Auto-Generated EGMO Solutions Treasury

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1 April 2025

Contents

1 Solutions for Angle Chasing

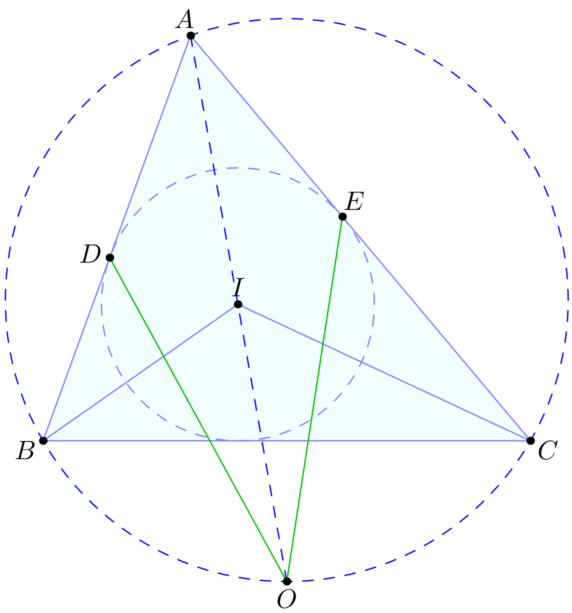
> I won't go easy on you, and I hope you won't go easy on me, either.
>
> Serral to Bunny before their semifinals match at *DreamHack Starcraft 2 Masters* Atlanta 2022

§1a CGMO 2012/5

Let ABC be a triangle. The incircle of $\triangle ABC$ has center I and is tangent to $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Let O denote the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$.

(Available online at https://aops.com/community/p2769872.)

By Fact 5, O is the midpoint of arc BC, and so it's immediate that $\triangle ADO \cong \triangle AEO$ which implies the result.



§1b Canada 1991/3

Let P be a point inside circle ω. Consider chords of ω passing through P. Prove that the midpoints of these chords all lie on a fixed circle.

(Available online at https://aops.com/community/p2445591.)

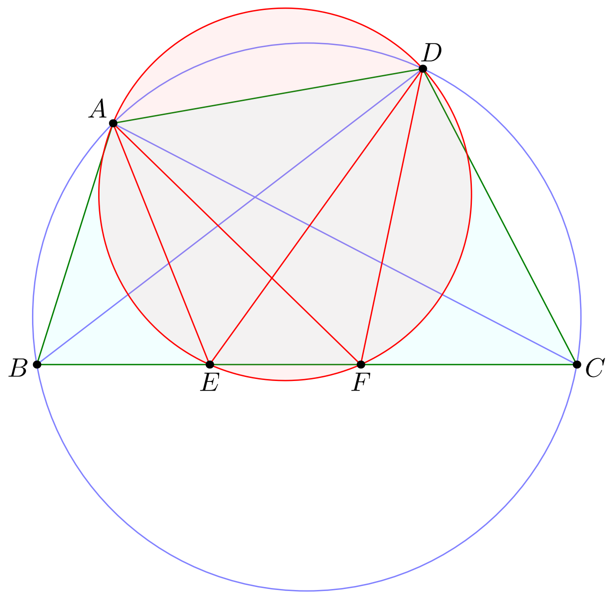
Letting O be the center of the circle, the midpoints lie on the circle with diameter $\overline{OP}$.

§1c Russia 1996/10.1

Points E and F are given on side BC of convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle FAC = \angle EDB$.

(Available online at https://aops.com/community/p3025732.)

This is a direct angle chase. First, the problem tells us that $AEFD$ is cyclic.



Claim — Quadrilateral $ABCD$ is cyclic too.

Proof. Note that

$$\begin{aligned} \measuredangle DCB = \measuredangle DCF &= \measuredangle CDF + \measuredangle DFC \\ &= \measuredangle EAB + \measuredangle DFE = \measuredangle EAB + \measuredangle DAE = \measuredangle DAB. \end{aligned}$$ □

To finish,

$$\measuredangle FAC = \measuredangle BAC - (\measuredangle BAE + \measuredangle EAF) = \measuredangle BDC - (\measuredangle FDC + \measuredangle EDF) = \measuredangle EDB.$$

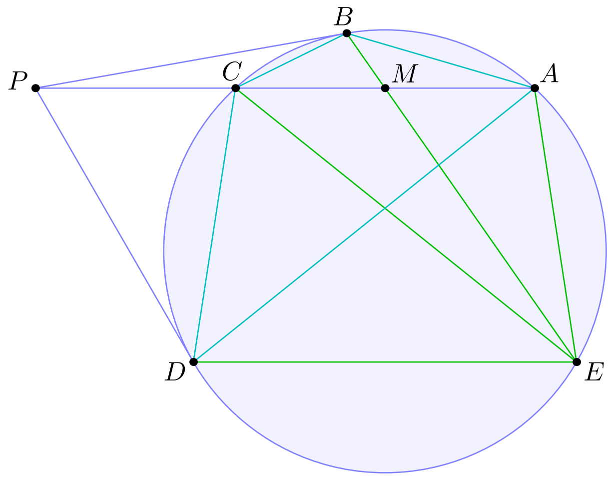
§1d JMO 2011/5

Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

(Available online at https://aops.com/community/p2254813.)

We present two solutions.

¶ First solution using harmonic bundles. Let $M = \overline{BE} \cap \overline{AC}$ and let ∞ be the point at infinity along $\overline{DE} \parallel \overline{AC}$.



Note that $ABCD$ is harmonic, so

$$-1 = (AC; BD) \overset{E}{=} (AC; M\infty)$$

implying M is the midpoint of $\overline{AC}$.

¶ Second solution using complex numbers (Cynthia Du). Suppose we let b, d, e be free on unit circle, so $p = \frac{2bd}{b+d}$. Then $d/c = a/e$, and $a + c = p + ac\overline{p}$. Consequently,

$$\begin{aligned} ac &= de \\ \frac{1}{2}(a+c) &= \frac{bd}{b+d} + de \cdot \frac{1}{b+d} = \frac{d(b+e)}{b+d}. \\ \frac{a+c}{2ac} &= \frac{(b+e)}{e(b+d)}. \end{aligned}$$

From here it's easy to see

$$\frac{a+c}{2} + \frac{a+c}{2ac} \cdot be = b + e$$

which is what we wanted to prove.

§1e IMO 2006/1

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

(Available online at https://aops.com/community/p571966.)

The condition rewrites as

$$\angle PBC + \angle PCB = (\angle B - \angle PBC) + (\angle C - \angle PCB) \implies \angle PBC + \angle PCB = \frac{\angle B + \angle C}{2}$$

which means that

$$\angle BPC = 180^\circ - \frac{\angle B + \angle C}{2} = 90^\circ + \frac{\angle A}{2} = \angle BIC.$$

Since P and I are both inside $\triangle ABC$ that implies P lies on the circumcircle of $\triangle BIC$.

It's well-known (by "Fact 5") that the circumcenter of $\triangle BIC$ is the arc midpoint M of $\widehat{BC}$. Therefore

$$AI + IM = AM \le AP + PM \implies AI \le AP$$

with equality holding iff A, P, M are collinear, or $P = I$.

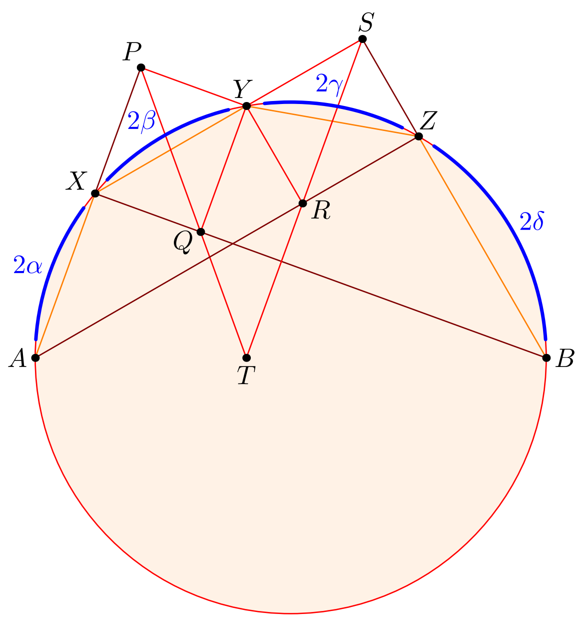
§1f USAMO 2010/1

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

(Available online at https://aops.com/community/p1860802.)

We present two possible approaches. The first approach is just "bare-hands" angle chasing. The second approach requires more insight but makes it clearer what is going on; it shows the intersection point of lines PQ and RS is the foot from the altitude from Y to AB using Simson lines. The second approach also has the advantage that it works even if $\overline{AB}$ is not a diameter of the circle.

¶ First approach using angle chasing. Define $T = \overline{PQ} \cap \overline{RS}$. Also, let 2α, 2β, 2γ, 2δ denote the measures of arcs $\widehat{AX}$, $\widehat{XY}$, $\widehat{YZ}$, $\widehat{ZB}$, respectively, so that $\alpha + \beta + \gamma + \delta = 90^\circ$.



We now compute the following angles:

$$\angle SRY = \angle SZY = 90^\circ - \angle YZA = 90^\circ - (\alpha + \beta)$$

$$\angle YQP = \angle YXP = 90^\circ - \angle BXY = 90^\circ - (\gamma + \delta)$$
$$\angle QYR = 180^\circ - \angle(\overline{ZR}, \overline{QX}) = 180^\circ - \frac{2\beta + 2\gamma + 180^\circ}{2} = 90^\circ - (\beta + \gamma).$$

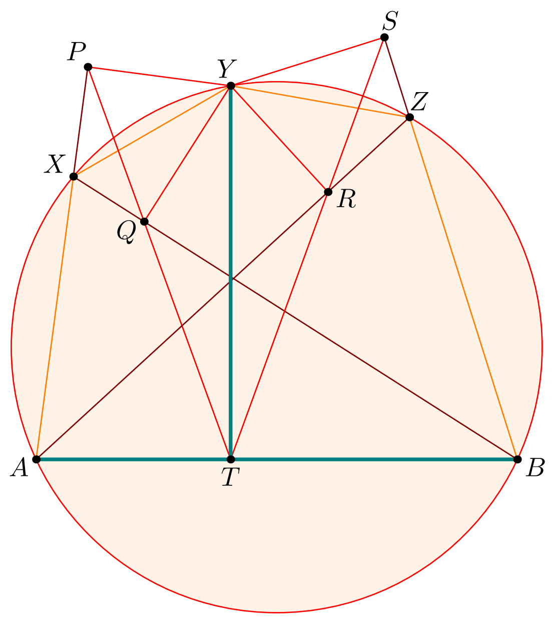
Hence, we can then compute

$$\begin{aligned}
\angle RTQ &= 360^\circ - (\angle QYR + (180^\circ - \angle SRY) + (180^\circ - \angle YQP)) \\
&= \angle SRY + \angle YQP - \angle QYR \\
&= (90^\circ - (\alpha + \beta)) + (90^\circ - (\gamma + \delta)) - (90^\circ - (\beta + \gamma)) \\
&= 90^\circ - (\alpha + \delta) \\
&= \beta + \gamma.
\end{aligned}$$

Since $\angle XOZ = \frac{2\beta + 2\gamma}{2} = \beta + \gamma$, the proof is complete.

¶ Second approach using Simson lines, ignoring the diameter condition. In this solution, we will ignore the condition that $\overline{AB}$ is a diameter; the solution works equally well without it, as long as O is redefined as the center of $(AXYZB)$ instead. We will again show the angle formed by lines PQ and RS is half the measure of $\widehat{XZ}$.

Unlike the previous solution, we instead define T to be the foot from Y to $\overline{AB}$. Then the Simson line of Y with respect to $\triangle XAB$ passes through P, Q, T. Similarly, the Simson line of Y with respect to $\triangle ZAB$ passes through R, S, T. Therefore, point T coincides with $\overline{PQ} \cap \overline{RS}$.



Now it's straightforward to see $APYRT$ is cyclic (in the circle with diameter $\overline{AY}$), and therefore

$$\angle RTY = \angle RAY = \angle ZAY.$$

Similarly,

$$\angle YTQ = \angle YBQ = \angle YBX.$$

Summing these gives $\angle RTQ$ is equal to half the measure of arc $\widehat{XZ}$ as needed.

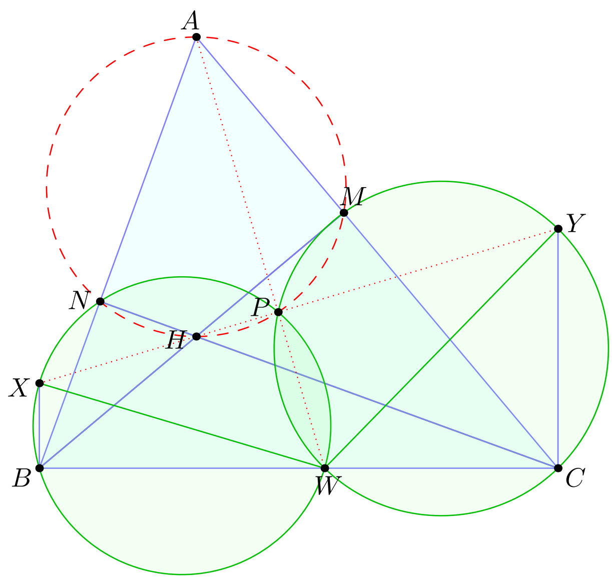
§1g IMO 2013/4

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.

(Available online at https://aops.com/community/p5720174.)

We present two solutions, an elementary one and then an advanced one by moving points.

¶ First solution, classical. Let P be the second intersection of ω_1 and ω_2; this is the Miquel point, so P also lies on the circumcircle of AMN, which is the circle with diameter $\overline{AH}$.



We now contend:

Claim — Points P, H, X collinear. (Similarly, points P, H, Y are collinear.)

Proof using power of a point. By radical axis on $BNMC$, ω_1, ω_2, it follows that A, P, W are collinear. We know that $\angle APH = 90^\circ$, and also $\angle XPW = 90^\circ$ by construction. Thus P, H, X are collinear. □

Proof using angle chasing. This is essentially Reim's theorem:

$$\measuredangle NPH = \measuredangle NAH = \measuredangle BAH = \measuredangle ABX = \measuredangle NBX = \measuredangle NPX$$

as desired. Alternatively, one may prove A, P, W are collinear by $\measuredangle NPA = \measuredangle NMA = \measuredangle NMC = \measuredangle NBC = \measuredangle NBW = \measuredangle NPW$. □

¶ Second solution, by moving points. Fix $\triangle ABC$ and vary W. Let ∞ be the point at infinity perpendicular to $\overline{BC}$ for brevity.

By spiral similarity, the point X moves linearly on $\overline{B\infty}$ as W varies linearly on $\overline{BC}$. Similarly, so does Y. So in other words, the map

$$X \mapsto W \mapsto Y$$

is linear. However, the map

$$X \mapsto Y' := \overline{XH} \cap \overline{C\infty}$$

is linear too.

To show that these maps are the same, it suffices to check it thus at two points.

- When $W = B$, the circle (BNW) degenerates to the circle through B tangent to $\overline{BC}$, and $X = \overline{CN} \cap \overline{B\infty}$. We have $Y = Y' = C$.
- When $W = C$, the result is analogous.
- Although we don't need to do so, it's also easy to check the result if W is the foot from A since then $XHWB$ and $YHWC$ are rectangles.

§1h IMO 1985/1

A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

(Available online at https://aops.com/community/p366584.)

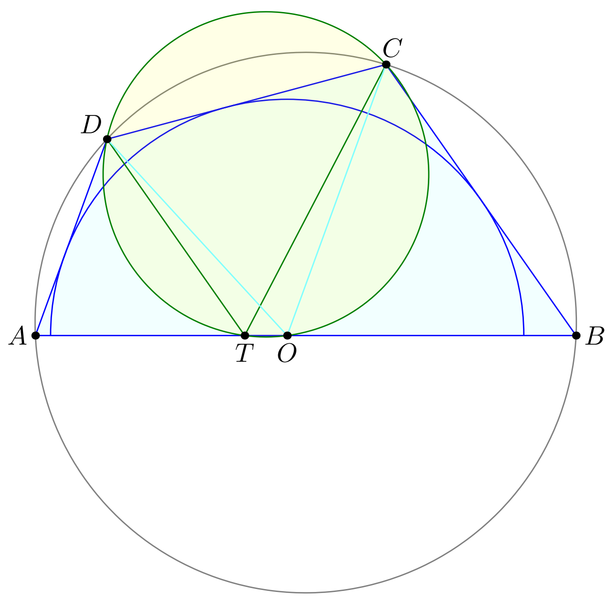
Let T be the point such that $DA = AT$.

Claim — T lies on (DOC).

Proof. Because

$$\angle DCO = \frac{1}{2}\angle DCB = \frac{1}{2}(180^\circ - \angle BAD) = 90^\circ - \frac{1}{2}\angle TAD = \angle DTA.$$

□



Reversing the previous proof on the other side gives $BC = BT$. So $AB = AT + TB = AD + BC$.

2 Solutions for Circles

이 자리에서 매일 기다렸지만
내일도 그럴 자신이 나는 없는 걸요

I've waited here every day
But I don't know if I can tomorrow as well

Lullaby, by Dreamcatcher

§2a USAMO 1990/5

An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that M, N, P, Q are concyclic.

(Available online at https://aops.com/community/c6h58273p356630.)

Let T be the foot of the altitude from A, and let H be the orthocenter. Apparently

$$HM \cdot HN = HA \cdot HT = HP \cdot HQ$$

so we're done by powerc of a point.

> **Remark.** Since $\overline{AB}$ and $\overline{AC}$ are the perpendicular bisectors of $\overline{MN}$ and $\overline{PQ}$ the circumcircle of $MNPQ$ coincides with the point A.

§2b JMO 2012/1

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

(Available online at https://aops.com/community/p2669111.)

Assume for contradiction that (PRS) and (QRS) are distinct. Then $\overline{RS}$ is the radical axis of these two circles. However, $\overline{AP}$ is tangent to (PRS) and $\overline{AQ}$ is tangent to (QRS), so point A has equal power to both circles, which is impossible since A does not lie on line BC.

