operatorname{Orb}(r, v(b), v(c), v(e), v(d-a)) \\
&\qquad + \partial \operatorname{Orb}(r, v(b), v(c), v(e)-1, v(d-a)) \Big) \\
\mathrm{GK}(r, v(b), v(c), v(e)-1, v(d-a)) &= \frac{(-1)^{r+v(c)}}{\log q} \Big(\partial \operatorname{Orb}(r, v(b), v(c), v(e)-1, v(d-a)) \\
&\qquad + \partial \operatorname{Orb}(r, v(b), v(c), v(e)-2, v(d-a)) \Big)
\end{aligned}$$

so subtraction (and recalling (15.2)) gives

$$\begin{aligned}
\operatorname{Int}^\circ((g,u), \mathbf{1}_{K,\le r}) &= \mathrm{GK}(r, v(b), v(c), v(e), v(d-a)) \\
&\qquad - \mathrm{GK}(r, v(b), v(c), v(e)-1, v(d-a)) \\
&= \frac{(-1)^{r+v(c)}}{\log q} \Big(\partial \operatorname{Orb}(r, v(b), v(c), v(e), v(d-a)) \\
&\qquad - \partial \operatorname{Orb}(r, v(b), v(c), v(e)-2, v(d-a)) \Big).
\end{aligned} \tag{15.3}$$

We now show that (15.3) implies Theorem 1.3.8. Because $r = 0$ is known already, it suffices to verify for $r > 0$.

Suppose we sum (15.3) with u replaced by u/ϖ^i for $i = 0, 1, \ldots$. The left-hand side equals $\mathrm{Int}((g,u), \mathbf{1}_{K,\le r})$ by (14.1). On the right-hand side this has the effect of decreasing $v(e)$ by 2 since $e = \mathrm{Nm}\, u$. Hence the sum of the right-hand sides telescopes and gives us the identity

$$\mathrm{Int}((g,u), \mathbf{1}_{K,\le r}) = \frac{(-1)^{v(c)+r}}{\log q} \partial\, \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}}). \tag{15.4}$$

Subtracting the same equation from itself with r replaced by $r-1$ gives

$$\begin{aligned}&\mathrm{Int}((g,u), \mathbf{1}_{K,\le r} - \mathbf{1}_{K,\le(r-1)})\\ &= \frac{(-1)^{v(c)+r}}{\log q} \partial\, \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}} + \mathbf{1}_{K'_{S,\le(r-1)}}).\end{aligned}$$

which, since $(-1)^{v(c)} = -\omega(\gamma, \mathbf{u}, \mathbf{v}^\top)$, becomes

$$\begin{aligned}&\mathrm{Int}((g,u), (-1)^r(\mathbf{1}_{K,\le r} - \mathbf{1}_{K,\le(r-1)}))\\ &= \frac{-\omega(\gamma, \mathbf{u}, \mathbf{v}^\top)}{\log q} \partial\, \mathrm{Orb}((\gamma, \mathbf{u}, \mathbf{v}^\top), \mathbf{1}_{K'_{S,\le r}} + \mathbf{1}_{K'_{S,\le(r-1)}}).\end{aligned} \tag{15.5}$$

But Lemma 15.3.1 says that $(-1)^r(\mathbf{1}_{K,\le r} - \mathbf{1}_{K,\le(r-1)}) \in \mathcal{H}(\mathrm{U}(\mathbb{V}_n^+))$ matches $\mathbf{1}_{K'_{S,\le r}} + \mathbf{1}_{K'_{S,\le(r-1)}} \in \mathcal{H}(S_2(F))$ for any $r \ge 0$. And hence from the $r = 0$ case we can inductively conclude Theorem 1.3.8 for $r > 0$, completing the proof.

15.7 A particularly clean formula for a certain intersection number

We mention in particular that the value of

$$\mathrm{Int}^\circ((g,u), 1_{K,r}) = \left\langle \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,r}} \Delta_{\mathcal{Z}(u)^\circ}, \Gamma_g \right\rangle_{\mathcal{N}_{2,2}}$$

(note the change from $\mathbf{1}_{K,\leq r}$ to $\mathbf{1}_{K,\leq r}$ here) has a particularly clean formula that seems worth mentioning. We phrase this entirely based on the quantities in the geometric side to keep in self-contained.

Theorem 15.7.1 ($\mathrm{Int}^\circ((g,u),1_{K,r})$)**.** *Let $r \geq 1$ and $v(\mathrm{Nm}\, u) > 0$ for $u \in \mathbb{V}_2^-$, and let*

$$g = \lambda^{-1}\begin{pmatrix} \alpha & \bar{\beta}\varpi \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathrm{U}(\mathbb{V}_2^-)$$

where $v(\lambda) = 0$. Then

$$\left\langle \mathbb{T}_{\mathbf{1}_K \otimes \mathbf{1}_{K,r}} \Delta_{\mathcal{Z}(u)^\circ}, \Gamma_g \right\rangle_{\mathcal{N}_{2,2}}$$

is equal to

$$\begin{cases} (C+1)q^N + (C+2)q^{N-1} & \text{if } v(\mathrm{Nm}\, u) - r = v(\alpha - \bar{\alpha}) \leq v(\beta) \\ 2q^N & \text{if } v(\beta) + r < \min(v(\mathrm{Nm}\, u), v(\alpha-\bar{\alpha}) + r) \\ q^N + q^{N-1} & \text{otherwise} \end{cases}$$

where

$$N = \min(v(\mathrm{Nm}\, u), v(\beta) + r, v(\alpha - \bar{\alpha}) + r)$$

and we write

$$C = v(\beta) - v(\alpha - \bar{\alpha}) \geq 0$$

in the first case.