§2c IMO 2008/1

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and

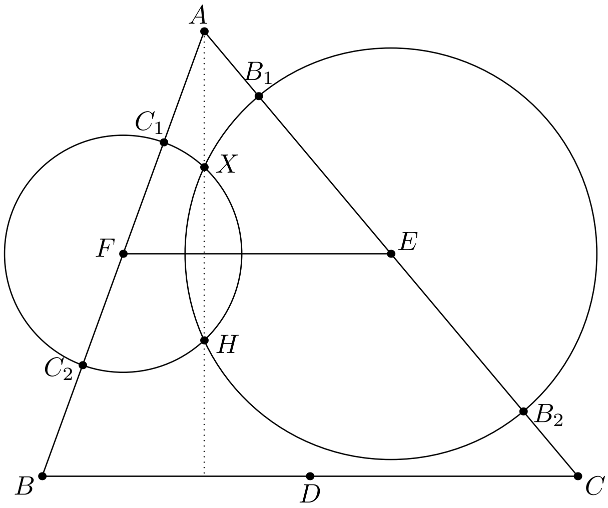
A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

(Available online at https://aops.com/community/p1190553.)

We show two solutions.

¶ First solution using power of a point. Let D, E, F be the centers of Γ_A, Γ_B, Γ_C (in other words, the midpoints of the sides).

We first show that B_1, B_2, C_1, C_2 are concyclic. It suffices to prove that A lies on the radical axis of the circles Γ_B and Γ_C.



Let X be the second intersection of Γ_B and Γ_C. Clearly $\overline{XH}$ is perpendicular to the line joining the centers of the circles, namely $\overline{EF}$. But $\overline{EF} \parallel \overline{BC}$, so $\overline{XH} \perp \overline{BC}$. Since $\overline{AH} \perp \overline{BC}$ as well, we find that A, X, H are collinear, as needed.

Thus, B_1, B_2, C_1, C_2 are concyclic. Similarly, C_1, C_2, A_1, A_2 are concyclic, as are A_1, A_2, B_1, B_2. Now if any two of these three circles coincide, we are done; else the pairwise radical axii are not concurrent, contradiction. (Alternatively, one can argue directly that O is the center of all three circles, by taking the perpendicular bisectors.)

¶ Second solution using length chase (Ritwin Narra). We claim the circumcenter O of $\triangle ABC$ is in fact the center of $(A_1A_2B_1B_2C_1C_2)$.

Define D, E, F as before. Then since $\overline{OD} \perp \overline{A_1A_2}$ and $DA_1 = DA_2$, which means $OA_1 = OA_2$. Similarly, we have $OB_1 = OB_2$ and $OC_1 = OC_2$.

Now since $DA_1 = DA_2 = DH$, we have $OA_1^2 = OD^2 + HD^2$. We seek to show

$$OD^2 + HD^2 = OE^2 + HE^2 = OF^2 + HF^2.$$

This is clear by Appollonius's Theorem since D, E, and F lie on the nine-point circle, which is centered at the midpoint of $\overline{OH}$.

§2d USAMO 1997/2

Let ABC be a triangle. Take noncollinear points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

(Available online at https://aops.com/community/p210283.)

The three lines are the radical axii of the three circles centered at D, E, F, so they concur.

§2e IMO 1995/1

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ meet at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

(Available online at https://aops.com/community/p365179.)

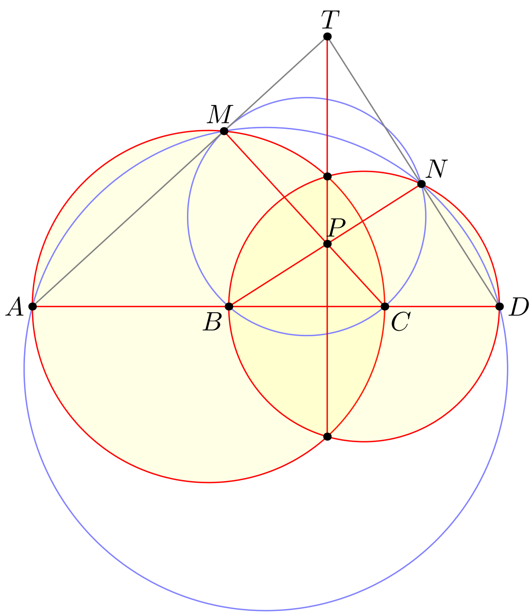
Note that:

Claim — $MBCN$ is cyclic.

Proof. From $PB \cdot PN = PX \cdot PY = PC \cdot PM$. □

Claim (Russia 1996/10.1) — $AMND$ is cyclic.

Proof. $\measuredangle DAM = \measuredangle CAM = 90^\circ - \measuredangle MCB = 90^\circ - \measuredangle MNB = 90^\circ + \measuredangle BNM = \measuredangle DNM$. □



Then the conclusion follows by radical axis on (AC), (BD), $(AMND)$.

§2f USAMO 1998/2

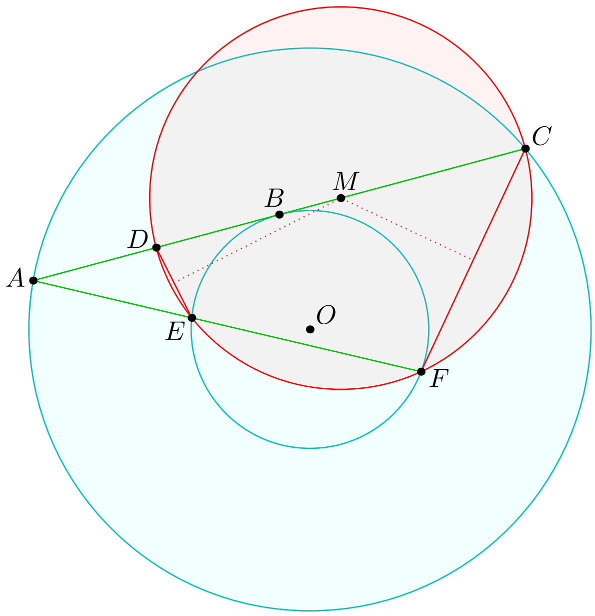
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of $\overline{DE}$ and $\overline{CF}$ intersect at a point M on line AB. Find, with proof, the ratio AM/MC.

(Available online at https://aops.com/community/p343866.)

By power of a point we have

$$AE \cdot AF = AB^2 = \left(\frac{1}{2}AB\right) \cdot (2AB) = AD \cdot AC$$

and hence $CDEF$ is cyclic. Then M is the circumcenter of quadrilateral $CDEF$.



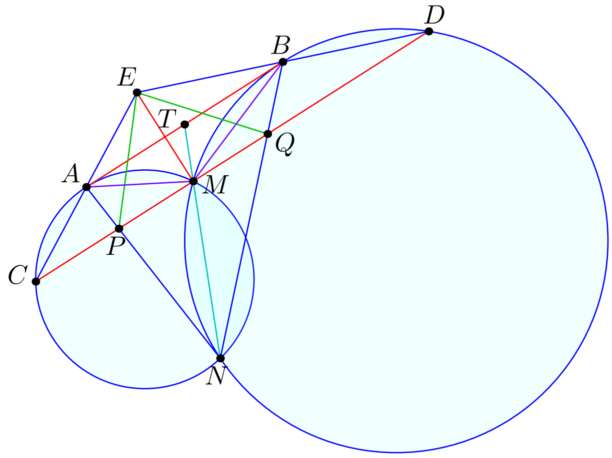
Thus M is the midpoint of $\overline{CD}$ (and we are given already that B is the midpoint of $\overline{AC}$, D is the midpoint of $\overline{AB}$). Thus a quick computation along $\overline{AC}$ gives $AM/MC = 5/3$.

§2g IMO 2000/1

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

(Available online at https://aops.com/community/p354110.)

First, we have $\measuredangle EAB = \measuredangle ACM = \measuredangle BAM$ and similarly $\measuredangle EBA = \measuredangle BDM = \measuredangle ABM$. Consequently, $\overline{AB}$ bisects $\angle EAM$ and $\angle EBM$, and hence $\triangle EAB \cong \triangle MAB$.



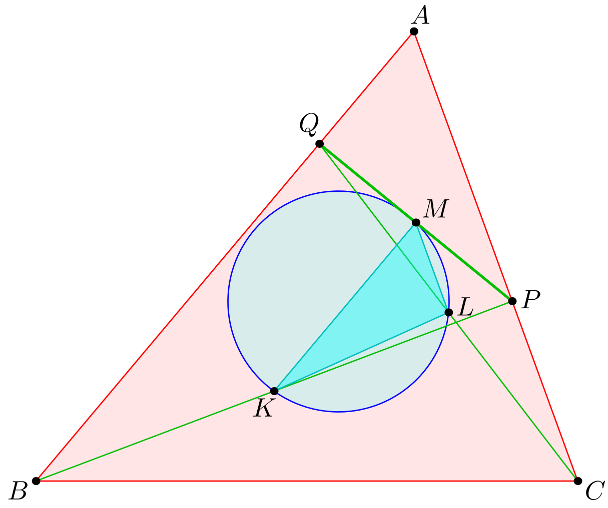
Now it is well-known that $\overline{MN}$ bisects $\overline{AB}$ and since $\overline{AB} \parallel \overline{PQ}$ we deduce that M is the midpoint of $\overline{PQ}$. As $\overline{AB}$ is the perpendicular bisector of $\overline{EM}$, it follows that $EP = EQ$ as well.

§2h IMO 2009/2

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

(Available online at https://aops.com/community/p1561572.)

By power of a point, we have $-AQ \cdot QB = OQ^2 - R^2$ and $-AP \cdot PC = OP^2 - R^2$. Therefore, it suffices to show $AQ \cdot QB = AP \cdot PC$.



As $\overline{ML} \parallel \overline{AC}$ and $\overline{MK} \parallel \overline{AB}$ we have that

$$\measuredangle APQ = \measuredangle LMP = \measuredangle LKM$$
$$\measuredangle PQA = \measuredangle KMQ = \measuredangle MLK$$

and consequently we have the (opposite orientation) similarity

$$\triangle APQ \overset{-}{\sim} \triangle MKL.$$

Therefore

$$\frac{AQ}{AP} = \frac{ML}{MK} = \frac{2ML}{2MK} = \frac{PC}{QB}$$

id est $AQ \cdot QB = AP \cdot PC$, which is what we wanted to prove.

§2i Canada 2007/5

Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let ω, ω_1, ω_2, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R. Show that lines PD, QE, and RF are concurrent.

(Available online at https://aops.com/community/p894696.)

We present two solutions, one just by angle chasing, and another tricky one using spiral similarity. Inversion at the incircle also works very well.

¶ First solution (angle chasing).

Claim — Quadrilaterals $PEDQ$, $QFER$, $PFDR$ are all cyclic.

Proof. Angle chase:

$$\begin{aligned}
\measuredangle QPE &= \measuredangle QPA + \measuredangle APE \\
&= \measuredangle QPA + \measuredangle AIE \\
&= \measuredangle QBA + \measuredangle ABI + \measuredangle IDE \\
&= \measuredangle QBI + \measuredangle IDE \\
&= \measuredangle QDI + \measuredangle IDE \\
&= \measuredangle QDE.
\end{aligned}$$

This is apparently much harder than I remember, seeing that it took me half an hour to write down. □

We're now done by radical axis.

¶ Second solution (spiral similarity, Ryan Kim). We note that:

Claim — Line PD bisects $\angle BPC$, and thus passes through the arc midpoint X of $\widehat{BC}$.

Proof. The spiral similarity gives $PB/PC = BF/EC = BD/DC$. □

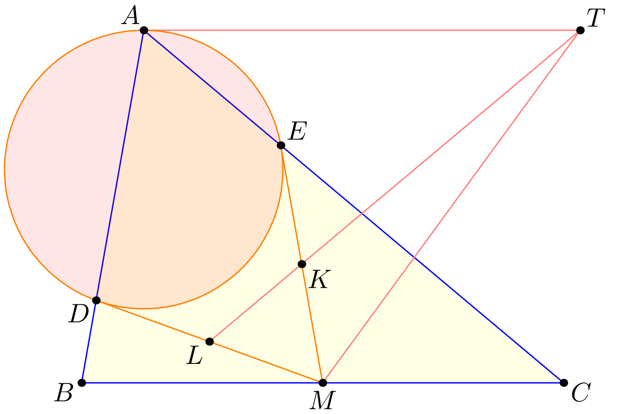
Now consider the positive homothety mapping the incircle to the circumcircle, centered at the so-called X_{56}. This homothety maps D to X, so we have X_{56} is collinear with DX. Hence $\overline{PD}$ passes through X_{56} as desired.

§2j Iran TST 2011/1

Let ABC be a triangle with $\angle B > \angle C$. Let M denote the midpoint of BC and let D and E denote the feet of the altitude from C and B respectively. Let K and L denote the midpoints of ME and MD respectively. If KL intersect the line through A parallel to BC at point T, prove that $TA = TM$.

(Available online at https://aops.com/community/p2266382.)

It's well-known that MD, ME, AT are all tangent to (ADE); see chapter 1 of the EGMO textbook, "three tangents" lemma.



Now line KL is the radical axis of (AED) and the circle centered at M of radius zero. So by power of a point,

$$TM^2 = \operatorname{Pow}_{(AED)}(T) = TA^2.$$

3 Solutions for Lengths and Ratios

> I don't know what's weirder — that you're fighting a stuffed animal, or that you seem to be losing.
>
> Susie Derkins, in *Calvin and Hobbes*

§3a Shortlist 2006 G3

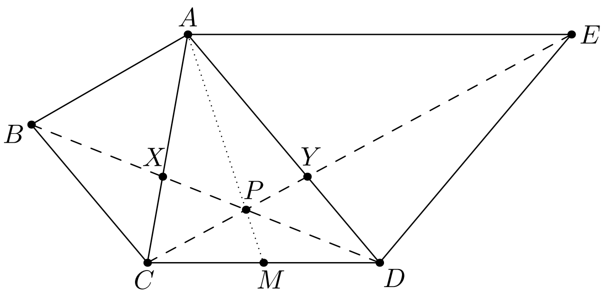
Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

(Available online at https://aops.com/community/p741369.)

Let X denote the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Let Y denote the intersection of diagonals $\overline{AD}$ and $\overline{CE}$.



The given conditions imply that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. From this it follows that quadrilaterals $ABCD$ and $ACDE$ are similar. In particular, we have that $\frac{AX}{XC} = \frac{AY}{YD}$.

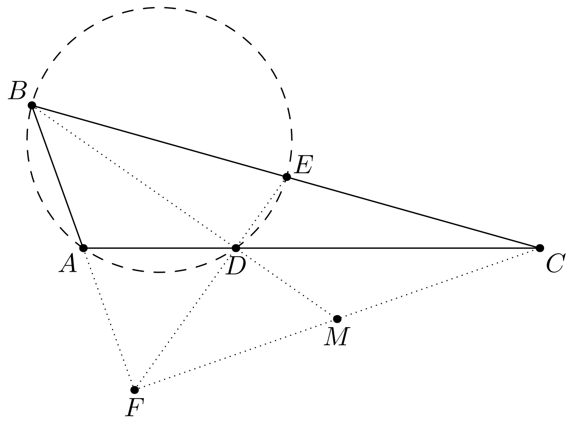
Now let ray AP meet $\overline{CD}$ at M. Then Ceva's theorem applied to triangle ACD implies that $\frac{AX}{XC} \cdot \frac{CM}{MD} \cdot \frac{DY}{YA} = 1$, so $CM = MD$.

§3b USAMO 2003/4

Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

(Available online at https://aops.com/community/p336205.)

Ceva theorem plus the similar triangles.



We know unconditionally that

$$\measuredangle CBD = \measuredangle EBD = \measuredangle EAD = \measuredangle EAC.$$

Moreover, by Ceva's theorem on $\triangle BCF$, we have $MF = MC \iff \overline{FC} \parallel \overline{AE}$. So we have the equivalences

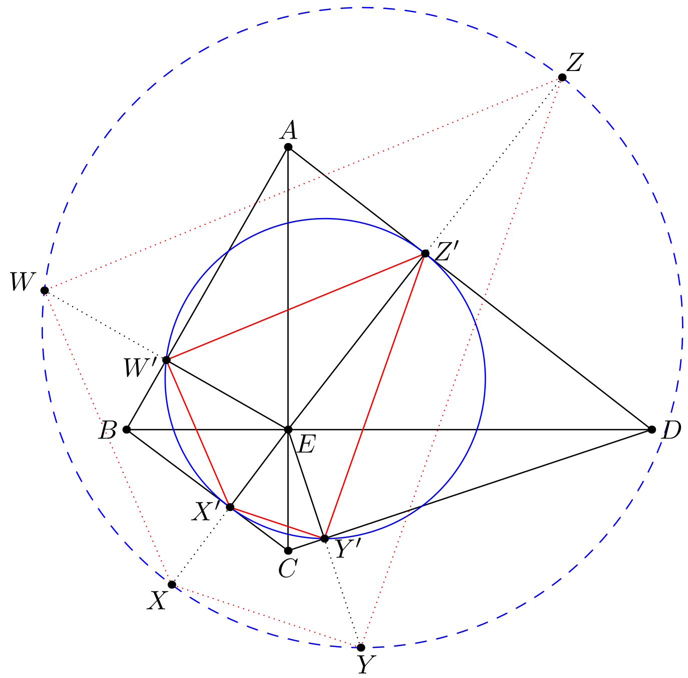
$$\begin{aligned} MF = MC &\iff \overline{FC} \parallel \overline{AE} \\ &\iff \measuredangle FCA = \measuredangle EAC \\ &\iff \measuredangle MCD = \measuredangle CBD \\ &\iff MC^2 = MB \cdot MD. \end{aligned}$$

§3c USAMO 1993/2

Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

(Available online at https://aops.com/community/p356408.)

Let W, X, Y, Z be the reflections across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ and let W', X', Y', Z' be the midpoints of $\overline{EW}$, $\overline{EX}$, $\overline{EY}$, $\overline{EZ}$; in other words, the feet of the perpendiculars from E to the respective sides. By a homothety, to prove that W, X, Y, Z are concyclic, it suffices to prove W', X', Y', Z' are concyclic.