Proof. Recall that

$$\begin{aligned} \mathrm{GK}(r, v(b), v(c), v(e), v(d-a)) &= \frac{(-1)^{r+v(c)}}{\log q}\Big(\partial\,\mathrm{Orb}(r, v(b), v(c), v(e), v(d-a)) \\ &\qquad + \partial\,\mathrm{Orb}(r, v(b), v(c), v(e)-1, v(d-a))\Big) \\ \mathrm{GK}(r-1, v(b), v(c), v(e), v(d-a)) &= \frac{(-1)^{r-1+v(c)}}{\log q}\Big(\partial\,\mathrm{Orb}(r-1, v(b), v(c), v(e), v(d-a)) \end{aligned}$$

$$- \partial \operatorname{Orb}(r-1, v(b), v(c), v(e)-1, v(d-a))\Big)$$

when we subtract we obtain that

$$\begin{aligned}
\operatorname{Int}^\circ((g,u), \mathbf{1}_{K,r}) = \frac{(-1)^{r+v(c)}}{\log q}\Big(&\partial \operatorname{Orb}(r, v(b), v(c), v(e), v(d-a)) \\
&+ \partial \operatorname{Orb}(r-1, v(b), v(c), v(e), v(d-a)) \\
&- \partial \operatorname{Orb}(r, v(b), v(c), v(e)-2, v(d-a)) \\
&+ \partial \operatorname{Orb}(r-1, v(b), v(c), v(e)-2, v(d-a))\Big).
\end{aligned}$$

Gathering the first two terms lets us apply the simpler formula Corollary 1.3.3 twice; doing so gives

$$\begin{aligned}
\operatorname{Int}^\circ((g,u), \mathbf{1}_{K,r}) &= \left((q^N + q^{N-1} + \cdots + 1) + Cq^N + C'q^{N-1}\right) \\
&\quad - \left((q^{N^\flat} + q^{N^\flat - 1} + \cdots + 1) + C^\flat q^{N^\flat} + (C')^\flat q^{N^\flat - 1}\right)
\end{aligned}$$

where N, C, C' are is in Corollary 1.3.3, and $N^\flat$, $C^\flat$, $(C')^\flat$ are the same quantities with $v(e)$ replaced by $v(e) - 2$. Let $\varkappa$ and $\varkappa^\flat = \varkappa - 2$ be also as in Corollary 1.3.3.

We consider cases now.

- Suppose first $v(e) \le \frac{v(b)+v(c)-1}{2} + r$ and $v(e) < v(d-a) + r$. Then $N = v(e)$ and $N^\flat = v(e) - 2$ and $C = C' = (C^\flat) = (C')^\flat = 0$, Hence in this case we have

$$\operatorname{Int}^\circ((g,u), \mathbf{1}_{K,r}) = q^N + q^{N-1}.$$

- Next suppose $2v(d-a) > v(b) + v(c)$ and consider cases on $v(e)$. We only need to consider $v(e) > \frac{v(b)+v(c)-1}{2} + r$.

– If $v(e) = \frac{v(b)+v(c)-1}{2} + r + 1$ then we have

$$N = \frac{v(b) + v(c) - 1}{2} + r, \qquad N^\flat = \frac{v(b) + v(c) - 1}{2} + r - 1$$

and $C = 1$, $C^\flat = 0$, and $C' = (C')^\flat = 0$. Consequently we get

$$\mathrm{Int}^\circ((g, u), \mathbf{1}_{K,r}) = 2q^N.$$

– Once $v(e) \geq \frac{v(b)+v(c)-1}{2} + r + 2$ we always have $N = N^\flat = \frac{v(b)+v(c)-1}{2} + r$,

$$C - C^\flat = (v(e) - N) - ((v(e) - 2) - N) = 2$$

and $C' = (C')^\flat = 0$. Hence in this case we have

$$\mathrm{Int}^\circ((g, u), \mathbf{1}_{K,r}) = 2q^N$$

as well.

- Finally suppose $2v(d - a) < v(b) + v(c)$ and consider cases on $v(e)$. We only need to consider $v(e) \geq v(d - a) + r$.

  – If $v(e) = v(d - a) + r$, then

$$N = v(d - a) + r, \qquad N^\flat = v(d - a) + r - 2.$$

In this case $\varkappa = 0$ (and $\varkappa^\flat = -2$). So $C^\flat = (C')^\flat = 0$ but we have larger terms

$$C = \frac{v(b) + v(c) - 2v(d - a) - 1}{2}$$
$$C' = \frac{v(b) + v(c) - 2v(d - a) + 1}{2}.$$

Hence, we get an exceptional case

$$\mathrm{Int}^\circ((g,u),\mathbf{1}_{K,r}) = \frac{v(b)+v(c)-2v(d-a)+1}{2}q^N + \frac{v(b)+v(c)-2v(d-a)+3}{2}q^{N-1}$$

– If $v(e) = v(d-a)+r+1$, then we have

$$N = v(d-a)+r, \qquad N^\flat = v(d-a)+r-1.$$

In this case $\varkappa = 1$ (and $\varkappa^\flat = -1$) so we have $C = 0$, $C' = 1$, $C^\flat = 0 = (C')^\flat = 0$.

Consequently we get

$$\mathrm{Int}^\circ((g,u),\mathbf{1}_{K,r}) = q^N + q^{N-1}.$$

– Once $v(e) \geq v(d-a)+r+2$, we always have $N = N^\flat = v(d-a)+r$ and

$$C - C^\flat = (C') - (C'^\flat) = 1$$

regardless of the parity of $\varkappa$. Hence in this case we also get

$$\mathrm{Int}^\circ((g,u),\mathbf{1}_{K,r}) = q^N + q^{N-1}.$$

Hence, in summary we get that

$$\mathrm{Int}^\circ((g,u),\mathbf{1}_{K,r}) = \begin{cases} (C+1)q^N + (C+2)q^{N-1} & \text{if } v(e)-r = v(d-a) \leq \frac{v(b)+v(c)-1}{2} \\ 2q^N & \text{if } \frac{v(b)+v(c)-1}{2} + r < \min(v(e), v(d-a)+r) \\ q^N + q^{N-1} & \text{otherwise.} \end{cases} \tag{15.6}$$

where

$$C = \frac{v(b) + v(c) - 2v(d - a) - 1}{2} \geq 0$$

in the first case. Then (15.6) translates via Lemma 15.1.1 into the desired claim. $\square$

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Appendix A

Sage implementations of formulas and test cases

Below we provide Sage implementations and test suites for some of the calculations this paper.

A.1 Orbital and Gross-Keating implementations and tests

A.1.1 Description of implemented functions

Semi-Lie orbital integral in $S_2(F) \times V_2'(F)$

All the formulas in Section 1.3.1 are implemented.

- `O(r, vb, vc, ve, vda)` implements Theorem 1.3.1.
- `delO(r, vb, vc, ve, vda)` implements Corollary 1.3.2, divided by a factor of $\log q$.
- `delO_combo(r, vb, vc, ve, vda)` implements Corollary 1.3.3, divided by a factor of $\log q$.

Inhomogeneous orbital integral for $S_3(F)$

The next part of the program implements the orbital integral that is alluded to in Theorem 1.3.9. The parameters are those described in Lemma 5.3.3 and Lemma 5.4.1.

- The ARCH function defined in Definition 5.5.1 is implemented as `ARCH(a0, a1, w1, w2, k)`.
- There are some common sums appearing in the formulas Theorems 5.5.2, 5.5.7 and 5.5.10 which are implemented as follows.
  - The function `ARCH_sum_n(a0, a1, w1, w2)` implements the sum
  $$\sum_{k=a_0}^{a_1} \left(1 + q + \cdots + q^{\mathrm{ARCH}_{[a_0,a_1]}(w_1,w_2)(k)}\right)(-q^s)^k \in \mathbb{Z}[q^s, q].$$
  - The function `ARCH_sum_c(a0, a1, w1, w2)` implements the sum
  $$\sum_{k=a_0}^{a_1} \underset{[a_0,a_1]}{\mathrm{ARCH}}(w_1, w_2)(k)(-q^s)^k \in \mathbb{Z}[q^s].$$
- Using these functions, we implement `O_for_S3(r, l, delta, lam)` as the full orbital integral for all the cases. Hence this function implements all there of Theorems 5.5.2, 5.5.7 and 5.5.10.

Derivative of the inhomogeneous orbital integral for $S_3(F)$

We also implement functions that can compute the derivative of the orbital integral for $S_3(F)$.

- `ARCH_deriv_n(r, C, W, H)` implements Lemma 7.7.1.
- `ARCH_deriv_c(r, C, W, H)` implements Lemma 7.7.3.
- `delO_for_S3_via_arch(r, l, delta, lam)` implements $\frac{1}{\log q}\partial\,\mathrm{Orb}(\gamma, \phi)$ by calling the previous two functions, using the arguments that are mentioned in the proof of

Theorem 11.5.1.

- `delO_for_S3(r, l, delta, lam)` implements the final formula for $\frac{1}{\log q}\partial \operatorname{Orb}(\gamma, \phi)$ detailed in Theorem 5.6.1. It is the result of directly substituting Lemma 7.7.1 and Lemma 7.7.3 with the parameters described in the proof of Theorem 5.6.1

The geometric side

- `gross_keating_sum(n1, n2)` implements the sum in Proposition 14.7.1.
- `GK(r, vb, vc, ve, vda)` implements the sum $\operatorname{GK}(r, v(b), v(c), v(e), v(d-a))$ that we introduced in (15.1).
- `clean_intersection(r, vb, vc, ve, vda)` implements the right-hand side of Theorem 15.7.1, although we use the argument names corresponding to the matched elements after the translation in Lemma 15.1.1.

A.1.2 Randomized test suite

The code then implements the following tests to check correctness. These tests are randomized tests where the parameters are randomly selected by a program which can then numerically compute them; they do not purport to be symbolic or formal proofs of the formulas.

1. `test_O` verifies that the formula Theorem 1.3.1 matches the casework described in Chapter 9. To speed up the test, rather than symbolically comparing, it selects the values $q = 17$ and $s = \log_q 1337$ in comparing the sides.

   Within this test, some auxiliary functions are defined.

   - The function `O_brute_odd(r, vb, vc, ve, vda)` is a naïve implementation of the casework in Chapter 9 when θ is odd.

- The function `O_brute_even(r, vb, vc, ve, vda)` is a naïve implementation of the casework in Chapter 9 when θ is even (encompassing both **Case 5**, **Case 6**$^{+}$ and **Case 6**$^{-}$).

2. `test_delO` verifies that Corollary 1.3.2 matches the derivative of Theorem 1.3.1.

3. `test_delO_combo` verifies that Corollary 1.3.3 matches a subtraction of Corollary 1.3.2.

4. `test_matrix_upper_triangular` verifies that the important entries of the matrix in Lemma 11.1.1 are computed correctly (that is, one indeed gets an upper triangular matrix when looking at the relevant rows).

5. `test_kernel_large_r` verifies Lemma 11.2.1 holds.

6. `test_kernel_full` verifies Theorem 11.3.1 holds.

7. `test_O_for_S3` verifies that the formulas Theorems 5.5.2, 5.5.7 and 5.5.10 match the casework described in Chapter 6. To speed up the test, rather than symbolically comparing, it selects the values $q = 17$ and $s = \log_q 1337$ in comparing the sides.

   There are many subfunctions in here used for the naïve implementations of the casework.

   - `O_zero(r, l, delta)` implements $I_{\leq 0}$ as in Proposition 7.1.1.
   - The function `vol_1disk(n, vxx, rho)` implements lemma 2.2.1. Here the argument `vxx` corresponds to $v(1 - \xi\bar{\xi})$.
   - The function `vol_2disk(n, vxx, rho)` implements lemma 2.2.3. Here the argument `vxx1` corresponds to $v(1 - \xi_1\bar{\xi}_1)$ and the argument `vxx2` corresponds to $v(1 - \xi_2\bar{\xi}_2)$.
   - The function `qs_weight(n, m)` implements the weight

$$\varkappa \cdot (-1)^n q^{s(2m-n)} q^{2n-2m} \Big(q^{2m}(1 - q^{-2}) \Big)$$

that occurs frequently throughout. This expression is the product of the factor $\varkappa \cdot (-1)^n q^{s(2m-n)} q^{2n-2m}$ from Section 6.1.2 and the volume factor $\mathrm{Vol}(t_1 : -v(t_1) = m) = q^{2m}(1 - q^{-2})$.

- The function `O_case_1_2_brute(r, l, delta, lam=None)` is a naïve implementation of **Case 1** and **Case 2** from Chapter 6. For odd ℓ, we combine it with $I_{\leq 0}$ in `O_ell_odd_brute(r, l, delta)` to get a naïve implemenattion of Theorem 5.5.2. For even ℓ we instead get (together with $I_{\leq 0}$) is Proposition 7.3.1.

- The function `O_case_3_4_brute(r, l, delta, lam)` is a naïve implementation of **Case 3**$^+$, **Case 3**$^-$, **Case 4**$^+$ and **Case 4**$^-$ (asserting it never occurs) from Chapter 6. Putting all the cases together gives `O_ell_odd_brute(r, l, delta, lam)` which is a naïve implementation of Theorem 5.5.7.

- The function `O_ell_neg_brute(r, l, delta, lam)` is a separate naïve implementation of Theorem 5.5.10. It re-does the cases separately in the same was as this paper rather than using the previous brute-force implementation.