We can do this with just angle chasing. Since $EW'BX'$ and $EX'CY'$ are cyclic,

$$\angle W'X'Y' = \angle W'X'E + \angle EX'Y' = \angle W'BE + \angle ECY' = \angle ABE + \angle ECD.$$

Similarly,

$$\angle Y'Z'W' = \angle BAE + \angle EDC.$$

Then,

$$\angle W'X'Y' + \angle Y'Z'W' = (\angle ABE + \angle BAE) + (\angle EDC + \angle EDC) = 90^\circ + 90^\circ = 180^\circ.$$

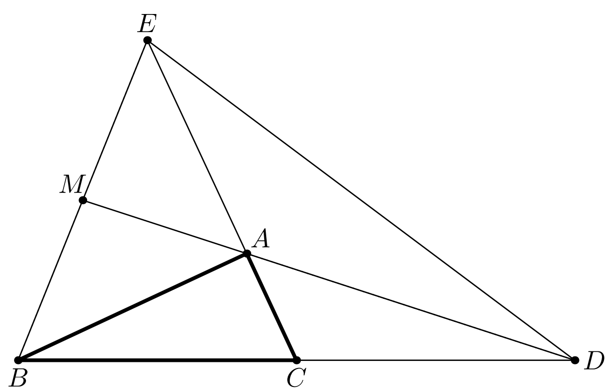
Hence W', X', Y', Z' are cyclic, as needed.

§3d EGMO 2013/1

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

(Available online at https://aops.com/community/p3013167.)

Let ray DA meet $\overline{BE}$ at M. Consider the triangle EBD. Since the point lies on median $\overline{EC}$, and $EA = 2AC$, it follows that A is the centroid of $\triangle EBD$.



So M is the midpoint of $\overline{BE}$. Moreover $MA = \frac{1}{2}AD = \frac{1}{2}BE$; so $MA = MB = ME$ and hence $\triangle ABE$ is inscribed in a circle with diameter $\overline{BE}$. Thus $\angle BAE = 90^\circ$, so $\angle BAC = 90^\circ$.

§3e APMO 2004/2

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

(Available online at https://aops.com/community/p15307.)

It's actually true with line OH replaced by any line ℓ through the centroid G; in that case the directed sum of distances from A, B, C to ℓ is equal to zero.

Indeed, assume ℓ intersects segments AB and AC. If M is the midpoint of $\overline{BC}$ then

$$d(B,\ell) + d(C,\ell) = 2d(M,\ell) = d(A,\ell)$$

by homothety. The end.

Tristan Shin points out that another way to see this is just directly by barycentric coordinates; indeed we have

$$\begin{aligned} [AOH] + [BOH] + [COH] &= \frac{1}{128K^2} \sum_{\text{cyc}} \det \begin{bmatrix} 1 & 0 & 0 \\ a^2S_A & b^2S_B & c^2S_C \\ S_{BC} & S_{CA} & S_{AB} \end{bmatrix} \\ &= \frac{1}{128K^2} \det \begin{bmatrix} 1 & 1 & 1 \\ a^2S_A & b^2S_B & c^2S_C \\ S_{BC} & S_{CA} & S_{AB} \end{bmatrix} \\ &= 0 \end{aligned}$$

again since the centroid lies on line OH.

§3f TSTST 2011/4

Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

(Available online at https://aops.com/community/p2374848.)

Let MH and NH meet the nine-point circle again at P' and Q', respectively. Recall that H is the center of the homothety between the circumcircle and the nine-point circle. From this we can see that P and Q are the images of this homothety, meaning that

$$HQ = 2HQ' \quad \text{and} \quad HP = 2HP'.$$

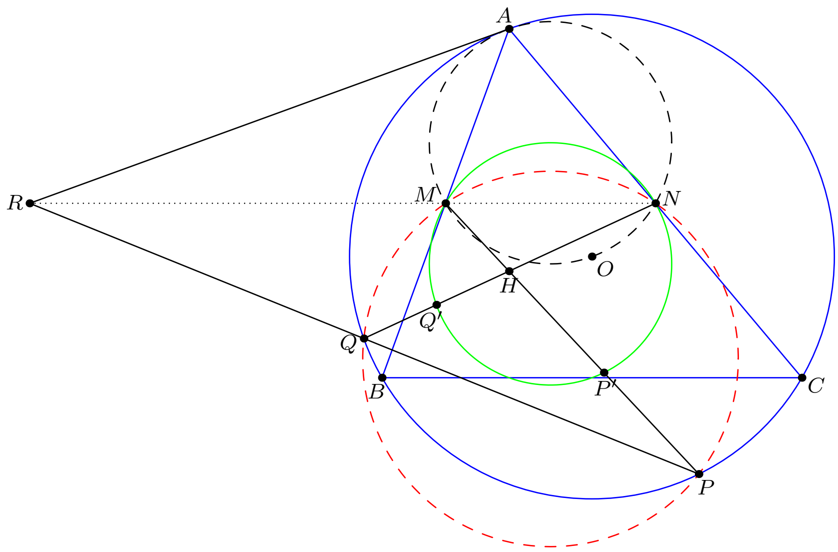
Since M, P', Q', N are cyclic, Power of a Point gives us

$$MH \cdot HP' = HN \cdot HQ'.$$

Multiplying both sides by two, we thus derive

$$HM \cdot HP = HN \cdot HQ.$$

It follows that the points M, N, P, Q are concyclic.



Let ω_1, ω_2, ω_3 denote the circumcircles of $MNPQ$, AMN, and ABC, respectively. The radical axis of ω_1 and ω_2 is line MN, while the radical axis of ω_1 and ω_3 is line PQ. Hence the line R lies on the radical axis of ω_2 and ω_3.

But we claim that ω_2 and ω_3 are internally tangent at A. This follows by noting the homothety at A with ratio 2 sends M to B and N to C. Hence the radical axis of ω_2 and ω_3 is a line tangent to both circles at A.

Hence $\overline{RA}$ is tangent to ω_3. Therefore, $\overline{RA} \perp \overline{OA}$.

§3g USAMO 2015/2

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

(Available online at https://aops.com/community/p4769957.)

We present three solutions, one by complex numbers, two more synthetic. (A fourth solution using median formulas is also possible.) Most solutions will prove that the center of the fixed circle is the midpoint of $\overline{AO}$ (with O the center of ω); this can be recovered empirically by letting

- X approach P (giving the midpoint of $\overline{BP}$)
- X approach Q (giving the point Q), and
- X at the midpoint of $\overline{PQ}$ (giving the midpoint of $\overline{BQ}$)

which determines the circle; this circle then passes through P by symmetry and we can find the center by taking the intersection of two perpendicular bisectors (which two?).

¶ Complex solution (Evan Chen). Toss on the complex unit circle with $a = -1$, $b = 1$, $z = -\frac{1}{2}$. Let s and t be on the unit circle. We claim Z is the center.

It follows from standard formulas that

$$x = \frac{1}{2}\left(s + t - 1 + s/t\right)$$

thus

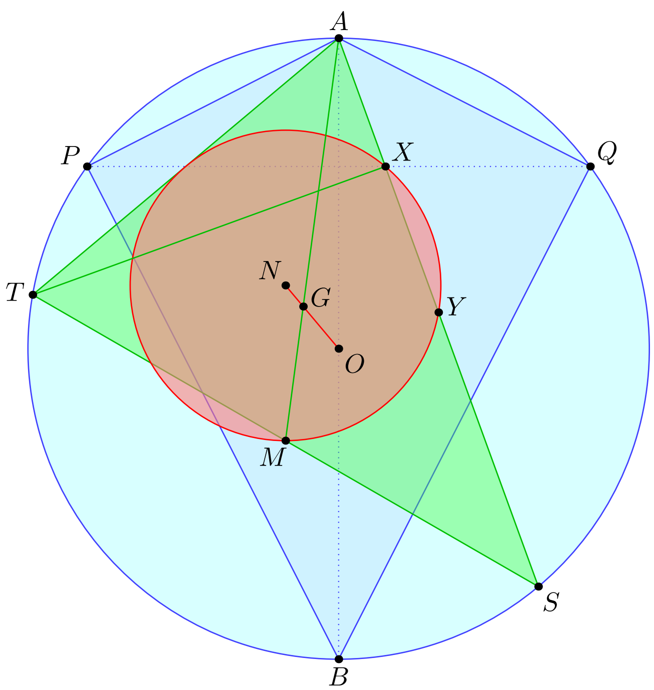
$$4\operatorname{Re} x + 2 = 2\left(x + \frac{1}{x}\right) + 2 = s + t + \frac{1}{s} + \frac{1}{t} + \frac{s}{t} + \frac{t}{s}$$

which depends only on P and Q, and not on X. Thus

$$4\left|z - \frac{s+t}{2}\right|^2 = |s+t+1|^2 = 3 + (4\operatorname{Re} x + 2)$$

does not depend on X, done.

¶ Homothety solution (Alex Whatley). Let G, N, O denote the centroid, nine-point center, and circumcenter of triangle AST, respectively. Let Y denote the midpoint of $\overline{AS}$. Then the three points X, Y, M lie on the nine-point circle of triangle AST, which is centered at N and has radius $\frac{1}{2}AO$.



Let R denote the radius of ω. Note that the nine-point circle of $\triangle AST$ has radius equal to $\frac{1}{2}R$, and hence is independent of S and T. Then the power of A with respect to the nine-point circle equals

$$AN^2 - \left(\frac{1}{2}R\right)^2 = AX \cdot AY = \frac{1}{2}AX \cdot AS = \frac{1}{2}AQ^2$$

and hence

$$AN^2 = \left(\frac{1}{2}R\right)^2 + \frac{1}{2}AQ^2$$

which does not depend on the choice of X. So N moves along a circle centered at A.

Since the points O, G, N are collinear on the Euler line of $\triangle AST$ with

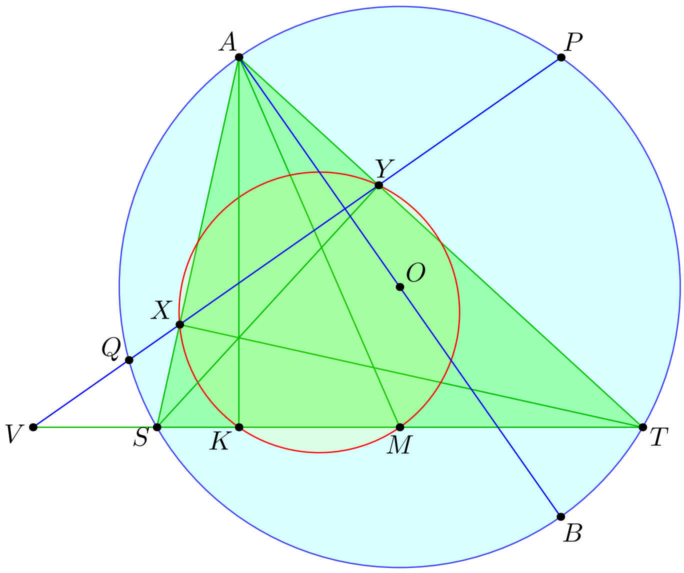
$$GO = \frac{2}{3} NO$$

it follows by homothety that G moves along a circle as well, whose center is situated one-third of the way from A to O. Finally, since A, G, M are collinear with

$$AM = \frac{3}{2} AG$$

it follows that M moves along a circle centered at the midpoint of $\overline{AO}$.

¶ Power of a point solution (Zuming Feng, official solution). We complete the picture by letting $\triangle KYX$ be the orthic triangle of $\triangle AST$; in that case line XY meets the ω again at P and Q.



The main claim is:

Claim — Quadrilateral $PQKM$ is cyclic.

Proof. To see this, we use power of a point: let $V = \overline{QXYP} \cap \overline{SKMT}$. One approach is that since $(VK;ST) = -1$ we have $VQ \cdot VP = VS \cdot VT = VK \cdot VM$. A longer approach is more elementary:

$$VQ \cdot VP = VS \cdot VT = VX \cdot VY = VK \cdot VM$$

using the nine-point circle, and the circle with diameter $\overline{ST}$. □

But the circumcenter of $PQKM$, is the midpoint of $\overline{AO}$, since it lies on the perpendicular bisectors of $\overline{KM}$ and $\overline{PQ}$. So it is fixed, the end.

4 Solutions for Assorted Configurations

> We should switch from 5 answer choices to 6 answer choices so we can just bubble a lot of F's to express our feelings.
>
> Evan's reaction to the AMC edVistas website

§4a Shortlist 2003 G2

Three distinct points A, B, and C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

(Available online at https://aops.com/community/p19089.)

Note that $\overline{QP}$ is a symmedian of $\triangle AQC$, so

$$\frac{AB}{BC} = \frac{AQ^2}{CQ^2}$$

so AQ/CQ is fixed, and done by angle bisector theorem.

§4b USAMO 1988/4

Let I be the incenter of triangle ABC, and let A', B', and C' be the circumcenters of triangles IBC, ICA, and IAB, respectively. Prove that the circumcircles of triangles ABC and $A'B'C'$ are concentric.

(Available online at https://aops.com/community/c6h420561p2375323.)

It's known that A' is the midpoint of minor arc BC along the circumcircle ABC. So not only are the desired circles obviously concentric, they are in fact the same circle...

§4c USAMO 1995/3

Given a scalene nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of the sides. Point A_2 is located on the ray OA_1 so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent.

(Available online at https://aops.com/community/p143328.)

As A_2 is the intersection of the tangents to the circumcircle at B and C, it follows line AA_2 is a symmedian. And the three symmedians concur at the symmedian point.

§4d USA TST 2014/1

Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point.

(Available online at https://aops.com/community/p3332310.)

The fixed point is the orthocenter, since ℓ is a Simson line. See Lemma 4.4 of *Euclidean Geometry in Math Olympiads*.

§4e USA TST 2011/1

In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on line EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

(Available online at https://aops.com/community/p2374795.)

The answer is 1.

To see this, focus just on triangle DEF. As H is the incenter and A is the D-excenter, the points Q and P are the respective contact points of the incircle and D-excircle. So R is the antipode of Q along the incircle.

§4f USAMO 2011/5

Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \qquad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \qquad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

(Available online at https://aops.com/community/p2254841.)

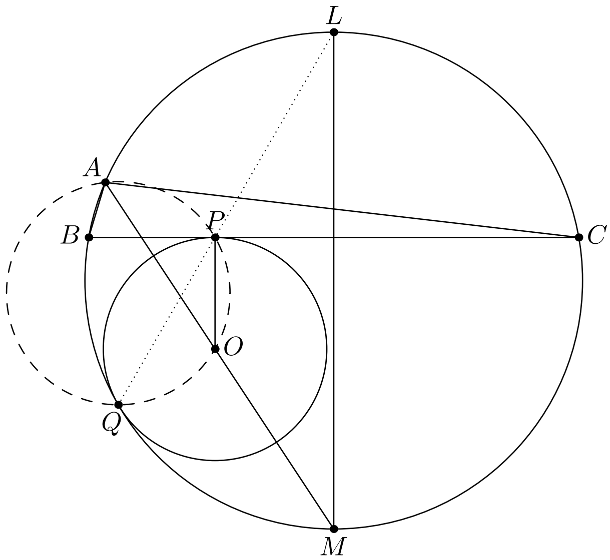
If $\overline{AB} \parallel \overline{CD}$ there is nothing to prove. Otherwise let $X = \overline{AB} \cap \overline{CD}$. Then the Q_1 and Q_2 are the isogonal conjugates of P with respect to triangles XBC and XAD. Thus X, Q_1, Q_2 are collinear, on the isogonal of $\overline{XP}$ with respect to $\angle DXA = \angle CXB$.

§4g Japan 2009

Triangle ABC has circumcircle Γ. A circle with center O is tangent to BC at P and internally to Γ at Q, so that Q lies on arc BC of Γ not containing A. Prove that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

We present two solutions.

¶ First solution by standard methods. Let M and L be the midpoints of the arcs BC of Γ where M lies on the opposite side of line BC as A.



We claim that the points P, Q, L are collinear. To see this, one could note that an inversion at L with radius $LB = LC$ swaps points P and Q. Alternatively, we take a homothety at Q mapping the circle with center O to Γ; since BC is a tangent, this necessarily takes Q to L.

In any case, we can now note that OP and LM are parallel (since they are both perpendicular to BC), and by assumption points A, O, M are collinear. It follows that $APOQ$ is cyclic, as

$$\angle AQP = \angle AQL = \angle AML = \angle AOP.$$

But $PO = QO$, so $\angle PAO = \angle QAO$.

¶ Second solution by inversion. A $\sqrt{bc}$ inversion swaps Γ and line BC. However, it also preserves line AO, since $\angle BAO = \angle CAO$. This is enough to imply that the circle (O) is preserved (not the point O itself), since its center remains on the $\angle A$-bisector, and it remains tangent to both Γ and line BC.

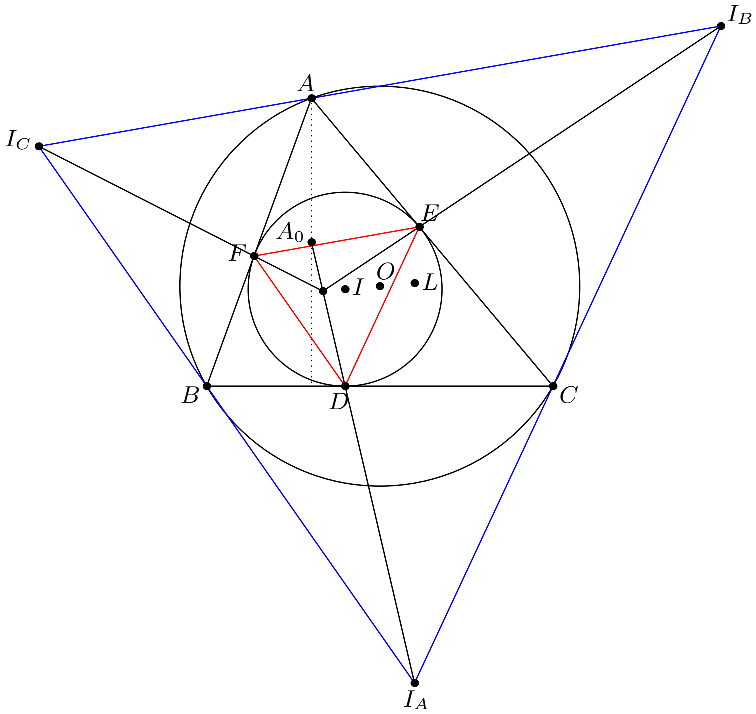
Thus, P and Q are swapped by $\sqrt{bc}$ inversion, as needed.