8. `test_delO_for_S3_via_arch` verifies that the derivatives of the formulas in Theorems 5.5.2, 5.5.7 and 5.5.10 match those predicted by Lemma 7.7.1 and Lemma 7.7.1. Hence it can be thought of as a verification of those two lemmas.

9. `test_delO_for_S3` verifies Theorem 5.6.1.

10. `test_ker_for_S3` verifies Theorem 11.5.1.

11. `test_GK_to_orbital` verifies Theorem 15.5.1.

12. `test_clean_intersection` verifies Theorem 15.7.1.

A.1.3 Code listing

```python
import argparse
import unittest

q = var("q")
qs = var("q_s") # = q^s


def irange(start, stop):
    return range(start, stop + 1)


def print_coeffs(expression) -> None:
    """
    If you're working in a Juptyer notebook, and you have a polynomial in q and qs,
    you can use this utility function to print out the coefficients of q^s each on their
    own line.

    :param expression: The polynomial to be pretty-printed
    """
    if expand(expression) == 0:
        show(0)
    else:
        for c in expand(expression).coefficients(qs, sparse=True):
            show(c[1], "." * 10, c[0])


# Semi-Lie Orbital and its derivatives
def O(r, vb, vc, ve, vda):
    assert vb % 2 != vc % 2, (vb, vc)
    assert r >= 0, r
    assert vb + vc >= 0, (vb, vc)
    S = 0
    for k in irange(-vb - r, 2 * ve + vc + r):
        n = min(
            (k + (vb + r)) // 2,
            (2 * ve + vc + r - k) // 2,
            ve,
            min((vb + vc) // 2, vda) + r,
        )
        S += (-qs) ** k * sum([q**i for i in irange(0, n)])
    if vda < ve - r and vb + vc > 2 * vda:
        for k in irange(2 * vda - vb + r, 2 * ve + vc - 2 * vda - r):
```

```
            c = min(k - (2 * vda - vb + r), 2 * ve + vc - 2 * vda - r - k, ve - vda - r)
            S += q ** (vda + r) * (-qs) ** k * c
    return S


def del0(r, vb, vc, ve, vda):
    assert r >= 0, r
    assert vb + vc >= 0 and vb % 2 != vc % 2, (vb, vc)
    varkappa = ve - vda - r
    N = min(ve, floor((vb + vc) // 2 + r), vda + r)
    j = var("j")
    S = sum(q**j * (floor((2 * ve + vb + vc + 1) / 2) + r - 2 * j), j, 0, N)

    if varkappa >= 0 and vb + vc > 2 * vda:
        if varkappa % 2 == 0:
            S += q ** (vda + r) * (-varkappa / 2)
        else:
            S += q ** (vda + r) * (varkappa / 2 - (ve + (vb + vc) / 2 - 2 * vda - r))
    return (-1) ** (r + vc) * S


def del0_combo(r, vb, vc, ve, vda):
    assert r >= 1
    N = min(ve, (vb + vc) // 2 + r, vda + r)
    j = var("j")
    S = sum(q**j, j, 0, N)

    if ve >= vda + r and vb + vc > 2 * vda:
        assert N > 0
        if (r + ve + vda) % 2 == 1:
            C = (ve - r - vda) // 2
        else:
            C = (ve + vb + vc - r - 3 * vda) // 2
        S += C * q**N + (C + 1) * q ** (N - 1)

    elif 2 * vda > vb + vc and ve >= (vb + vc) // 2 + r:
        C = ve - r - (vb + vc) // 2
        S += C * q**N

    else:
        pass