§4h Vietnam TST 2003/2

Let ABC be a scalene triangle, and denote by O and I the circumcenter and incenter. Let A_0 be the midpoint of the A-altitude, and define B_0 and C_0 similarly. Suppose the incircle is tangent to the sides BC, CA, AB at points D, E, F. Prove that lines A_0D, B_0E, C_0F are concurrent with line OI.

(Available online at https://aops.com/community/p268390.)

Let I_A, I_B, I_C be the excenters of the triangle. It's known that I_AD passes through the midpoint A_0, and thus we can consider the problem in terms of this triangle instead.



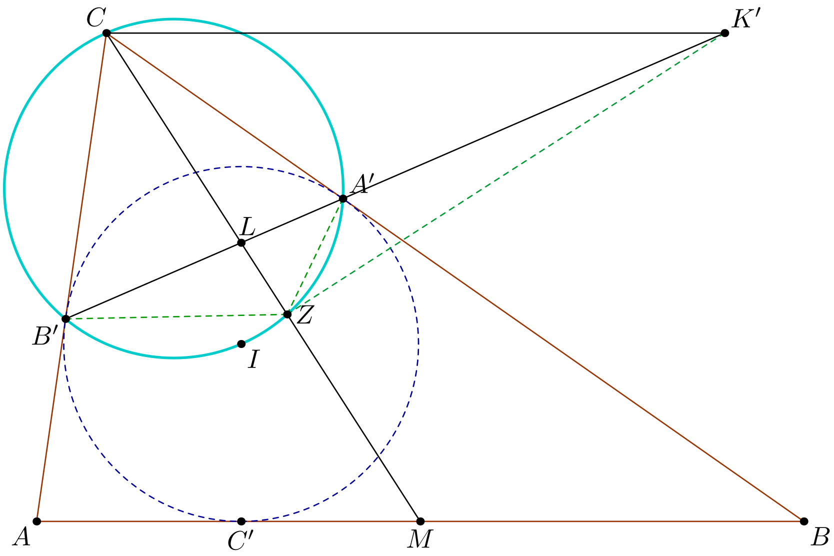
Let L be the circumcenter of $I_AI_BI_C$. Note that DEF and $I_AI_BI_C$ are homothetic, since $\overline{EF}$ and $\overline{I_BI_C}$ are both perpendicular to the A-bisector. Therefore, the lines DI_A, EI_B, FI_C concur at a single point X. Moreover, X, I, L are collinear. (In fact X is the exsimilicenter of the circumcircles.)

It remains to show I, O, L are collinear, but this follows by noting that they are the orthocenter, nine-point center, and circumcenter of triangle $I_AI_BI_C$, respectively.

§4i Sharygin 2013/16

The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points A', B' and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$.

Let ω be the circumcircle of $\triangle A'B'C$ and let K' be the intersection of line $A'B'$ with the line through C parallel to AB. Furthermore, let Z be the foot of the perpendicular from I to CM and observe that $Z \in \omega$. It suffices to prove that $\angle K'ZL$ is right, because this will imply $K' = K$.



Let P_∞ be the point at infinity on line AB. Then the quadruple $(A, B; M, P_\infty)$ is clearly harmonic. Taking perspectivity from C onto line $A'B'$ we observe that $(B', A'; L, K')$ is harmonic.

Now consider point Z. Observe that ZL is an angle bisector of $\angle BZA'$, since $B'C = A'C$ implies the arcs $B'C$ and $A'C$ are equal. Since we have a harmonic bundle, we conclude that $LZ \perp K'Z$ as desired.

§4j APMO 2012/4

Let ABC be an acute triangle. Denote by D the foot of the A-altitude, by M the midpoint of BC, and by H the orthocenter of triangle ABC. Ray MH meets the circumcircle Γ of triangle ABC again at E. Line ED meets Γ again at F. Prove that

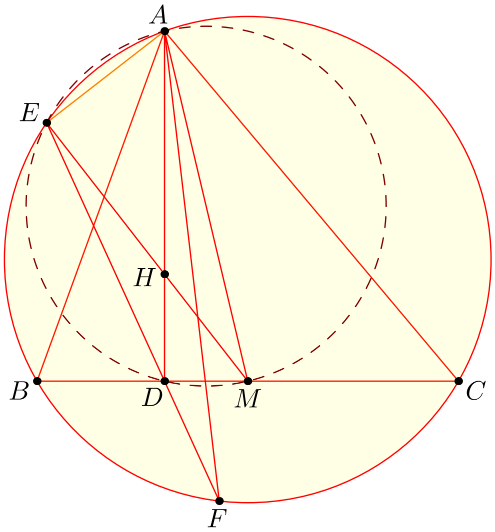
$$\frac{BF}{CF} = \frac{AB}{AC}.$$

(Available online at https://aops.com/community/p2648114.)

The conclusion is a contrived way of saying:

Claim — $\overline{AF}$ is the A-symmedian of $\triangle ABC$.

Proof of main claim. It's well known that $\angle AEM = 90°$, since the second intersection of $\overline{EHM}$ is the A-antipode. That means $MDEA$ is cyclic.



Now,

$$\begin{aligned} \measuredangle BAF = \measuredangle BEF &= \measuredangle EBC + \measuredangle BDE = \measuredangle EBC - \measuredangle EDM \\ &= \measuredangle EAC - \measuredangle EAM = \measuredangle MAC. \end{aligned}$$

□

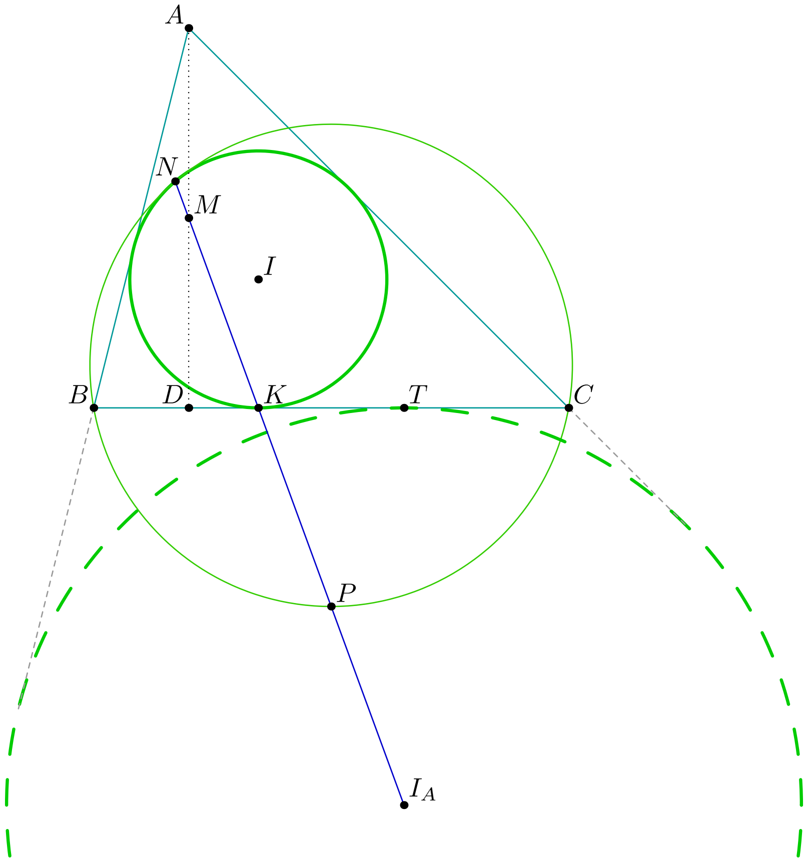
§4k Shortlist 2002 G7

The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of $\overline{AD}$. If N is the common point of the circle Ω and $\overline{KM}$ (distinct from K), then prove Ω and the circumcircle of triangle BCN are tangent to each other.

(Available online at `https://aops.com/community/p118682`.)

We present three solutions, two synthetic and one harmonic.

¶ First solution (from EGMO). Let I_A be the A-excenter tangent to line BC at T. Define P to be the midpoint of $\overline{KI_A}$. Let r be the radius of the incircle and r_a the radius of the A-excircle.



It is well-known that M, K and I_A are collinear. We claim that $NBPC$ is cyclic; it suffices to prove that $2BK \cdot KC = 2KP \cdot KN = KN \cdot KI_A$. On the other hand, by Power of a Point we have that

$$I_AK\,(I_AK + KN) = II_A^2 - r^2 \implies KN \cdot KI_A = II_A^2 - r^2 - I_AK^2.$$

Now we need only simplify the right-hand side using the Pythagorean Theorem; it is

$$\left((r + r_a)^2 + KT^2\right) - r^2 - \left(r_a^2 + KT^2\right) = 2rr_a.$$

So it suffices to prove $rr_a = (s - b)(s - c)$, which is not hard.

Now, since P is the midpoint of minor arc $\widehat{BC}$ of (NBC) (via $BK = CT$), while the incircle is tangent to segment BC at K, the conclusion follows readily.

¶ Second solution using power of a point (Haroon Khan). Define P as the midpoint of $\overline{KI_A}$ as before. As noted already, N, M, K, P, I_A are collinear.

Claim — We have

$$PB^2 = PK \cdot PN = PC^2$$

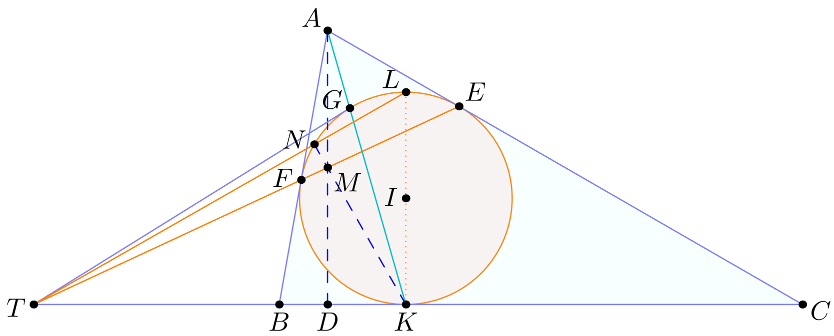
or equivalently that P is the radical center of (I), (B), (C) (the latter two circles having radius zero).

Proof. Consider the K-midline of $\triangle KBI_A$, which we denote ℓ. We claim it is the radical axis of (B) and (I). Indeed, $\ell \parallel \overline{BI_A} \perp \overline{BI}$, and the midpoint of $\overline{BK}$ clearly lies on this radical axis, as needed.

So P lies on the radical axis of (B) and (I); symmetrically it lies on the radical axis of (C) and (I), done. □

This implies P is the arc midpoint of $\widehat{BC}$ in (BCN). Since the incircle is tangent to $\overline{BC}$ at K, it follows that N is the common tangency point requested.

¶ Third solution (harmonic). As before it would be sufficient to show that $\angle BNC$ is bisected by $\overline{NK}$. Let L be the antipode of K on the incircle and let G be the second intersection of $\overline{AK}$ with the incircle. Moreover let E and F be the contact points of the incircle on $\overline{AC}$, $\overline{AB}$.



Note that:

- $GFEK$ is harmonic, since $\overline{AF}$ and $\overline{AE}$ are tangent.
- $GNKL$ is harmonic, if ∞ is the infinity point on $\overline{AD}$ then $-1 = (AD; M\infty) \overset{K}{=} (GK; NL)$.

Thus lines LN, EF, BC concur at $T = \overline{GG} \cap \overline{KK}$, the pole of $\overline{AGK}$ with respect to the incircle.

Moreover $(TK; BC) = -1$, and so since $\angle LKN = 90^\circ$ we get the desired bisection.

5 Solutions for Computational Geometry

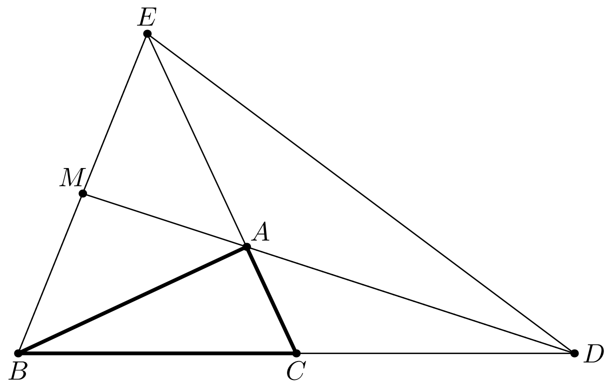
> We both know we don't want to be here, so let's get this over with.
>
> Xiaoyu He, during a MOP 2013 test review

§5a EGMO 2013/1

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

(Available online at https://aops.com/community/p3013167.)

Let ray DA meet $\overline{BE}$ at M. Consider the triangle EBD. Since the point lies on median $\overline{EC}$, and $EA = 2AC$, it follows that A is the centroid of $\triangle EBD$.



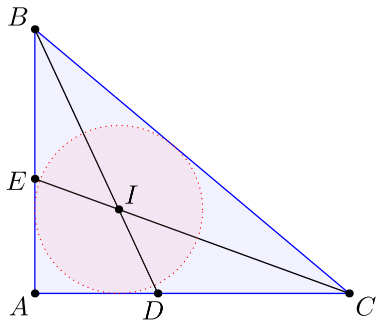
So M is the midpoint of $\overline{BE}$. Moreover $MA = \frac{1}{2}AD = \frac{1}{2}BE$; so $MA = MB = ME$ and hence $\triangle ABE$ is inscribed in a circle with diameter $\overline{BE}$. Thus $\angle BAE = 90^\circ$, so $\angle BAC = 90^\circ$.

§5b USAMO 2010/4

Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

(Available online at https://aops.com/community/p1860753.)

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135^\circ$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90^\circ}{2} = 45^\circ.$$

So, $\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 135^\circ$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 + BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

which is irrational, and this produces the desired contradiction.

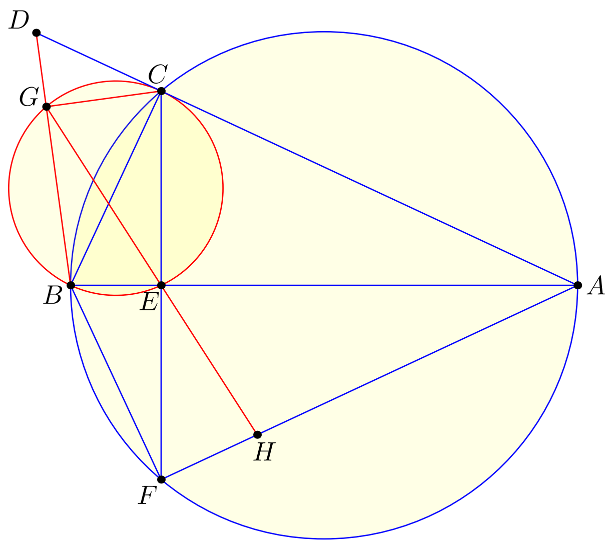
§5c CGMO 2002/4

Circles Γ_1 and Γ_2 intersect at two points B and C, and $\overline{BC}$ is the diameter of Γ_1. Construct a tangent line to circle Γ_1 at C intersecting Γ_2 at another point A. Line AB meets Γ_1 again at E and line CE meets Γ_2 again at F. Let H be an arbitrary point on segment AF. Line HE meets Γ_1 again at G, and $\overline{BG}$ meets $\overline{AC}$ at D. Prove that

$$\frac{AH}{HF} = \frac{AC}{CD}.$$

(Available online at https://aops.com/community/p1358427.)

Note that $BCAF$ is a kite with perpendicular diagonals, from the right angles given in the problem statement.



On the one hand, by law of sines repeatedly (and power of a point as $EB \cdot EA = EC \cdot EF$), we have

$$\frac{AH}{HF} = \frac{EA}{EF}\frac{\sin\angle AEH}{\sin\angle FEH} = \frac{EB}{EC}\frac{\sin\angle GEB}{\sin\angle CEG} = \frac{EB}{EC}\cdot\frac{GB}{GC}.$$

This takes care of the points F and H which we can now erase.

On the other hand, from similar right triangles $\triangle BCE \sim \triangle BCA$ and $\triangle BGC \sim \triangle BCD$, we get

$$AC = EC \cdot \frac{CB}{EB}$$
$$DC = GC \cdot \frac{CB}{GB}.$$

Taking the quotient then completes the problem.

§5d IMO 2007/4

In triangle ABC the bisector of $\angle BCA$ meets the circumcircle again at R, the perpendicular bisector of $\overline{BC}$ at P, and the perpendicular bisector of $\overline{AC}$ at Q. The midpoint of $\overline{BC}$ is K and the midpoint of $\overline{AC}$ is L. Prove that the triangles RPK and RQL have the same area.

(Available online at `https://aops.com/community/p894655`.)

We first begin by proving the following claim.

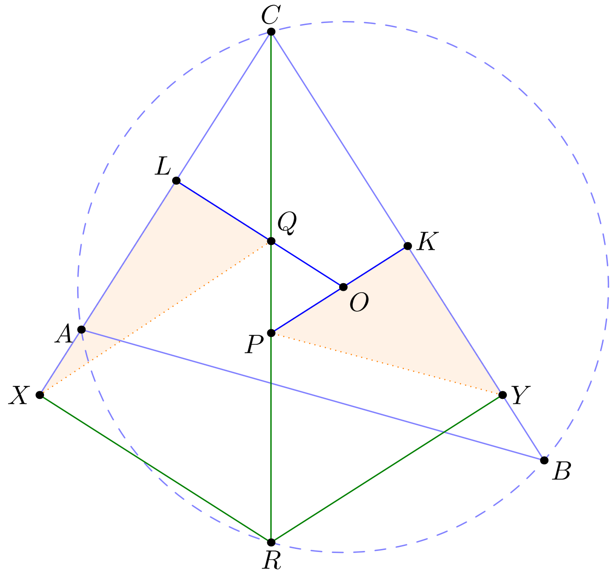
Claim — We have $CQ = PR$ (equivalently, $CP = QR$).