```

```
    return (-1) ** (r + vc) * S


# Formulas for the group AFL on S3(F)
def ARCH(a0, a1, w1, w2, k):
    assert a0 <= a1
    assert w1 + w2 <= (a1 - a0) / 2
    assert a0 <= k <= a1

    if a0 <= k <= a0 + w1:
        return k - a0
    elif a0 + w1 <= k <= a0 + w1 + w2:
        return w1 + floor((k - (a0 + w1)) / 2)
    elif a0 + w1 + w2 <= k <= a1 - (w1 + w2):
        return w1 + floor(w2 / 2)
    elif a1 - (w1 + w2) <= k <= a1 - w1:
        return w1 + floor(((a1 - w1) - k) / 2)
    elif a1 - w1 <= k <= a1:
        return a1 - k
    else:
        raise ValueError


def ARCH_sum_n(a0, a1, w1, w2):
    j = var("j")
    S = 0
    for k in irange(a0, a1):
        S += sum(q**j, j, 0, ARCH(a0, a1, w1, w2, k)) * (-qs) ** k
    return S


def ARCH_sum_c(a0, a1, w1, w2):
    S = 0
    for k in irange(a0, a1):
        S += ARCH(a0, a1, w1, w2, k) * (-qs) ** k
    return S


# Orbital for S3
def O_for_S3(r, l, delta, lam):
    if l % 2 == 1:
        assert lam == l, (l, lam)
    else:
        assert l < lam, (l, lam)
    if l < 0 or delta < 0:
```

```
        assert l == delta and l % 2 == 0, (l, delta)
    else:
        assert l <= 2 * delta, (l, delta)

    S = 0
    if l % 2 == 1:
        return ARCH_sum_n(-2 * r, 2 * delta + l + 2 * r, r, l)
    elif l % 2 == 0 and l >= 0:
        n_sum = ARCH_sum_n(-2 * r, 2 * delta + lam + 2 * r, r, l)
        c_sum = ARCH_sum_c(
            l - r, 2 * delta + lam - l + r, delta - l / 2, min(lam - l - 1, 2 * r)
        )
        return n_sum + q ** (r + l / 2) * c_sum
    else:
        assert (avd := -l / 2) > 0 # avd := abs(vd)
        if r < avd:
            return 0
        n_sum = ARCH_sum_n(-2 * r, lam + 2 * (r - 2 * avd), r - avd, 0)
        c_sum = ARCH_sum_c(-r - avd, lam + r - 3 * avd, 0, min(lam - 1, 2 * (r - avd)))
        return n_sum + q ** (r - avd) * c_sum


# Derivatives of the orbital integral
def ARCH_deriv_n(r, C, W, H):
    j = var("j")
    assert W > 4 * H >= 0, (W, H)
    assert W % 2 == 1, W
    return -(
        (-1) ** (r + C) * sum(((W + 1) / 2 + r - 2 * (j - r)) * q**j, j, r + 1, r + H)
        + sum((-1) ** (j + C) * ((W + 1) / 2 + 2 * r - j) * q**j, j, 0, r)
    )


def ARCH_deriv_c(r, C, W, L):
    k = var("k")
    assert L >= 1, L
    assert W >= 0, W
    assert L % 2 == 1, L
    return (-1) ** (r + W + C) * (
        W / 2 - (L - 1) / 2 * r if W % 2 == 0 else -(W + L) / 2 - (L + 1) / 2 * r
```

```
    )


def del0_for_S3_via_arch(r, l, delta, lam):
    if l % 2 == 1:
        assert lam == l, (l, lam)
    else:
        assert l < lam, (l, lam)
    if l < 0 or delta < 0:
        assert l == delta and l % 2 == 0, (l, delta)
    else:
        assert l <= 2 * delta, (l, delta)

    if l % 2 == 1:
        return ARCH_deriv_n(r, C=0, W=l + 2 * delta, H=(l - 1) / 2)
    elif l >= 0:
        n_sum = ARCH_deriv_n(r, C=0, W=lam + 2 * delta, H=l / 2)
        c_sum = ARCH_deriv_c(r, C=0, W=delta - l / 2, L=lam - l)
        return n_sum + q ** (r + l / 2) * c_sum
    else:
        assert (avd := -l / 2) > 0 # avd := abs(vd)
        if r < avd:
            return 0
        n_sum = ARCH_deriv_n(r - avd, C=-2 * avd, W=lam, H=0)
        c_sum = ARCH_deriv_c(r - avd, C=0, W=0, L=lam)
        return n_sum + q ** (r - avd) * c_sum


def del0_for_S3(r, l, delta, lam):
    if l % 2 == 1:
        assert lam == l, (l, lam)
    else:
        assert l < lam, (l, lam)
    if l < 0 or delta < 0:
        assert l == delta and l % 2 == 0, (l, delta)
    else:
        assert l <= 2 * delta, (l, delta)

    j = var("j")
    if l % 2 == 1:
        S = (-1) ** (r + 1) * sum(
            ((l + 2 * delta + 1) / 2 + 3 * r - 2 * j) * q**j, j, r + 1,
                r + (l - 1) / 2
        )
        S += sum(
```

```
            (-1) ** (j + 1) * ((l + 2 * delta + 1) / 2 + 2 * r - j) * q**j, j, 0, r
        )
        return S
    elif l >= 0:
        frac = (lam + 2 * delta + 1) / 2
        S = (-1) ** (r + 1) * sum((frac + 3 * r - 2 * j) * q**j, j, r + 1, r + l / 2)
        S += sum((-1) ** (j + 1) * (frac + 2 * r - j) * q**j, j, 0, r)
        S += (
            (-1) ** (r + delta - l / 2)
            * q ** (r + l / 2)
            * (
                (delta - l / 2) / 2 - (lam - l - 1) / 2 * r
                if (delta - l // 2) % 2 == 0
                else -(delta - 3 * l / 2 + lam) / 2 - (lam - l + 1) / 2 * r
            )
        )
        return S
    else:
        assert (avd := -l / 2) > 0 # avd := abs(vd)
        t = r - avd # top-level exponent (if t < 0, then S = 0)
        S = sum((-1) ** (j + 1) * ((lam + 1) / 2 + 2 * t - j) * q**j, j, 0, t)
        S += (-1) ** (t + 1) * (lam - 1) / 2 * max(t, 0) * q**t
        return S


# Geometric side --- Gross-Keating and friends
def gross_keating_sum(n1, n2):
    j = var("j")
    assert 0 <= n1 <= n2
    if n1 % 2 == 1:
        return sum((n1 + n2 - 4 * j) * q**j, j, 0, (n1 - 1) / 2)
    else:
        S = sum((n1 + n2 - 4 * j) * q**j, j, 0, n1 / 2 - 1)
        S += (n2 - n1 + 1) / 2 * q ^ (n1 / 2)
        return S


def GK(r, vb, vc, ve, vda):
    assert r >= 0
    assert vb + vc >= 0 and vb % 2 != vc % 2
    vxx = 2 * ve
```

```
    vxy = r + vda + ve
    vdet = 2 * r + 2 * ve + vb + vc
    vyy = min(2 * vxy, vdet) - vxx

    return gross_keating_sum(min(vxx, vxy, vyy), vdet - min(vxx, vxy,
        vyy))


def clean_intersection(r, vb, vc, ve, vda):
    vbeta = (vb + vc - 1) / 2
    N = min(ve, vbeta + r, vda + r)
    C = vbeta - vda
    if ve - r == vda <= vbeta:
        return (C + 1) * q**N + (C + 2) * q ** (N - 1)
    elif vbeta + r < min(ve, vda + r):
        return 2 * q**N
    else:
        return q**N + q ** (N - 1)


class RandThesisTest(unittest.TestCase):
    def get_semi_lie_params(self, r_min=0, r_max=15):
        params = {
            "r": randint(r_min, r_max),
            "vb": randint(-5, 15),
            "vda": randint(0, 15),
            "ve": randint(0, 15),
        }
        params["vc"] = randrange(1 - params["vb"], 15, 2)
        assert params["vb"] % 2 != params["vc"] % 2
        assert params["vb"] + params["vc"] >= 0
        return params

    def test_O(self):
        params = self.get_semi_lie_params()

        def O_brute_odd(r, vb, vc, ve, vda):
            assert vb % 2 != vc % 2, (vb, vc)
            assert vda >= 0, vda
            assert r >= 0, r
            assert vda * 2 > vb + vc, (vda, vb, vc)

            voffset = vda - vc
            S = 0