Proof. Let $O = \overline{LQ} \cap \overline{KP}$ be the circumcenter. Then

$$\measuredangle OPQ = \measuredangle KPC = 90^\circ - \measuredangle PCK = 90^\circ - \measuredangle LCQ = \measuredangle\measuredangle CQL = \measuredangle PQO.$$

Thus $OP = OQ$. Since $OC = OR$ as well, we get the conclusion. □

Denote by X and Y the feet from R to $\overline{CA}$ and $\overline{CB}$, so $\triangle CXR \cong \triangle CYR$. Then, let $t = \frac{CQ}{CR} = 1 - \frac{CP}{CR}$.



Then it follows that

$$[RQL] = [XQL] = t(1-t) \cdot [XRC] = t(1-t) \cdot [YCR] = [YKP] = [RKP]$$

as needed.

Remark. Trigonometric approaches are very possible (and easier to find) as well: both areas work out to be $\frac{1}{8}ab\tan\frac{1}{2}C$.

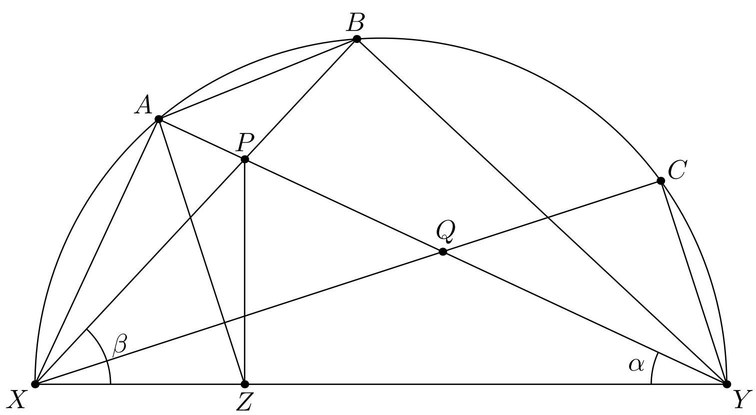
§5e JMO 2013/5

Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line $\overline{XY}$. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

(Available online at https://aops.com/community/p3043750.)

Let $\beta = \angle YXP$ and $\alpha = \angle PYX$ and set $XY = 1$. We do not direct angles in the following solution.



Observe that

$$\angle AZX = \angle APX = \alpha + \beta$$

since $APZX$ is cyclic. In particular, $\angle CXY = 90^\circ - (\alpha + \beta)$. It is immediate that

$$BY = \sin\beta, \quad CY = \cos(\alpha+\beta), \quad AY = \cos\alpha, \quad AX = \sin\alpha.$$

The Law of Sines on $\triangle XPY$ gives $XP = XY\frac{\sin\alpha}{\sin(\alpha+\beta)}$, and on $\triangle XQY$ gives $XQ = XY\frac{\sin\alpha}{\sin(90+\beta)} = \frac{\sin\alpha}{\cos\beta}$. So, the given is equivalent to

$$\frac{\sin\beta}{\frac{\sin\alpha}{\sin(\alpha+\beta)}} + \frac{\cos(\alpha+\beta)}{\frac{\sin\alpha}{\cos\beta}} = \frac{\cos\alpha}{\sin\alpha}$$

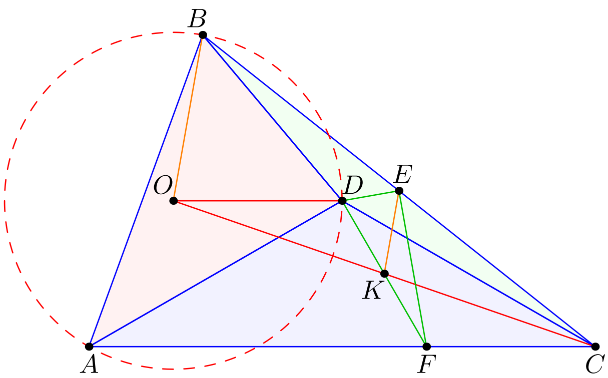
which is equivalent to $\cos\alpha = \cos\beta\cos(\alpha+\beta) + \sin\beta\sin(\alpha+\beta)$. This is obvious, because the right-hand side is just $\cos((\alpha+\beta)-\beta)$.

§5f CGMO 2007/5

Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment BC. Point F lies on segment AC with $AF = 2FC$. Prove that $\overline{DE} \perp \overline{EF}$.

(Available online at https://aops.com/community/p1358815.)

Without loss of generality, $AC = 3$; thus $AD = DC = \sqrt{3}$, and $DF = CF = 1$. Let O be the circumcenter of triangle BAD.



We have $\overline{OD} \parallel \overline{FC}$ since $\angle ODA = 30^\circ = \angle DAF$, and $OD = AD/\sqrt{3} = 1 = CF$. So $ODCF$ is a parallelogram, so diagonals $\overline{DF}$ and $\overline{OC}$ bisect each other say at K. Then $DK = KF = \frac{1}{2}$.

But, $EK = \frac{1}{2}BO = \frac{1}{2}OD = \frac{1}{2}$ too. Thus from $KD = KE = KF$ we conclude the desired result.

§5g Shortlist 2011 G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcenter O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

(Available online at https://aops.com/community/p2739318.)

First, use the fact that

$$90^\circ + \frac{1}{2}\angle A = \angle B'OC' > \angle BOC = 2\angle A$$

to obtain $\angle A < 60^\circ$.

Now M be the midpoint of BC. Then

$$OL \ge OM = R\cos A > R/2$$

so we are done.

§5h IMO 2001/1

Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \ge \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

(Available online at https://aops.com/community/p119192.)

The conclusion rewrites as

$$\begin{aligned} \angle COP &< 90^\circ - \angle A = \angle OCP \\ \iff PC &< PO \\ \iff PC^2 &< PO^2 \\ \iff PC^2 &< R^2 - PB \cdot PC \\ \iff PC \cdot BC &< R^2 \\ \iff ab\cos C &< R^2 \\ \iff \sin A \sin B \cos C &< \frac{1}{4}. \end{aligned}$$

Now

$$\cos C \sin B = \frac{1}{2}\left(\sin(C+B) - \sin(C-B)\right) \le \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

which finishes when combined with $\sin A < 1$.

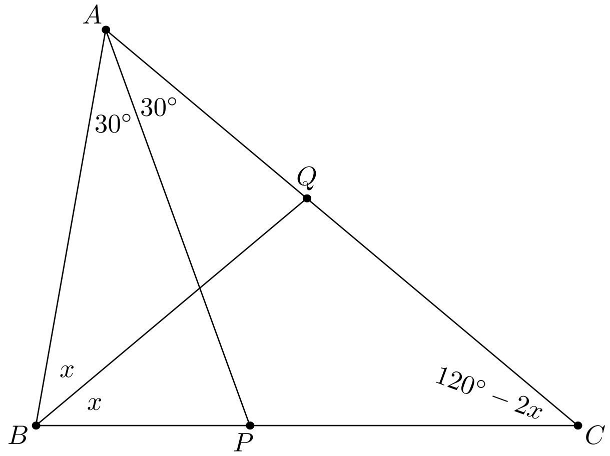
> **Remark.** If we allow ABC to be right then equality holds when $\angle A = 90^\circ$, $\angle C = 60^\circ$, $\angle B = 30^\circ$. This motivates the choice of estimates after reducing to a trig inequality.

§5i IMO 2001/5

Let ABC be a triangle. Let $\overline{AP}$ bisect $\angle BAC$ and let $\overline{BQ}$ bisect $\angle ABC$, with P on $\overline{BC}$ and Q on $\overline{AC}$. If $AB + BP = AQ + QB$ and $\angle BAC = 60^\circ$, what are the angles of the triangle?

(Available online at https://aops.com/community/p119207.)

The answer is $\angle B = 80^\circ$ and $\angle C = 40^\circ$. Set $x = \angle ABQ = \angle QBC$, so that $\angle QCB = 120^\circ - 2x$. We observe $\angle AQB = 120^\circ - x$ and $\angle APB = 150^\circ - 2x$.



Now by the law of sines, we may compute

$$\begin{aligned} BP &= AB \cdot \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} \\ AQ &= AB \cdot \frac{\sin x}{\sin(120^\circ - x)} \\ QB &= AB \cdot \frac{\sin 60^\circ}{\sin(120^\circ - x)}. \end{aligned}$$

So, the relation $AB + BP = AQ + QB$ is exactly

$$1 + \frac{\sin 30^\circ}{\sin(150^\circ - 2x)} = \frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)}.$$

This is now a trig problem, and we simply solve for x. There are many possible approaches and we just present one.

First of all, we can write

$$\sin x + \sin 60^\circ = 2 \sin\left(\frac{1}{2}(x + 60^\circ)\right) \cos\left(\frac{1}{2}(x - 60^\circ)\right).$$

On the other hand, $\sin(120^\circ - x) = \sin(x + 60^\circ)$ and

$$\sin(x + 60^\circ) = 2 \sin\left(\frac{1}{2}(x + 60^\circ)\right) \cos\left(\frac{1}{2}(x + 60^\circ)\right)$$

so

$$\frac{\sin x + \sin 60^\circ}{\sin(120^\circ - x)} = \frac{\cos\left(\frac{1}{2}x - 30^\circ\right)}{\cos\left(\frac{1}{2}x + 30^\circ\right)}.$$

Let $y = \frac{1}{2}x$ for brevity now. Then

$$\begin{aligned} \frac{\cos(y - 30^\circ)}{\cos(y + 30^\circ)} - 1 &= \frac{\cos(y - 30^\circ) - \cos(y + 30^\circ)}{\cos(y + 30^\circ)} \\ &= \frac{2\sin(30^\circ)\sin y}{\cos(y + 30^\circ)} \\ &= \frac{\sin y}{\cos(y + 30^\circ)}. \end{aligned}$$

Hence the problem is just

$$\frac{\sin 30^\circ}{\sin(150^\circ - 4y)} = \frac{\sin y}{\cos(y + 30^\circ)}.$$

Equivalently,

$$\begin{aligned}\cos(y+30^\circ) &= 2\sin y\sin(150^\circ-4y)\\ &= \cos(5y-150^\circ)-\cos(150^\circ-3y)\\ &= -\cos(5y+30^\circ)+\cos(3y+30^\circ).\end{aligned}$$

Now we are home free, because $3y+30^\circ$ is the average of $y+30^\circ$ and $5y+30^\circ$. That means we can write

$$\frac{\cos(y+30^\circ)+\cos(5y+30^\circ)}{2}=\cos(3y+30^\circ)\cos(2y).$$

Hence

$$\cos(3y+30^\circ)\left(2\cos(2y)-1\right)=0.$$

Recall that

$$y=\frac{1}{2}x=\frac{1}{4}\angle B<\frac{1}{4}\left(180^\circ-\angle A\right)=30^\circ.$$

Hence it is not possible that $\cos(2y)=\frac{1}{2}$, since the smallest positive value of y that satisfies this is $y=30^\circ$. So $\cos(3y+30^\circ)=0$.

The only permissible value of y is then $y=20^\circ$, giving $\angle B=80^\circ$ and $\angle C=40^\circ$.

§5j IMO 2001/6

Let $a>b>c>d>0$ be integers satisfying

$$ac+bd=(b+d+a-c)(b+d-a+c).$$

Prove that $ab+cd$ is not prime.

(Available online at https://aops.com/community/p119217.)

The problem condition is equivalent to

$$ac+bd=(b+d)^2-(a-c)^2$$

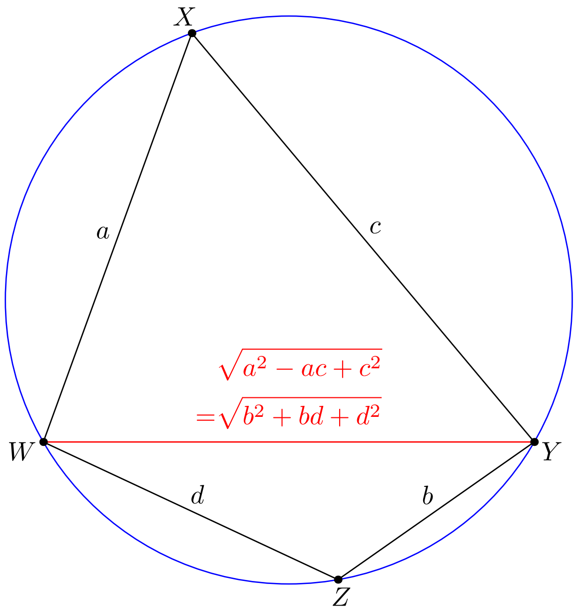
or

$$a^2-ac+c^2=b^2+bd+d^2.$$

Let us construct a quadrilateral $WXYZ$ such that $WX=a$, $XY=c$, $YZ=b$, $ZW=d$, and

$$WY=\sqrt{a^2-ac+c^2}=\sqrt{b^2+bd+d^2}.$$

Then by the law of cosines, we obtain $\angle WXY=60^\circ$ and $\angle WZY=120^\circ$. Hence this quadrilateral is cyclic.



By the more precise version of Ptolemy's theorem, we find that

$$WY^2 = \frac{(ab+cd)(ad+bc)}{ac+bd}.$$

Now assume for contradiction that that $ab + cd$ is a prime p. Recall that we assumed $a > b > c > d$. It follows, for example by rearrangement inequality, that

$$p = ab + cd > ac + bd > ad + bc.$$

Let $y = ac + bd$ and $x = ad + bc$ now. The point is that

$$p \cdot \frac{x}{y}$$

can never be an integer if p is prime and $x < y < p$. But $WY^2 = a^2 - ac + c^2$ is clearly an integer, and this is a contradiction.

Hence $ab + cd$ cannot be prime.

> **Remark.** It may be tempting to try to apply the more typical form of Ptolemy to get $ab + cd = WY \cdot XZ$; the issue with this approach is that WY and XZ are usually not integers.

6 Solutions for Complex Numbers

> The real fun of living wisely is that you get to be smug about it.
>
> Hobbes, in *Calvin and Hobbes*

§6a USAMO 2015/2

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

(Available online at https://aops.com/community/p4769957.)

We present three solutions, one by complex numbers, two more synthetic. (A fourth solution using median formulas is also possible.) Most solutions will prove that the center of the fixed circle is the midpoint of $\overline{AO}$ (with O the center of ω); this can be recovered empirically by letting

- X approach P (giving the midpoint of $\overline{BP}$)
- X approach Q (giving the point Q), and
- X at the midpoint of $\overline{PQ}$ (giving the midpoint of $\overline{BQ}$)

which determines the circle; this circle then passes through P by symmetry and we can find the center by taking the intersection of two perpendicular bisectors (which two?).

¶ Complex solution (Evan Chen). Toss on the complex unit circle with $a = -1$, $b = 1$, $z = -\frac{1}{2}$. Let s and t be on the unit circle. We claim Z is the center.

It follows from standard formulas that

$$x = \frac{1}{2}\left(s + t - 1 + s/t\right)$$

thus

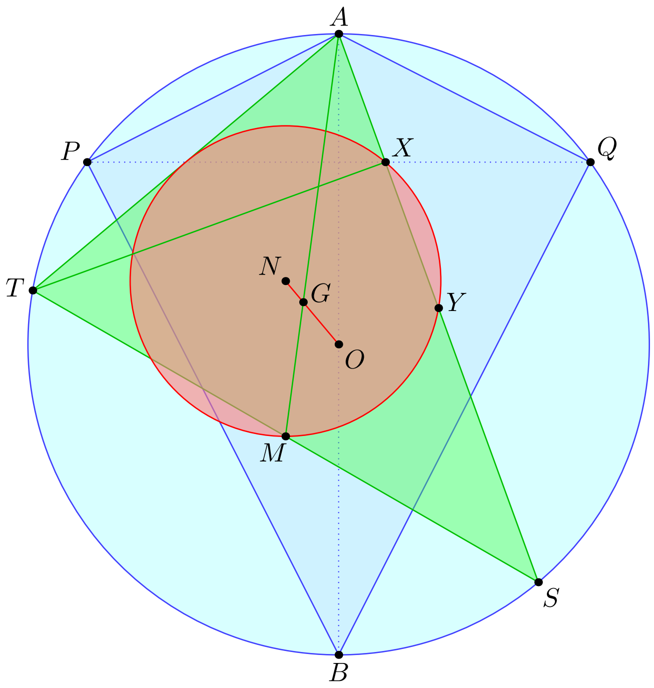
$$4\operatorname{Re} x + 2 = 2\left(x + \frac{1}{x}\right) + 2 = s + t + \frac{1}{s} + \frac{1}{t} + \frac{s}{t} + \frac{t}{s}$$

which depends only on P and Q, and not on X. Thus

$$4\left|z - \frac{s+t}{2}\right|^2 = |s+t+1|^2 = 3 + (4\operatorname{Re} x + 2)$$

does not depend on X, done.

¶ Homothety solution (Alex Whatley). Let G, N, O denote the centroid, nine-point center, and circumcenter of triangle AST, respectively. Let Y denote the midpoint of $\overline{AS}$. Then the three points X, Y, M lie on the nine-point circle of triangle AST, which is centered at N and has radius $\frac{1}{2}AO$.



Let R denote the radius of ω. Note that the nine-point circle of $\triangle AST$ has radius equal to $\frac{1}{2}R$, and hence is independent of S and T. Then the power of A with respect to the nine-point circle equals

$$AN^2 - \left(\frac{1}{2}R\right)^2 = AX \cdot AY = \frac{1}{2}AX \cdot AS = \frac{1}{2}AQ^2$$

and hence

$$AN^2 = \left(\frac{1}{2}R\right)^2 + \frac{1}{2}AQ^2$$

which does not depend on the choice of X. So N moves along a circle centered at A.