```

```python
        for n2 in irange(0, ve):
            for n1 in irange(n2 - vb - r, n2 + vc + r):
                rho1 = max(-n1, -r - vc)
                rho2 = ceil((n2 - n1 - vc - r) / 2)
                if voffset < min(rho1, rho2):
                    continue
                else:
                    S += (
                        q ** (n2 - n1)
                        * q ** (-max(rho1, rho2))
                        * (-1) ** (n1 + n2)
                        * qs ** (n1 + n2)
                    )
        return S

    def O_brute_even(r, vb, vc, ve, vda):
        assert vb % 2 != vc % 2, (vb, vc)
        assert vda >= 0, vda
        assert r >= 0, r
        assert vda * 2 < vb + vc, (vda, vb, vc)
        theta = vda * 2
        vt = theta // 2 - vc

        S = 0

        for n2 in irange(0, ve):
            # Case 5
            for m in irange(0, theta + 2 * r):
                n1 = n2 + vc + r - m
                S += (
                    q ** (m - max(m - n2, ceil(m / 2)))
                    * (-1) ** (n1 + n2)
                    * qs ** (n1 + n2)
                )

            # Case 6+/6-
            center_plus = vb - theta // 2
            center_minus = theta // 2 - vc

            for m in irange(
                theta + 2 * r + 1,
                max(n2 + vc + r, 2 * vc + 2 * r + vt) + center_plus,
            ):
                n1 = n2 + vc + r - m
                rho1 = max(m - n2, 0) - vc - r
```

```
                    rho2 = m - 2 * vc - 2 * r - vt
                    if center_plus >= min(rho1, rho2):
                        S += (
                            q ** (n2 - n1)
                            * q ** (-max(rho1, rho2))
                            * (-1) ** (n1 + n2)
                            * qs ** (n1 + n2)
                        )
                    if center_minus >= min(rho1, rho2):
                        S += (
                            q ** (n2 - n1)
                            * q ** (-max(rho1, rho2))
                            * (-1) ** (n1 + n2)
                            * qs ** (n1 + n2)
                        )

            return S

        brute_res = (
            O_brute_odd(**params)
            if params["vb"] + params["vc"] < 2 * params["vda"]
            else O_brute_even(**params)
        )

        orb = O(**params)
        self.assertEqual(orb.subs(q_s=1), 0)
        self.assertEqual(brute_res.subs(q=17, q_s=1337), orb.subs(q=17, q_s=1337))

    def test_delO(self):
        params = self.get_semi_lie_params()
        self.assertEqual(
            delO(**params),
            derivative(O(**params), qs).subs(q_s=1),
        )

    def test_delO_combo(self):
        params = self.get_semi_lie_params(r_min=1)
        r = params.pop("r")
        self.assertEqual(
            delO(r, **params) + delO(r - 1, **params),
            delO_combo(r, **params),
        )

    def test_matrix_upper_triangular(self):
```

```
        N = 7
        params = self.get_semi_lie_params(r_min=0, r_max=N)
        del params["ve"]
        del params["r"]
        vb, vc, vda = params["vb"], params["vc"], params["vda"]
        theta = min(vb + vc, 2 * vda)

        # Here M0 = M, M1 = M', M2 = M''
        M0 = matrix(
            [
                vector(
                    (-1) ** (r + vc) * del0(r, vb, vc, i, vda) for r in range(0, N + 1)
                )
                for i in irange(0, N + theta // 2 + 1)
            ]
        )
        M1 = matrix([M0[0]] + [M0[i + 1] - M0[i] for i in range(M0.nrows() - 1)])
        M2 = matrix([M1[0], M1[1]] + [M1[i + 2] - M1[i] for i in range(M1.nrows() - 2)])

        for r in range(0, N + 1):
            t = r + theta // 2
            for i in irange(t + 2, N + theta // 2 + 1):
                self.assertEqual(M2[i][r], 0)
            C = (vb + vc - 1 - 2 * vda) / 2
            if theta % 2 == 1:
                self.assertEqual(M2[t + 1][r], q**t - (q ** (t - 1) if t > 0 else 0))
            else:
                self.assertEqual(
                    M2[t + 1][r], -C * q**t - (C + 1) * (q ** (t - 1) if t > 0 else 0)
                )

    def test_kernel_large_r(self):
        params = self.get_semi_lie_params(r_min=2)
        r = max(params.pop("r"), params["ve"] + 2)
        self.assertEqual(
            del0(r, **params) + 2 * del0(r - 1, **params) + del0(r - 2, **params), 0
        )

    def test_kernel_full(self):
```

```
        params = self.get_semi_lie_params(r_min=5)
        vb, vc, vda, ve = params["vb"], params["vc"], params["vda"],
            params["ve"]
        r = params.pop("r")

        def delPhi(r):
            return (
                del0(r, **params)
                + del0(r - 1, **params)
                - q**2 * del0(r - 2, **params)
                - q**2 * del0(r - 3, **params)
            )

        left_bound = ve - min((vb + vc - 1) / 2, vda) + 2
        right_bound = left_bound + 2
        expr = expand(delPhi(r) + (q + 1) * delPhi(r - 1) + q *
            delPhi(r - 2))
        if not left_bound <= r <= right_bound:
            self.assertEqual(expr, 0)
        # in fact I think the following is the exact criteria for it to
            be nonzero
        # although I don't claim this in the paper
        # but we'll put it in the test case just for kicks
        elif left_bound <= r <= right_bound - 1:
            self.assertNotEqual(expr, 0)
        elif r == right_bound - 1 and vb + vc < 2 * vda:
            self.assertNotEqual(expr, 0)
        elif r == right_bound - 1 and vb + vc >= 2 * vda:
            self.assertEqual(expr, 0)

    def get_S3_params(self, r_min=0, r_max=10):
        l = randint(-5, 10)
        if l < 0:
            l *= 2

        params = {
            "r": randint(r_min, r_max),
            "l": l,
            "lam": l if l % 2 == 1 else randrange(max(0, l + l % 2) +
                1, 17, 2),
            "delta": l if l < 0 else randint((l + 1) // 2, 10),
        }
        return params

    def test_0_for_S3(self):
```

```
        # Brute force auxiliary functions for the inhomogeneous group AFL orbital
        def O_zero(r, l, delta):
            j = var("j")
            return sum(qs ** (2 * j), j, -r, delta + r)

        def vol_1disk(n, vxx, rho):
            assert n >= 1 and n >= rho
            if vxx < rho:
                return 0
            elif rho <= 0:
                return q ** (-n) * (1 - q ** (-2))
            else:
                return q ** (-(n + rho)) * (1 - q ** (-1))

        def vol_2disk(n, vxx1, vx12, rho1, rho2):
            assert rho1 >= rho2
            assert n >= 1 and n >= rho1

            if vxx1 >= rho1 and vx12 >= rho2:
                return (
                    q ** (-(n + rho1)) * (1 - q ** (-1))
                    if rho1 >= 1
                    else q ** (-n) * (1 - q ** (-2))
                )
            else:
                return 0

        def qs_weight(n, m):
            kappa = 1 / ((1 - q ** (-1)) * (1 - q ** (-2)))
            return (
                kappa
                * (-1) ** n
                * qs ** (2 * m - n)
                * q ** (2 * n - 2 * m)
                * q ** (2 * m)
                * (1 - q ** (-2))
            )

        def O_case_1_2_brute(r, l, delta, lam=None):
            assert 0 <= l <= 2 * delta, (l, delta)
            assert r >= 0, r
            if lam is None:
                assert l % 2 == 1
                lam = l
```

```
        else:
            assert l % 2 == 0

        S = 0
        for n in irange(1, l + r):
            for m in irange(n - r, n + delta + r):
                r_n = ceil((n - r) / 2)
                r_m = m - delta - r
                if r_n >= r_m:
                    S += vol_2disk(
                        n, vxx1=min(l, delta), vxx12=ceil(l / 2),
                            rho1=r_n, rho2=r_m
                    ) * qs_weight(n, m)
                else:
                    S += vol_2disk(
                        n, vxx1=lam, vx12=ceil(l / 2), rho1=r_m,
                            rho2=r_n
                    ) * qs_weight(n, m)
        return S

    def O_ell_odd_brute(r, l, delta):
        return O_zero(r, l, delta) + O_case_1_2_brute(r, l, delta)

    def O_case_3_4_brute(r, l, delta, lam):
        assert 0 <= l <= 2 * delta, (l, delta)
        assert r >= 0, r
        assert l % 2 == 0, l
        assert lam % 2 == 1, lam
        INFINITY = abs(r) + abs(l) + abs(delta) + abs(lam) + 1

        S = 0
        for n in irange(l + r + 1, INFINITY):
            for m in irange(n - r, n + delta + r):
                r_n = n - l / 2 - r
                r_m = m - delta - r

                # Case 3+ and 4+
                if r_n > r_m: # Case 3+
                    S += vol_2disk(
                        n,
                        vxx1=lam + delta - l,
                        vx12=lam - l / 2,
                        rho1=r_n,
                        rho2=r_m,
                    ) * qs_weight(n, m)
```

```
                else: # Case 4+
                    S += vol_2disk(
                        n,
                        vxx1=lam,
                        vx12=lam - l / 2,
                        rho1=r_m,
                        rho2=r_n,
                    ) * qs_weight(n, m)

                # Cases 3- and 4-
                if r_n > r_m: # Case 3-
                    S += vol_2disk(
                        n,
                        vxx1=delta,
                        vx12=l / 2,
                        rho1=r_n,
                        rho2=r_m,
                    ) * qs_weight(n, m)
                else: # Case 4-
                    assert (
                        vol_2disk(
                            n,
                            vxx1=lam,
                            vx12=l / 2,
                            rho1=r_m,
                            rho2=r_n,
                        )
                        == 0
                    )

        return S

    def O_ell_even_brute(r, l, delta, lam):
        return (
            O_zero(r, l, delta)
            + O_case_1_2_brute(r, l, delta, lam)
            + O_case_3_4_brute(r, l, delta, lam)
        )

    def O_ell_neg_brute(r, vb, lam):
        assert r >= 0
        assert vb < 0
        assert lam % 2 == 1
        l = 2 * vb
        delta = 2 * vb
```

```
            INFINITY = abs(r) + abs(l) + abs(delta) + abs(lam) + 1

            S = 0
            for n in irange(1, INFINITY):
                for m in irange(n - r, n + delta + r):
                    if n <= l + r:
                        S += vol_1disk(n, vxx=lam, rho=m - delta - r) * qs_weight(n, m)
                    elif n > l + r:
                        rho1 = max(n - l / 2 - r, m - delta - r)
                        rho2 = min(n - l / 2 - r, m - delta - r)
                        S += vol_2disk(
                            n, vxx1=lam, vx12=lam, rho1=rho1, rho2=rho2
                        ) * qs_weight(n, m)

            return S + O_zero(r, l, delta)

        params = self.get_S3_params()
        l = params["l"]
        orb = O_for_S3(**params)
        if l < 0:
            brute_res = O_ell_neg_brute(r=params["r"], vb=l // 2, lam=params["lam"])
        elif l % 2 == 0:
            brute_res = O_ell_even_brute(**params)
        elif l % 2 == 1:
            del params["lam"]
            brute_res = O_ell_odd_brute(**params)

        self.assertEqual(orb.subs(q_s=1), 0)
        self.assertEqual(brute_res.subs(q=17, q_s=1337), orb.subs(q=17, q_s=1337))

    def test_delO_for_S3_via_arch(self):
        params = self.get_S3_params()
        self.assertEqual(
            derivative(O_for_S3(**params), qs).subs(q_s=1),
            delO_for_S3_via_arch(**params),
        )

    def test_delO_for_S3(self):
        params = self.get_S3_params()
        self.assertEqual(delO_for_S3_via_arch(**params), delO_for_S3(**params))