Since the points O, G, N are collinear on the Euler line of $\triangle AST$ with

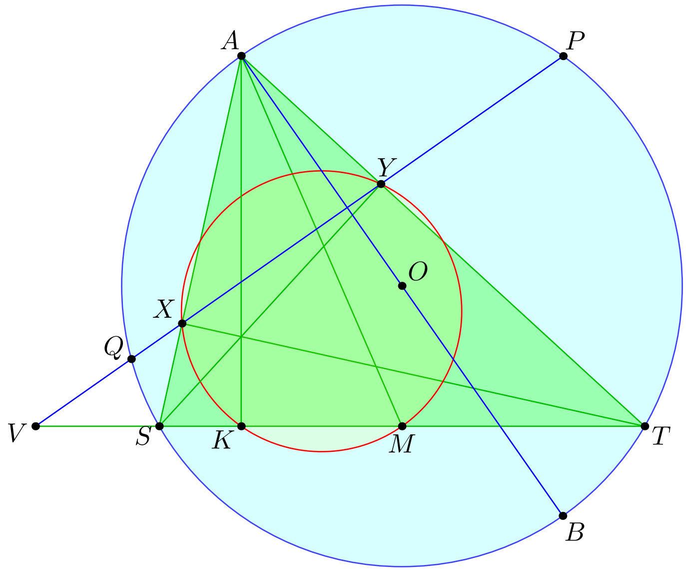
$$GO = \frac{2}{3}NO$$

it follows by homothety that G moves along a circle as well, whose center is situated one-third of the way from A to O. Finally, since A, G, M are collinear with

$$AM = \frac{3}{2}AG$$

it follows that M moves along a circle centered at the midpoint of $\overline{AO}$.

¶ Power of a point solution (Zuming Feng, official solution). We complete the picture by letting $\triangle KYX$ be the orthic triangle of $\triangle AST$; in that case line XY meets the ω again at P and Q.



The main claim is:

Claim — Quadrilateral $PQKM$ is cyclic.

Proof. To see this, we use power of a point: let $V = \overline{QXYP} \cap \overline{SKMT}$. One approach is that since $(VK;ST) = -1$ we have $VQ \cdot VP = VS \cdot VT = VK \cdot VM$. A longer approach is more elementary:

$$VQ \cdot VP = VS \cdot VT = VX \cdot VY = VK \cdot VM$$

using the nine-point circle, and the circle with diameter $\overline{ST}$. □

But the circumcenter of $PQKM$, is the midpoint of $\overline{AO}$, since it lies on the perpendicular bisectors of $\overline{KM}$ and $\overline{PQ}$. So it is fixed, the end.

§6b China TST 2006/4/1

Let H be the orthocenter of triangle ABC. Let D, E, F lie on the circumcircle of ABC such that $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$. Let S, T, U respectively denote the reflections of D, E, F across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Prove that points S, T, U, H are concyclic.

(Available online at https://aops.com/community/p550632.)

Let (ABC) be the unit circle and $h = a+b+c$. WLOG, $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are perpendicular to the real axis (rotate appropriately); thus $d = \overline{a}$ and so on.

Thus $s = b + c - bc\overline{d} = b + c - abc$ and so on; we now have

$$\frac{s-t}{s-u} = \frac{b-a}{c-a} \quad \text{and} \quad \frac{h-t}{h-u} = \frac{b+abc}{c+abc}.$$

Compute

$$\frac{s-t}{s-u} : \frac{h-t}{h-u} = \frac{(b-a)(c+abc)}{(c-a)(b+abc)} = \frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{abc}\right)}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{b}+\frac{1}{abc}\right)}$$

and thus

$$\frac{s-t}{s-u} : \frac{h-t}{h-u} \in \mathbb{R}$$

as desired.

> **Remark.** In fact, the problem remains true if the all-parallel condition is replaced by $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ merely being concurrent at some point. The calculation in this case is more involved though.

§6c USA TST 2014/5

Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

(Available online at https://aops.com/community/p3476291.)

The following solution is due to Grace Wang.
We begin with:

Claim — Point W has coordinates $\frac{1}{2}(2a+b+d)$.

Proof. The orthocenter of $\triangle DAB$ is $d+a+b$, and $\triangle AHE$ is homothetic to $\triangle DAB$ through A with ratio $1/2$. Hence $w = \frac{1}{2}(a+(d+a+b))$ as needed. □

By symmetry, we have

$$\begin{aligned} w &= \frac{1}{2}(2a+b+d) \\ x &= \frac{1}{2}(2b+c+a) \\ y &= \frac{1}{2}(2c+d+b) \\ z &= \frac{1}{2}(2d+a+c). \end{aligned}$$

We see that $w-y=a-c$, $x-z=b-d$. So the diagonals of $WXYZ$ have the same length as those of $ABCD$ as well as the same directed angle between them. This implies the areas are equal, too.

§6d OMO 2013 F26

Let ABC be an acute triangle with circumcenter O. Denote the reflections of B and C across $\overline{AC}$, $\overline{AB}$ by D, E, respectively. Let P be a point such that $\triangle DPO \sim \triangle PEO$ with the same orientation, and let X and Y be the midpoints of the major and minor arcs $\widehat{BC}$ of the circumcircle of triangle ABC. Calculate $PX \cdot PY$ in terms of the side lengths of ABC.

(Available online at https://aops.com/community/p3261431.)

We will prove that

$$PX \cdot PY = BC^2.$$

We apply complex numbers with (ABC) the unit circle. Observe that $x+y=0$ and $xy+bc=0$. Moreover, the condition $\triangle DPO \sim \triangle PEO$ is just

$$\frac{d-p}{p-0} = \frac{p-e}{e-0} \iff p^2 - pe = de - pe \iff p^2 = de.$$

Now we can compute

$$\begin{aligned}
(PX \cdot PY)^2 &= |p-x|^2 \, |p-y|^2 \\
&= (p-x)\left(\overline{p}-\overline{x}\right)(p-y)\left(\overline{p}-\overline{y}\right) \\
&= \left(p^2 - (x+y)p + xy\right)\left(\overline{p}^2 - \left(\overline{x}+\overline{y}\right)\overline{p} + \overline{xy}\right) \\
&= \left(p^2 + xy\right)\left(\overline{p}^2 + \overline{xy}\right) \\
&= (de - bc)\left(\overline{de} - \overline{bc}\right) \\
&= |de - bc|^2 .
\end{aligned}$$

Thus $PX \cdot PY = |de - bc|$. Now

$$d = a + c - \frac{ac}{b}, \quad e = a + b - \frac{ab}{c}.$$

Therefore,

$$\begin{aligned}
de &= \left(a + c - \frac{ac}{b}\right)\left(a + b - \frac{ab}{c}\right) \\
&= a^2 + ab + ac + bc - \frac{a^2c}{b} - ac - \frac{a^2b}{c} - ab + a^2 \\
&= 2a^2 - \frac{a^2c}{b} - \frac{a^2b}{c} + bc.
\end{aligned}$$

Hence

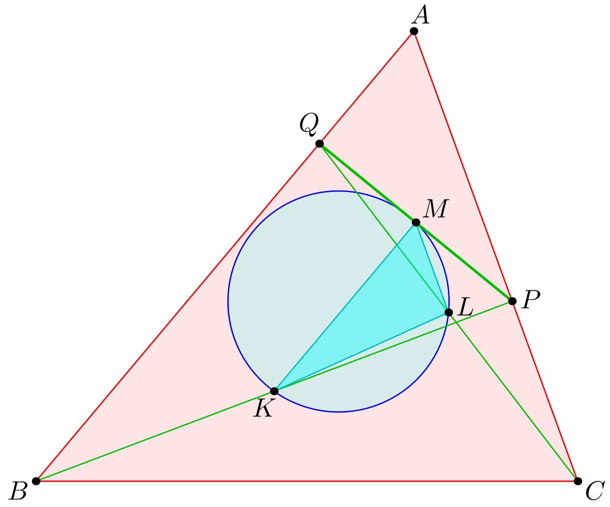
$$\begin{aligned}
PX \cdot PY &= |de - bc| \\
&= \left|2a^2 - \frac{a^2c}{b} - \frac{a^2b}{c}\right| \\
&= \left|-\frac{a^2}{bc}(b-c)^2\right| \\
&= \left|-\frac{a^2}{bc}\right| |b-c|^2 \\
&= BC^2.
\end{aligned}$$

§6e IMO 2009/2

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

(Available online at https://aops.com/community/p1561572.)

By power of a point, we have $-AQ \cdot QB = OQ^2 - R^2$ and $-AP \cdot PC = OP^2 - R^2$. Therefore, it suffices to show $AQ \cdot QB = AP \cdot PC$.



As $\overline{ML} \parallel \overline{AC}$ and $\overline{MK} \parallel \overline{AB}$ we have that

$$\measuredangle APQ = \measuredangle LMP = \measuredangle LKM$$
$$\measuredangle PQA = \measuredangle KMQ = \measuredangle MLK$$

and consequently we have the (opposite orientation) similarity

$$\triangle APQ \overset{-}{\sim} \triangle MKL.$$

Therefore

$$\frac{AQ}{AP} = \frac{ML}{MK} = \frac{2ML}{2MK} = \frac{PC}{QB}$$

id est $AQ \cdot QB = AP \cdot PC$, which is what we wanted to prove.

§6f APMO 2010/4

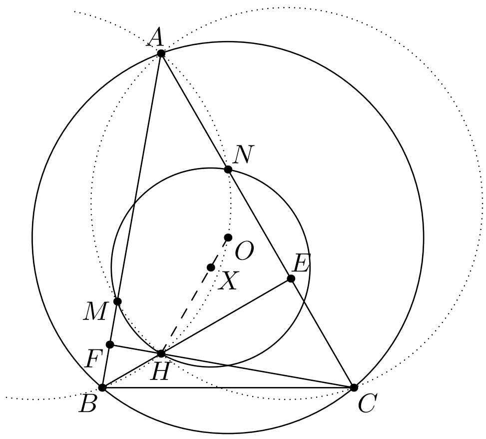
Let ABC be an acute triangle with $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter of ABC. Suppose that the circumcircle of triangle AHC intersects the line AB at M (other than A), and the circumcircle of triangle AHB intersects the line AC at N (other than A). Prove that the circumcenter of triangle MNH lies on line OH.

(Available online at https://aops.com/community/p1868946.)

Inversion solution: Perform a negative inversion at H mapping the circumcircle to the nine-point circle. Then look at $\triangle DEF$.

The problem reduces to the $\overline{DE} \perp \overline{IO}$ lemma (in the style of EGMO 2014/2).

Complex numbers solution: Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$.



First, we claim that M is the reflection of B over F. Indeed, we have that

$$\measuredangle BMH = \measuredangle AMH = \measuredangle ACH = \measuredangle ECF = \measuredangle EBF = \measuredangle HBM$$

implying that $\triangle MHB$ is isosceles. As $\overline{HF} \perp \overline{MB}$, the conclusion follows. Similarly, we can see that N is the reflection of C over E.

Now we can apply complex numbers with (ABC) as the unit circle. Hence we have $f = \frac{1}{2}(a + b + c - ab\overline{c})$, and hence

$$m = 2f - b = a + c - ab\overline{c}.$$

Similarly,

$$n = a + b - ac\overline{b}.$$

Now we wish to compute the circumcenter X of $\triangle HMN$, where $h = a + b + c$. Let M' be the point corresponding to $m - h = -b - ab\overline{c}$ and N' be the point corresponding to $n - h = -c - ac\overline{b}$, noting that O corresponds to $h - h = 0$. Then the circumcenter of $\triangle M'N'O$ corresponds to the point $x - h$. But we can compute the circumcenter of $\triangle M'N'O$; it is

$$\begin{aligned} x - h &= \frac{(m-h)(n-h)\left(\overline{(m-h)} - \overline{(n-h)}\right)}{\overline{(m-h)}(n-h) - (m-h)\overline{(n-h)}} \\ &= \frac{\left(-b - \frac{ab}{c}\right)\left(-c - \frac{ac}{b}\right)\left(\left(-\frac{1}{b} - \frac{c}{ab}\right) - \left(-\frac{1}{c} - \frac{b}{ac}\right)\right)}{\left(-\frac{1}{b} - \frac{c}{ab}\right)\left(-c - \frac{ac}{b}\right) - \left(-b - \frac{ab}{c}\right)\left(-\frac{1}{c} - \frac{b}{ac}\right)} \\ &= \frac{\left(b + \frac{ab}{c}\right)\left(c + \frac{ac}{b}\right)\left(\left(\frac{1}{b} + \frac{c}{ab}\right) - \left(\frac{1}{c} + \frac{b}{ac}\right)\right)}{\left(\frac{1}{b} + \frac{c}{ab}\right)\left(c + \frac{ac}{b}\right) - \left(b + \frac{ab}{c}\right)\left(\frac{1}{c} + \frac{b}{ac}\right)}. \end{aligned}$$

Multiplying the numerator and denominator by ab^2c^2,

$$\begin{aligned} x - h &= \frac{bc\,(a+b)\,(a+c)\,(c(a+c) - b(a+b))}{c^3(a+b)(a+c) - b^3(a+b)(a+c)} \\ &= \frac{bc\left(c^2 - b^2 + a(c-b)\right)}{c^3 - b^3} \\ &= \frac{bc(c-b)(a+b+c)}{(c-b)(b^2+bc+c^2)} \\ &= \frac{bc(a+b+c)}{b^2+bc+c^2}. \end{aligned}$$

So
$$x = h + \frac{bc(a+b+c)}{b^2+bc+c^2} = h\left[1 + \frac{bc}{b^2+bc+c^2}\right].$$

Finally, to show X, H, O are collinear, we only need to prove $\frac{x}{h} = \frac{bc}{b^2+bc+c^2} + 1$ is real. It is equivalent to show $\frac{bc}{b^2+bc+c^2}$ is real, but its conjugate is

$$\overline{\left(\frac{bc}{b^2+bc+c^2}\right)} = \frac{\frac{1}{bc}}{\frac{1}{b^2}+\frac{1}{bc}+\frac{1}{c^2}} = \frac{bc}{b^2+bc+c^2}$$

and the proof is complete.

§6g Shortlist 2006 G9

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

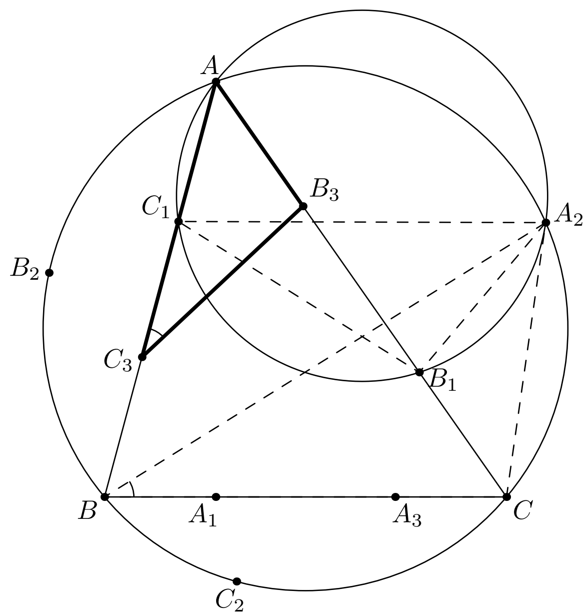
(Available online at https://aops.com/community/p875036.)

We will prove the following claim, after which only angle chasing remains.

Claim — We have $\measuredangle AC_3B_3 = \measuredangle A_2BC$.

Proof. By spiral similarity at A_2, we deduce that $\triangle A_2C_1B \sim \triangle A_2B_1C$, hence

$$\frac{A_2B}{A_2C} = \frac{C_1B}{B_1C} = \frac{AC_3}{AB_3}.$$



It follows that

$$\triangle A_2BC \sim \triangle AC_3B_3$$

since we also have $\measuredangle BA_2C = \measuredangle BAC = \measuredangle C_3AB_3$. (Configuration issues: we can check that A_2 lies on the same side of A as $\overline{BC}$ since B_1 and C_1 are constrained to lie on the sides of the triangle. So we can deduce $\angle C_3AB_3 = \angle BA_2C$.)

Thus $\measuredangle AC_3B_3 = \measuredangle A_2BC$, completing the proof. □

Similarly, $\measuredangle BC_3A_3 = \measuredangle B_2AC$

The rest is angle chasing; we have

$$\begin{aligned}\measuredangle A_3C_3B_3 &= \measuredangle A_3C_3A + \measuredangle AC_3B_3 \\ &= \measuredangle A_3C_3B + \measuredangle AC_3B_3 \\ &= \measuredangle CAB_2 + \measuredangle A_2BC \\ &= \measuredangle A_2C_2C + \measuredangle CC_2B_2 \\ &= \measuredangle A_2C_2B_2.\end{aligned}$$

§6h MOP 2006/4/1

Given a cyclic quadrilateral $ABCD$ with circumcenter O and a point P on the plane, let O_1, O_2, O_3, O_4 denote the circumcenters of triangles PAB, PBC, PCD, PDA respectively. Prove that the midpoints of segments O_1O_3, O_2O_4, and OP are collinear.

We apply complex numbers with $(ABCD)$ as the unit circle. The problem is equivalent to proving that

$$\frac{\frac{1}{2}p - \frac{1}{2}(o_1 + o_3)}{\frac{1}{2}\overline{p} - \frac{1}{2}(\overline{o_1} + \overline{o_3})} = \frac{\frac{1}{2}p - \frac{1}{2}(o_2 + o_4)}{\frac{1}{2}\overline{p} - \frac{1}{2}(\overline{o_2} + \overline{o_4})}.$$

First, we compute

$$\begin{aligned} o_1 &= \begin{vmatrix} a & a\overline{a} & 1 \\ b & b\overline{b} & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ p & p\overline{p} & 1 \end{vmatrix} \div \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 1 \\ b & 0 & 1 \\ p & p\overline{p} - 1 & 1 \end{vmatrix} \div \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ p & \overline{p} & 1 \end{vmatrix} \\ &= \frac{(p\overline{p} - 1)(b - a)}{\frac{a}{b} - \frac{b}{a} + p(\frac{1}{a} - \frac{1}{b}) + \overline{p}(b - a)} \\ &= \frac{p\overline{p} - 1}{\frac{p}{ab} + \overline{p} - \frac{a+b}{ab}}. \end{aligned}$$

The conjugate of this expression is easier to work with; we have

$$\overline{o_1} = \frac{p\overline{p} - 1}{ab\overline{p} + p - (a + b)}.$$

Similarly,

$$\overline{o_3} = \frac{p\overline{p} - 1}{cd\overline{p} + p - (c + d)}.$$

In what follows, we let $s_1 = a+b+c+d$, $s_2 = ab+bc+cd+da+ac+bd$, $s_3 = abc+bcd+cda+dab$, and $s_4 = abcd$ for brevity. Then,

$$\begin{aligned} &\overline{o_1} + \overline{o_3} - \overline{p} \\ &= (p\overline{p}-1)\left(\frac{1}{ab\overline{p}+p-(a+b)} + \frac{1}{cd\overline{p}+p-(c+d)}\right) - \overline{p} \\ &= \frac{(p\overline{p}-1)\left(2p+(ab+cd)\overline{p}-s_1\right)}{\left(ab\overline{p}+p-(a+b)\right)\left(cd\overline{p}+p-(c+d)\right)} - \overline{p}. \end{aligned}$$

Consider the fraction in the above expansion. One can check that the denominator expands as

$$\mathcal{D} = s_4\overline{p}^2 + (ab+cd)\,p\overline{p} + p^2 - s_3\overline{p} - s_1 p + (ac+ad+bc+bd).$$

On the other hand, the numerator is equal to

$$\mathcal{N} = (2p-s_1)(p\overline{p}-1) + (ab+cd)\overline{p}(p\overline{p}-1).$$

Thus,

$$\overline{o_1} + \overline{o_3} - \overline{p} = \frac{\mathcal{N} - \overline{p}\mathcal{D}}{\mathcal{D}}.$$

We claim that the expression $\mathcal{N} - \overline{p}\mathcal{D}$ is symmetric in a, b, c, d. To see this, we need only look at the terms of $\mathcal{N}$ and $\mathcal{D}$ that are not symmetric in a, b, c, d. These are $(ab+cd)\overline{p}(p\overline{p}-1)$ and $(ab+cd)p\overline{p} + (ac+ad+bd+bc)$, respectively. Subtracting $\overline{p}$ times the latter from the former yields $-s_2\overline{p}$. Hence $\mathcal{N} - \overline{p}\mathcal{D}$ is symmetric in a, b, c, d, as claimed.[1] Now we may set $\mathcal{S} = \mathcal{N} - \overline{p}\mathcal{D}$.

Thus

$$\begin{aligned} \frac{o_1+o_3-p}{\overline{o_1}+\overline{o_3}-\overline{p}} &= \frac{\overline{\mathcal{S}}/\overline{\mathcal{D}}}{\mathcal{S}/\mathcal{D}} \\ &= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot \frac{\mathcal{D}}{\overline{\mathcal{D}}} \\ &= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot \frac{(ab\overline{p}+p-(a+b))(cd\overline{p}+p-(c+d))}{(\frac{1}{ab}p+\overline{p}-\frac{1}{a}-\frac{1}{b})(\frac{1}{cd}p+\overline{p}-\frac{1}{c}-\frac{1}{d})} \\ &= \frac{\overline{\mathcal{S}}}{\mathcal{S}} \cdot abcd. \end{aligned}$$

Hence, we deduce

$$\frac{o_1+o_3-p}{\overline{o_1}+\overline{o_3}-\overline{p}}$$

is in fact symmetric in a, b, c, d. Hence if we repeat the same calculation with $\frac{o_2+o_4-p}{\overline{o_2}+\overline{o_4}-\overline{p}}$, we must obtain exactly the same result. This completes the solution.

§6i Shortlist 1998 G6

Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

[1] In fact, if you really want to do the computation you can check that $\mathcal{N} - \overline{p}\mathcal{D} = -s_4\overline{p}^3 + p^2\overline{p} + s_3\overline{p}^2 - s_2\overline{p} + \overline{p} + 2p + s - 1$. But we will not need to do anything with this expression other than notice that it is symmetric.

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

(Available online at https://aops.com/community/p3488.)

We use complex numbers, since the condition in its given form is an abomination. Consider the quantity

$$\frac{b-a}{f-a} \cdot \frac{d-c}{b-c} \cdot \frac{f-e}{d-e}.$$

By the first condition, its argument is 360°, so it is a positive real However, the second condition implies that it has norm 1. We deduce that it is actually equal to 1.

So, we are given that

$$0 = (a-b)(c-d)(e-f) + (b-c)(d-e)(f-a)$$

and wish to show that

$$|(b-c)(a-e)(f-d)| = |(c-a)(e-f)(d-b)|.$$

But in fact one can check they are equal.

§6j ELMO SL 2013 G7

Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to $\overline{AC}$ at C, and let O_2 be the center of the circle through E tangent to $\overline{AB}$ at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

(Available online at https://aops.com/community/p3151965.)

We use complex numbers with (ABC) the unit circle.

To compute D, note that since the midpoint of $\overline{AC}$ lies on chord $\overline{BD}$, we should have

$$b + d = \frac{a+c}{2} + bd \cdot \frac{a+c}{2ac} \implies d = \frac{\frac{a+c}{2} - b}{1 - \frac{b(a+c)}{2ac}} = \frac{ac(a+c-2b)}{2ac - b(a+c)}.$$

We now seek to compute O_1. Let O denote the circumcircle. Note that since $\triangle AOD \sim \triangle DCO_1$ we have

$$\frac{o_1 - d}{c-d} = \frac{-d}{a-d}$$

so

$$\begin{aligned} o_1 &= \frac{d(a-d) - d(c-d)}{a-d} = \frac{d(a-c)}{a-d} \\ &= \frac{ac(a+c-2b)(a-c)}{a(2ac - b(a+c)) - ac(a+c-2b)} \\ &= \frac{c(a+c-2b)(a-c)}{ac - ab + bc - c^2} = \frac{c(a+c-2b)}{c-b}. \end{aligned}$$

Similarly $o_2 = \frac{b(a+b-2c)}{b-c}$. We now find that

$$\frac{o_1 + o_2}{2} = \frac{b(a+b-2c) - c(a+c-2b)}{2(b-c)} = \frac{a+b+c}{2}$$

so in fact the nine-point center is the midpoint of O_1 and O_2.

7 Solutions for Barycentric Coordinates

> I don't care if you're a devil in disguise! I love you all the same!
>
> Misa Amane, in *Death Note: The Last Name*

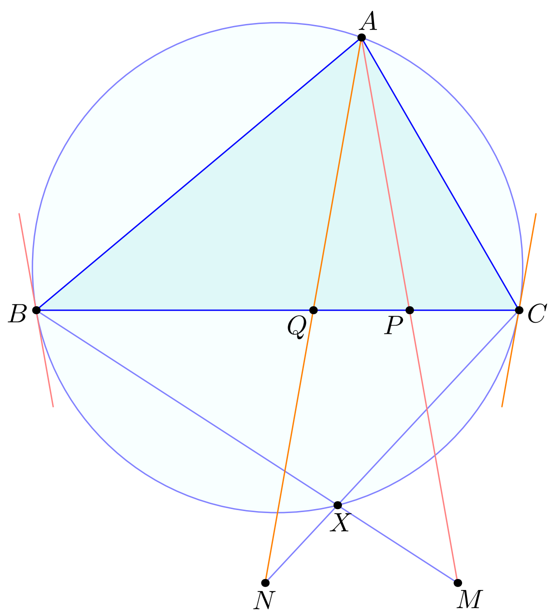
§7a IMO 2014/4

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

(Available online at https://aops.com/community/p3543136.)

We give three solutions.

¶ First solution by harmonic bundles. Let $\overline{BM}$ intersect the circumcircle again at X.



The angle conditions imply that the tangent to (ABC) at B is parallel to $\overline{AP}$. Let ∞ be the point at infinity along line AP. Then

$$-1 = (AM; P\infty) \overset{B}{=} (AX; BC).$$

Similarly, if $\overline{CN}$ meets the circumcircle at Y then $(AY; BC) = -1$ as well. Hence $X = Y$, which implies the problem.

¶ Second solution by similar triangles. Once one observes $\triangle CAQ \sim \triangle CBA$, one can construct D the reflection of B across A, so that $\triangle CAN \sim \triangle CBD$. Similarly, letting E be the reflection of C across A, we get $\triangle BAP \sim \triangle BCA \implies \triangle BAM \sim \triangle BCE$. Now to show $\angle ABM + \angle ACN = 180^\circ$ it suffices to show $\angle EBC + \angle BCD = 180^\circ$, which follows since $BCDE$ is a parallelogram.

¶ Third solution by barycentric coordinates. Since $PB = c^2/a$ we have

$$P = (0 : a^2 - c^2 : c^2)$$

so the reflection $\vec{M} = 2\vec{P} - \vec{A}$ has coordinates

$$M = (-a^2 : 2(a^2 - c^2) : 2c^2).$$

Similarly $N = (-a^2 : 2b^2 : 2(b^2 - a^2))$. Thus

$$\overline{BM} \cap \overline{CN} = (-a^2 : 2b^2 : 2c^2)$$

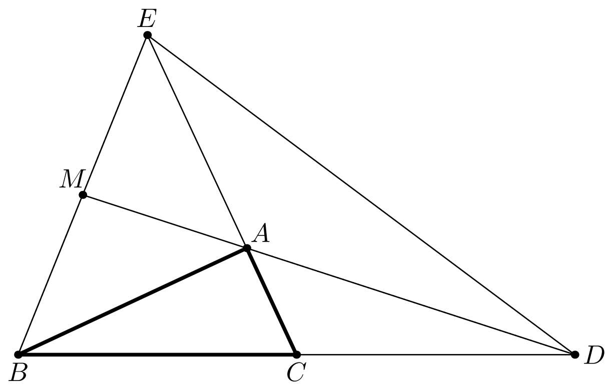
which clearly lies on the circumcircle, and is in fact the point identified in the first solution.

§7b EGMO 2013/1

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

(Available online at https://aops.com/community/p3013167.)

Let ray DA meet $\overline{BE}$ at M. Consider the triangle EBD. Since the point lies on median $\overline{EC}$, and $EA = 2AC$, it follows that A is the centroid of $\triangle EBD$.



So M is the midpoint of $\overline{BE}$. Moreover $MA = \frac{1}{2}AD = \frac{1}{2}BE$; so $MA = MB = ME$ and hence $\triangle ABE$ is inscribed in a circle with diameter $\overline{BE}$. Thus $\angle BAE = 90^\circ$, so $\angle BAC = 90^\circ$.

§7c ELMO SL 2013 G3

In non-right triangle ABC, a point D lies on line $\overline{BC}$. The circumcircle of ABD meets $\overline{AC}$ at F (other than A), and the circumcircle of ADC meets $\overline{AB}$ at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to $\overline{BC}$.

(Available online at https://aops.com/community/p3151962.)

After a $\sqrt{bc}$ inversion around A, it suffices to prove that for variable D^* on (ABC), the line through $E^* = \overline{BD^*} \cap \overline{AC}$ and $F^* = \overline{CD^*} \cap \overline{AB}$ passes through a fixed point on the A-symmedian. By Brocard's theorem this is the pole of $\overline{BC}$.

Alternatively, use barycentric coordinates with $A = (1, 0, 0)$, etc. Let $D = (0 : m : n)$ with $m + n = 1$. Then the circle ABD has equation $-a^2yz - b^2zx - c^2xy + (x + y + z)\left(a^2m \cdot z\right)$. To intersect it with side AC, put $y = 0$ to get $(x + z)(a^2mz) = b^2zx \implies \frac{b^2}{a^2m} \cdot x = x + z \implies \left(\frac{b^2}{a^2m} - 1\right) x = z$, so

$$F = (a^2m : 0 : b^2 - a^2m)$$

Similarly,

$$G = (a^2n : c^2 - a^2n : 0).$$

Then, the circle (AFG) has equation

$$-a^2yz - b^2zx - c^2xy + a^2(x + y + z)(my + nz) = 0.$$

Upon picking $y = z = 1$, we easily see there exists a t such that $(t : 1 : 1)$ is on the circle, implying the conclusion.

One can also use trigonometry directly. Let M be the midpoint of BC. By power of a point, $c \cdot BE + b \cdot CF = a \cdot BD + a \cdot CD = a^2$ is constant. Fix a point D_0; and let $P_0 = AM \cap (AE_0F_0)$. For any other point D, we have $\frac{E_0E}{F_0F} = \frac{b}{c} = \frac{\sin\angle BAM}{\sin\angle CAM} = \frac{P_0E_0}{P_0F_0}$ from the extended law of sines, so triangles P_0E_0E and P_0F_0F are directly similar, whence AEP_0F is cyclic, as desired.

§7d IMO 2012/1

Let ABC be a triangle and J the center of the A-excircle. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of $\overline{ST}$.

(Available online at https://aops.com/community/p2736397.)

We employ barycentric coordinates with reference $\triangle ABC$. As usual $a = BC$, $b = CA$, $c = AB$, $s = \frac{1}{2}(a + b + c)$.

It's obvious that $K = (-(s - c) : s : 0)$, $M = (0 : s - b : s - c)$. Also, $J = (-a : b : c)$. We then obtain

$$G = \left(-a : b : \frac{-as + (s - c)b}{s - b}\right).$$

It follows that

$$T = \left(0 : b : \frac{-as + (s-c)}{s-b}\right) = (0 : b(s-b) : b(s-c) - as).$$

Normalizing, we see that $T = \left(0, -\frac{b}{a}, 1 + \frac{b}{a}\right)$, from which we quickly obtain $MT = s$. Similarly, $MS = s$, so we're done.

§7e USA TST 2008/7

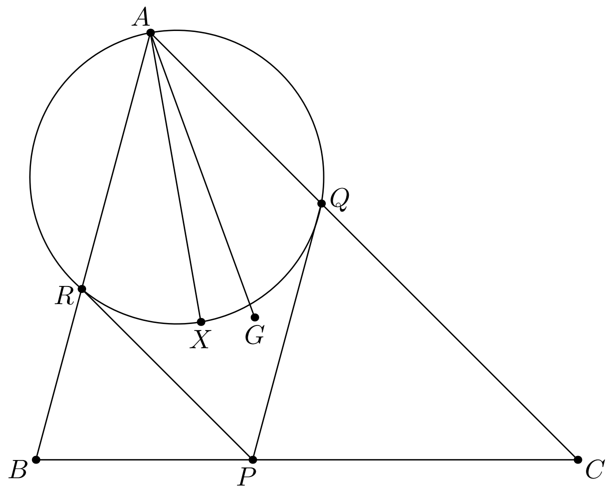
Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{PR} \parallel \overline{AC}$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

(Available online at https://aops.com/community/p1247506.)

Let $P = (0, s, t)$ where $s + t = 1$. One can check that $Q = (s, 0, t)$. Similarly, $R = (t, s, 0)$. So the circumcircle of $\triangle AQR$ is given by

$$-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz) = 0$$

where u, v, w are some real numbers.



Plugging in the point A gives $u = 0$. Plugging in the point Q gives $wt = b^2st$, so $w = b^2s$. Plugging in the point R gives $vs = c^2st$, so $v = c^2t$. Thus the circumcircle has equation

$$-a^2yz - b^2zx - c^2xy + (x+y+z)\left(c^2ty + b^2sz\right) = 0.$$

Now let us consider the intersection of the A-symmedian with this circumcircle. Let the intersection be $X = (k : b^2 : c^2)$. We aim to show the value of k does not depend on s or t. But this is obvious, as substitution gives

$$-a^2b^2c^2 - 2b^2c^2k + (k + b^2 + c^2)(b^2c^2)(s+t) = 0.$$

Since $s + t = 1$ and the equation is linear in k, we have exactly one solution for k. The proof ends here; there is no need to compute the value of k explicitly. (For the curious, the actual value of k is $k = -a^2 + b^2 + c^2$.)

§7f USAMO 2001/2

Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

(Available online at https://aops.com/community/p337870.)

We have that P is the Nagel point

$$P = (s-a : s-b : s-c).$$

Therefore,

$$\frac{PD_2}{AD_2} = \frac{s-a}{(s-a)+(s-b)+(s-c)} = \frac{s-a}{s}.$$

Meanwhile, Q is the antipode of D_1. The classical homothety at A mapping Q to D_1 (by mapping the incircle to the A-excircle) has ratio $\frac{s-a}{s}$ as well (by considering the length of the tangents from A), so we are done.

§7g TSTST 2012/7

Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

(Available online at https://aops.com/community/p2745857.)

By angle chasing, equivalent to show $\overline{MN} \parallel \overline{AD}$, so discard the point H. We now present a three solutions.

¶ First solution using vectors. We first contend that:

Claim — We have $QB = PC$.

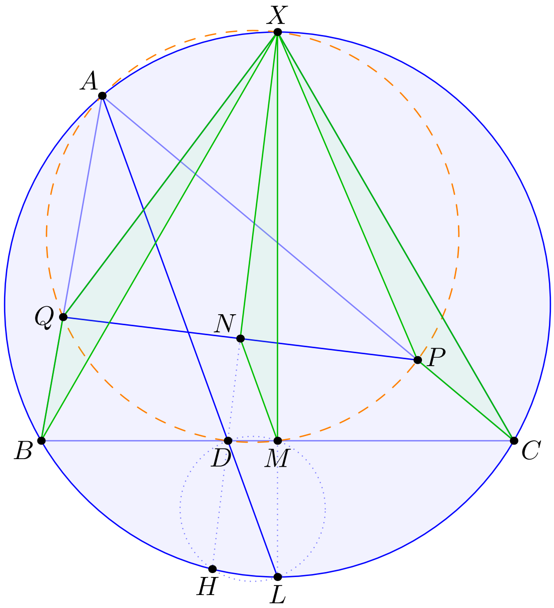
Proof. Power of a Point gives $BM \cdot BD = AB \cdot QB$. Then use the angle bisector theorem. □

Now notice that the vector

$$\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{BQ} + \overrightarrow{CP}\right)$$

which must be parallel to the angle bisector since $\overrightarrow{BQ}$ and $\overrightarrow{CP}$ have the same magnitude.

¶ Second solution using spiral similarity. let X be the arc midpoint of BAC. Then $ADMX$ is cyclic with diameter $\overline{XD}$, and hence X is the Miquel point of $QBPC$ and thus the midpoint of arc BAC. Moreover $\overline{XND}$ collinear (as $XP = XQ$, $DP = DQ$) on (APQ).



Then $\triangle XNM \sim \triangle XPC$ spirally, and

$$\measuredangle XMN = \measuredangle XCP = \measuredangle XCA = \measuredangle XLA$$

thus done.