```

```
    def test_ker_for_S3(self):
        params = self.get_S3_params(r_min=3)
        l = params["l"]
        r = params.pop("r")
        deriv = (
            (del0_for_S3(r, **params) - del0_for_S3(r - 1, **params))
            + 2 * q * (del0_for_S3(r - 1, **params) - del0_for_S3(r - 2, **params))
            + q**2 * (del0_for_S3(r - 2, **params) - del0_for_S3(r - 3, **params))
        )
        if l < 0:
            if r < -l // 2:
                self.assertEqual(deriv, 0)
            elif r >= -l // 2 + 3:
                self.assertEqual(deriv, -2 * q - 2)
        else:
            self.assertEqual(deriv, -2 * q - 2)

    def test_GK_to_orbital(self):
        params = self.get_semi_lie_params(r_min=1)
        omega = (-1) ** (params["r"] + params["vc"]) # transfer factor
        ve = params.pop("ve")
        self.assertEqual(
            omega * (del0(ve=ve, **params) + del0(ve=ve - 1, **params)),
            GK(ve=ve, **params),
        )

    def test_clean_intersection(self):
        params = self.get_semi_lie_params(r_min=1)
        if params["ve"] == 0:
            params["ve"] += 1
        r, vb, vc, ve, vda = (
            params["r"],
            params["vb"],
            params["vc"],
            params["ve"],
            params["vda"],
        )
        self.assertEqual(
            (GK(r, vb, vc, ve, vda) - GK(r - 1, vb, vc, ve, vda))
            - (GK(r, vb, vc, ve - 1, vda) - GK(r - 1, vb, vc, ve - 1, vda)),
            clean_intersection(**params),
        )
```

```python


if __name__ == "__main__":
    parser = argparse.ArgumentParser(
        "checkthesis",
        description="Checks the formulas in Evan’s thesis for consistency",
    )
    parser.add_argument(
        "--trials", default=5, type=int, help="Number of trials to run."
    )
    parser.add_argument("--seed", type=int, help="Random seed passed to Sage")
    parser.add_argument("--failfast", action="store_true", help="Stop on 1st failure.")
    group = parser.add_mutually_exclusive_group()
    group.add_argument("--verbose", action="store_true", help="Set verbosity to 2.")
    group.add_argument("--quiet", action="store_true", help="Set verbosity to 0.")
    parser.add_argument(
        "--test",
        default="",
        type=str,
        help="The name of a specific test to run (if empty, runs all).",
    )
    args = parser.parse_args()
    if args.verbose is True:
        verbosity = 2
    elif args.quiet is True:
        verbosity = 0
    else:
        verbosity = 1

    suite = unittest.TestSuite()
    loader = unittest.TestLoader()
    if args.seed:
        set_random_seed(args.seed)
    print(f"Using random seed {initial_seed()}")
    for _ in range(args.trials):
        if args.test:
            suite.addTest(RandThesisTest(args.test))
        else:
            suite.addTest(loader.loadTestsFromTestCase(RandThesisTest))
```

```
    runner = unittest.TextTestRunner(failfast=args.failfast, verbosity=verbosity)
    runner.run(suite)
```

A.2 Quaternion implementations and tests

As an afterthought we also provide the following short self-contained file verifying the quaternion calculations done in Chapter 14. Unlike the previous code, it is symbolic.

```
import unittest


def show_quaternion(expr, **kwargs) -> None:
    """If you're using a Jupyter notebook you can use this utility function to
    pretty-print the four coefficients of the quaternion, each on its own line.

    :param expr: the quaternion to print
    :param **kwargs: passed to c.subs for each of the four coefficients c
    """
    coeffs = [c.subs(**kwargs) for c in expr.coefficient_tuple()]
    if coeffs[1] == 0 and coeffs[2] == 0 and coeffs[3] == 0:
        show(coeffs[0])
    else:
        show(1, "." * 12, coeffs[0])
        show(LatexExpr(r"\sqrt{\epsilon}"), "." * 10, coeffs[1])
        show(LatexExpr(r"\Pi"), "." * 11, coeffs[2])
        show(LatexExpr(r"\sqrt{\epsilon}\Pi"), "." * 9, coeffs[3])


def project_to_trace_zero(expr):
    return expr - expr.coefficient_tuple()[0]


def hermitian_form(x, y):
    coeffs = (x * y.conjugate()).coefficient_tuple()
    return coeffs[0] + coeffs[1] * sqrt_eps


```

```
def symmetric_form(x, y):
    return (x * y.conjugate() + y * x.conjugate()) / 2


# Variables
R = QQ["s0", "s1", "t0", "t1", "a0", "a1", "b0", "b1", "eps", "varpi",
    "z0", "z1"]
s0, s1, t0, t1, a0, a1, b0, b1, eps, varpi, z0, z1 = R.gens()
DD = QuaternionAlgebra(Frac(R), eps, varpi, names=("sqrt_eps", "Pi",
    "sqrt_eps_Pi"))
sqrt_eps, Pi, sqrt_eps_Pi = DD.gens()

# Main variables
alpha = a0 + a1 * sqrt_eps
beta = b0 + b1 * sqrt_eps
s = s0 + s1 * sqrt_eps
t = t0 + t1 * sqrt_eps
lam_inv = z0 + z1 * sqrt_eps

# Their conjugates
alphac = a0 - a1 * sqrt_eps
betac = b0 - b1 * sqrt_eps
sc = s0 - s1 * sqrt_eps
tc = t0 - t1 * sqrt_eps
lamc_inv = z0 - z1 * sqrt_eps

r = 100


# The pair (g,u)
def g(x):
    return lam_inv * x * (alpha + beta * Pi)


u = s + t * Pi


# Derived quantities
uu = hermitian_form(u, u)
x = project_to_trace_zero(u.conjugate() * sqrt_eps * u)
y = project_to_trace_zero(varpi ^ r * (alpha + beta * Pi))
xx = symmetric_form(x, x)
xy = symmetric_form(x, y)
yy = symmetric_form(y, y)

```

```

class QuaternionTestCase(unittest.TestCase):
    def assertQtrnsEqualWhen(self, expr1, expr2, **kwargs):
        coeffs1 = [c.subs(**kwargs) for c in expr1.coefficient_tuple()]
        coeffs2 = [c.subs(**kwargs) for c in expr2.coefficient_tuple()]
        for i in range(4):
            self.assertEqual(coeffs1[i], coeffs2[i])

    def assertQtrnsEqual(self, expr1, expr2):
        self.assertQtrnsEqualWhen(expr1, expr2, s0=0, s1=0)
        self.assertQtrnsEqualWhen(expr1, expr2, t0=0, t1=0)

    def assertQtrnsEqualExactly(self, expr1, expr2):
        self.assertQtrnsEqualWhen(expr1, expr2)

    def test_uu(self):
        self.assertQtrnsEqualExactly(uu, s * sc - t * tc * varpi)

    def test_gu_u(self):
        self.assertQtrnsEqualWhen(
            hermitian_form(g(u), u), lam_inv * alphac * uu, s0=0, s1=0
        )
        self.assertQtrnsEqualWhen(
            hermitian_form(g(u), u), lam_inv * alpha * uu, t0=0, t1=0
        )

    def test_x(self):
        self.assertQtrnsEqual(x, (s * sc + t * tc * varpi) * sqrt_eps)

    def test_xx(self):
        self.assertQtrnsEqual(xx, -eps * uu**2)

    def test_xy(self):
        self.assertQtrnsEqual(xy, -varpi ^ r * a1 * eps * (s * sc + t *
            tc * varpi))

    def test_yy(self):
        self.assertQtrnsEqual(
            yy, varpi ^ (2 * r) * (-(a1**2) * eps - beta * betac *
                varpi)
        )

    def test_det(self):
        self.assertQtrnsEqual(
            xx * yy - xy**2,
```

```
            eps * uu ^ 2 * varpi ^ 200 * (varpi * beta * betac),
        )


if __name__ == "__main__":
    unittest.main()
```