¶ Third solution using barycentrics (mine). Once reduced to $\overline{MN} \parallel \overline{AB}$, straight bary will also work. By power of a point one obtains

$$\begin{aligned} P &= \left(a^2 : 0 : 2b(b+c) - a^2\right) \\ Q &= \left(a^2 : 2c(b+c) - a^2 : 0\right) \\ \implies N &= \left(a^2(b+c) : 2bc(b+c) - ba^2 : 2bc(b+c) - ca^2\right). \end{aligned}$$

Now the point at infinity along $\overline{AD}$ is $(-(b+c) : b : c)$ and so we need only verify

$$\det \begin{bmatrix} a^2(b+c) & 2bc(b+c) - ba^2 & 2bc(b+c) - ca^2 \\ 0 & 1 & 1 \\ -(b+c) & b & c \end{bmatrix} = 0$$

which follows since the first row is $-a^2$ times the third row plus $2bc(b+c)$ times the second row.

§7h December TST 2012/1

In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side $\overline{BC}$, the circumcircle of triangle ADE passes through a fixed point other than A.

(Available online at https://aops.com/community/p3195787.)

Use reference ABC. Let $P = (0, s, t)$ with $s + t = 1$.
Then we have that:

$$BD = 2BP\cos B = 2(at)\cos B = t \cdot 2c \in S_B$$

Subtracting, $AD = c - BD = c - t \cdot 2c^{-1}S_B$, so

$$D = \left(t \cdot 2c^{-1}S_A : c - t \cdot 2c^{-1}S_B : 0\right) = \left(t \cdot 2S_A : c^2 - t \cdot 2S_B : 0\right).$$

Analogously,

$$E = \left(s \cdot 2S_C : 0 : b^2 - s \cdot 2S_C\right).$$

Claim — The circumcircle of $\triangle ADE$ has equation

$$-a^2yz - b^2zx - c^2xy + 2(x + y + z)(tS_By + sS_Cz) = 0.$$

Proof. Circle formula applied to A gives $u = 0$. Plugging in D and E:

$$\begin{aligned} c^2(t \cdot 2S_B)(c^2 - t \cdot 2S_B) &= c^2(v \cdot (c^2 - t \cdot 2S_B)) \\ \implies v &= 2t \cdot S_B \\ \implies w &= 2s \cdot S_C. \end{aligned}$$

□

From here one can check that the fixed point turns out to be $H = (\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C})$, which is the orthocenter of triangle ABC.

Remark. One does not even need to compute the point H. Instead, by inspection one observes there is a unique real number λ for which $(\lambda : \frac{1}{S_B} : \frac{1}{S_C})$ lies on the circle, since one obtains a linear equation in λ whose linear coefficient is $\frac{-b^2}{S_B} + \frac{-c^2}{S_C} + 2 \neq 0$, and that yields a fixed point.

§7i Sharygin 2013/20

Let C_1 be an arbitrary point on side AB of $\triangle ABC$. Points A_1 and B_1 are on rays BC and AC such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

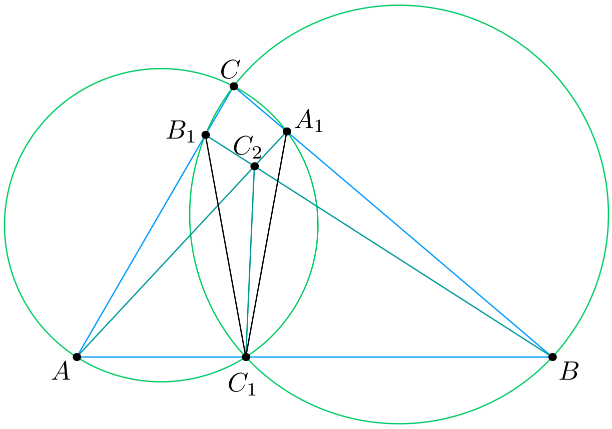
Here are two approaches.

¶ First DDIT solution. Use dual Desargues' involution theorem from C_1 to complete quadrilateral $ABA_1B_1CC_2$; the involution corresponds to reflection over $\overline{AB}$ so we find that C_1C_2 passes through the reflection of C over $\overline{AB}$.

¶ Second barycentric solution. We use barycentric coordinates. Let $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$. Denote $a = BC$, $b = CA$, and $c = AB$. We claim that the common point is

$$K = \left(a^2 - b^2 + c^2 : b^2 - a^2 + c^2 : -c^2\right).$$

Let $C_1 = (u, v, 0)$ with $u + v = 1$.



By power of a point, we observe that $BA_1 = \frac{uc^2}{a}$. Therefore, we obtain that

$$A_1 = \left(0 : a - \frac{uc^2}{a} : \frac{uc^2}{a}\right) = \left(0 : a^2 - uc^2 : uc^2\right).$$

Similarly,

$$B_1 = \left(b^2 - vc^2 : 0 : vc^2\right).$$

Therefore,

$$C_2 = \left(u(b^2 - vc^2) : v(a^2 - uc^2) : uvc^2\right).$$

Now we show that C_1, C_2, and K are collinear. Expand

$$\begin{aligned}
\begin{vmatrix} u(b^2 - vc^2) & v(a^2 - uc^2) & uvc^2 \\ u & v & 0 \\ a^2 - b^2 + c^2 & b^2 - a^2 + c^2 & -c^2 \end{vmatrix} &= uvc^2 \begin{vmatrix} b^2 - vc^2 & a^2 - uc^2 & uv \\ 1 & 1 & 0 \\ \frac{a^2 - b^2 + c^2}{u} & \frac{b^2 - a^2 + c^2}{v} & -1 \end{vmatrix} \\
&= uvc^2 \Big[(a^2 - uc^2) - (b^2 - vc^2) \\
&\qquad + u(b^2 - a^2 + c^2) - v(a^2 - b^2 + c^2)\Big] \\
&= uvc^2(b^2 - a^2)(u + v - 1) = 0
\end{aligned}$$

which implies that C_1, C_2, and K are collinear, as desired.

§7j APMO 2013/5

Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

(Available online at https://aops.com/community/p3046946.)

¶ First solution. Let E' be the second intersection of $\overline{BR}$ with ω. Then

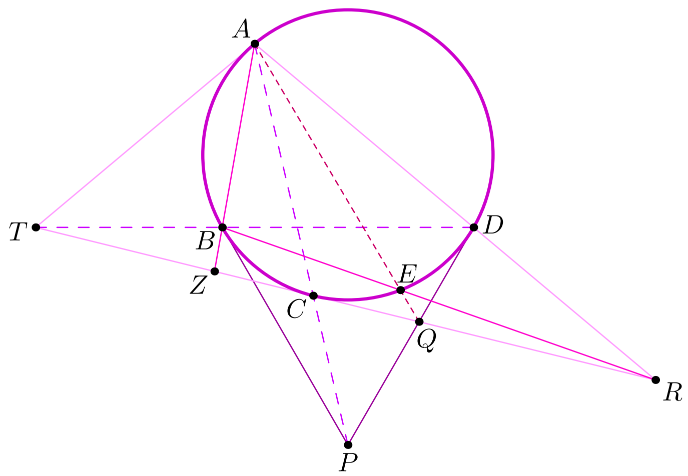
$$-1 = (AC;BD) \overset{R}{=} (DC;AE').$$

But $DACE$ is harmonic, so $E = E'$.

¶ Second solution. Define E' as before. Set $T = \overline{AA} \cap \overline{CR}$, $Z = \overline{AB} \cap \overline{CR}$. Then

$$-1 = (AC;BD) \overset{A}{=} (TC;ZR) \overset{B}{=} (DC;AE').$$

So again $E = E'$.



¶ Third solution using Pascal. After defining T as before, use Pascal on $AAEBDD$.

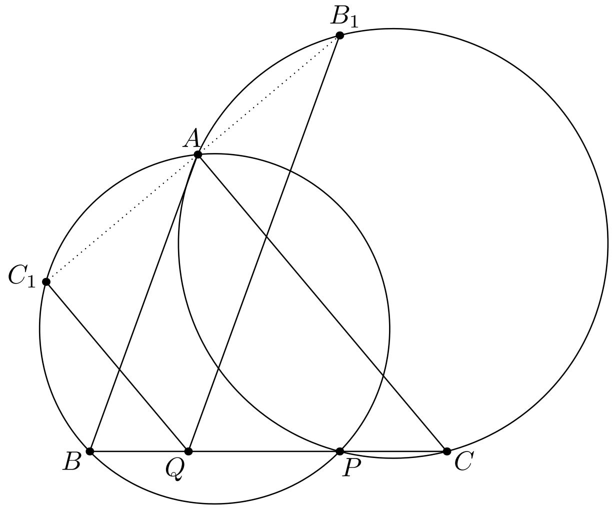
¶ Third solution with homography. Note that $ABCD$ is harmonic. Thus we can take a homography which preserves ω and sends $ABCD$ to a square (i.e. harmonic rectangle), and then coordinate bash.

§7k USAMO 2005/3

Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $\overline{QC_1} \parallel \overline{CA}$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $\overline{QB_1} \parallel \overline{BA}$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

(Available online at https://aops.com/community/p213011.)

It is enough to prove that A, B_1, and C_1 are collinear, since then $\measuredangle C_1QP = \measuredangle ACP = \measuredangle AB_1P = \measuredangle C_1B_1P$.



¶ First solution. Let T be the second intersection of $\overline{AC_1}$ with (APC). Then readily $\triangle PC_1T \sim \triangle ABC$. Consequently, $\overline{QC_1} \parallel \overline{AC}$ implies TC_1QP cyclic. Finally, $\overline{TQ} \parallel \overline{AB}$ now follows from the cyclic condition, so $T = B_1$ as desired.

¶ Second solution. One may also use barycentric coordinates. Let $P = (0, m, n)$ and $Q = (0, r, s)$ with $m + n = r + s = 1$. Once again,

$$(APB) : -a^2yz - b^2zx - c^2xy + (x + y + z)(a^2m \cdot z) = 0.$$

Set $C_1 = (s - z, r, z)$, where $C_1Q \parallel AC$ follows by $(s - z) + r + z = 1$. We solve for this z.

$$\begin{aligned} 0 &= -a^2rz + (s - z)(-b^2z - c^2r) + a^2mz \\ &= b^2z^2 + (-sb^2 + rc^2)z - a^2rz + a^2mz - c^2rs \\ &= b^2z^2 + (-sb^2 + rc^2 + a^2(m - r))z - c^2rs \\ \implies 0 &= rb^2\left(\frac{z}{r}\right)^2 + (-sb^2 + rc^2 + a^2(m - r))\left(\frac{z}{r}\right) - c^2s. \end{aligned}$$

So the quotient of the z and y coordinates of C_1 satisfies this quadratic. Similarly, if $B_1 = (r - y, y, s)$ we obtain that

$$0 = sc^2\left(\frac{y}{s}\right)^2 + (-rc^2 + sb^2 + a^2(n - s))\left(\frac{y}{s}\right) - b^2r$$

Since these two quadratics are the same when one is written backwards (and negated), it follows that their roots are reciprocals. But the roots of the quadratics represent $\frac{z}{y}$ and $\frac{y}{z}$ for the points C_1 and B_1, respectively. This implies (with some configuration blah) that the points B_1 and C_1 are collinear with $A = (1, 0, 0)$ (in some line of the form $\frac{y}{z} = k$), as desired.

§71 Shortlist 2011 G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. For $1 \le i \le 4$, let O_i and r_i be the circumcenter and the circumradius of triangle $A_{i+1}A_{i+2}A_{i+3}$ (where $A_{i+4} = A_i$). Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

(Available online at https://aops.com/community/p2739321.)

Let ω_i be the circle with center O_i and radius r_i. Set $A_1 = (1,0,0)$, $A_2 = (0,1,0)$, $A_3 = (0,0,1)$, and as usual let $a = A_2A_3$ and so on. Let $A_4 = (p,q,r)$, where $p+q+r = 1$. Let $T = a^2qr + b^2rp + c^2pq$ for brevity.

The circumcircle of $\triangle A_2A_3A_4$ can be seen to have equation

$$-a^2yz - b^2zx - c^2xy + (x+y+z)\left(\frac{T}{p}x\right) = 0.$$

By power of a point, we thus have that

$$O_1A_1^2 - r_1^2 = (1+0+0)\cdot\frac{T}{p}\cdot 1 = \frac{T}{p}.$$

Similarly,

$$O_2A_2^2 - r_2^2 = \frac{T}{q} \text{ and } O_3A_3^2 - r_3^2 = \frac{T}{r}.$$

Finally, we obtain $O_4A_4^2 - r_4^2$ by plugging in A_4 into $(A_1A_2A_3)$, which gives a value of $-T$. Hence the left-hand side of our expression is

$$\frac{p}{T} + \frac{q}{T} + \frac{r}{T} - \frac{1}{T} = 0$$

since $p+q+r = 1$.

§7m Romania TST 2010/6/2

Let ABC be a scalene triangle, let I be its incenter, and let A_1, B_1, and C_1 be the points of contact of the excircles with the sides BC, CA, and AB, respectively. Prove that the circumcircles of the triangles AIA_1, BIB_1, and CIC_1 have a common point different from I.

Let $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$ and define a,b,c in the usual fashion. Then, we get

$$A_1 = (0 : s-b : s-c)$$

and its cyclic variants, as well as $I = (a : b : c)$.

Let us calculate $\omega_A = (AIA_1)$ and its cyclic variants. Upon using the generic circle form $-a^2yz - b^2zx - c^2xy + (x+y+z)(ux+vy+wz)$ we find $u = 0$ and the system

$$abc = vb + wc$$
$$a(s-b)(s-c) = v(s-b) + w(s-c)$$

Solving, we find that $v = \frac{ac(s-c)(2b-s)}{s(b-c)}$ and $w = \frac{ab(s-b)(2c-s)}{s(c-b)}$. In summary:

$$\begin{aligned}\omega_A : \quad 0 &= -a^2yz - b^2zx - c^2xy \\ &+ (x+y+z)\left(\frac{ac(s-c)(2b-s)}{s(b-c)}y + \frac{ab(s-b)(2c-s)}{s(c-b)}z\right)\end{aligned}$$

One can then apply symmetry and compute the pairwise radical axes. However, a nice trick, due to Anant Mudgal, is to instead compute the radical axis with the circumcircle instead.

We define ℓ_A as the radical axis of the circumcircle of $\triangle ABC$ and ω_A. Consequently,

$$\ell_A : \quad c(s-c)(2b-s)y + b(s-b)(2c-s)z = 0.$$

If we define ℓ_B and ℓ_C similarly, then we find that ℓ_A, ℓ_B, ℓ_C concur at a point P (by Ceva, since $\prod_{\text{cyc}} \frac{c(s-c)(2b-s)}{b(s-b)(2c-s)} = 1$). Then line PI is the common radical axis of the three circles.

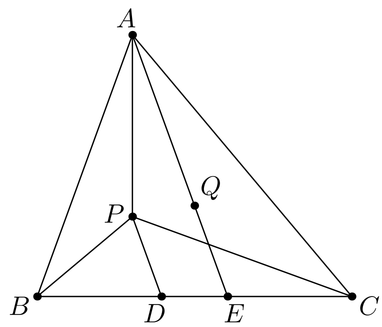
> **Remark** (Ryan Li)**.** Technically, we need to also show that the three circles are not all tangent.

§7n ELMO 2012/5

Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $\overline{PD} \parallel \overline{AE}$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

(Available online at https://aops.com/community/p2728469.)

¶ First solution (barycentric coordinates). Suppose that $D = (0 : 1 : t)$ and $E = (0 : t : 1)$. Let Q be the isogonal conjugate of P; evidently Q lies on $\overline{AE}$, so $Q = (k : t : 1)$ for some k. Moreover, $P = \left(\frac{a^2}{k} : \frac{b^2}{t} : c^2\right)$.



So the condition that $\overline{PD} \parallel \overline{AE}$ implies that P and D are collinear with the point at infinity $(-(1+t) : t : 1)$ along line AE, so we find

$$0 = \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -(1+t) & t & 1 \end{vmatrix}$$

which can be rewritten as

$$0 = \det \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -(1+t) & 1+t & 1+t \end{vmatrix} = (1+t) \begin{vmatrix} a^2/k & b^2/t & c^2 \\ 0 & 1 & t \\ -1 & 1 & 1 \end{vmatrix}.$$

Expanding the determinant, we derive that

$$0 = a^2(1-t) + k(c^2 - b^2)$$

and applying the perpendicular bisector formula, we derive that $BQ = QC$. So $\angle QBC = \angle QCB$, implying $\angle PBA = \angle PCA$.

¶ Second solution (isogonality lemma). Let R be the reflection of P across the midpoint of $\overline{BC}$, so $PBRC$ is a parallelogram. The conditions $BD = CE$ and $\overline{PD} \parallel \overline{AE}$ imply that R lies on $\overline{AE}$. Then since $\overline{AP}$ and $\overline{AR}$ are isogonal, isogonality lemma implies that B, C, $\overline{BP} \cap \overline{AC}$ and $\overline{CP} \cap \overline{AB}$ are concyclic, done.

§7o USA TST 2004/4

Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at $E \neq B$ and $F \neq C$, respectively. Let $X = \overline{DF} \cap \overline{AB}$ and $Y = \overline{DE} \cap \overline{AC}$. Show that $\overline{XY} \parallel \overline{BC}$.

(Available online at https://aops.com/community/p456576.)

The following solution is with Mason Fang. We use barycentrics on $\triangle DBC$, with $a = BC$, $b = DC$, $c = DB$. Let's write the circles as

$$\omega_1 : -a^2yz - b^2zx - c^2xy + (x+y+z)(mz) = 0$$
$$\omega_2 : -a^2yz - b^2zx - c^2xy + (x+y+z)(ny) = 0$$

for constants $m, n \in \mathbb{R}$. Then

$$E = (0 : m : a^2 - m)$$
$$F = (0 : a^2 - n : n).$$

Then A lies on the radical axis $mz - ny = 0$, so we may let

$$A = (u : m : n).$$

Thus, intersecting cevians,

$$X = (u : a^2 - n : n)$$
$$Y = (u : m : a^2 - m).$$

Then XY is the line $\frac{y+z}{x} = \frac{a^2}{u}$ which is parallel to $\overline{BC}$ (it passes through $(0 : 1 : -1)$).

§7p TSTST 2012/2

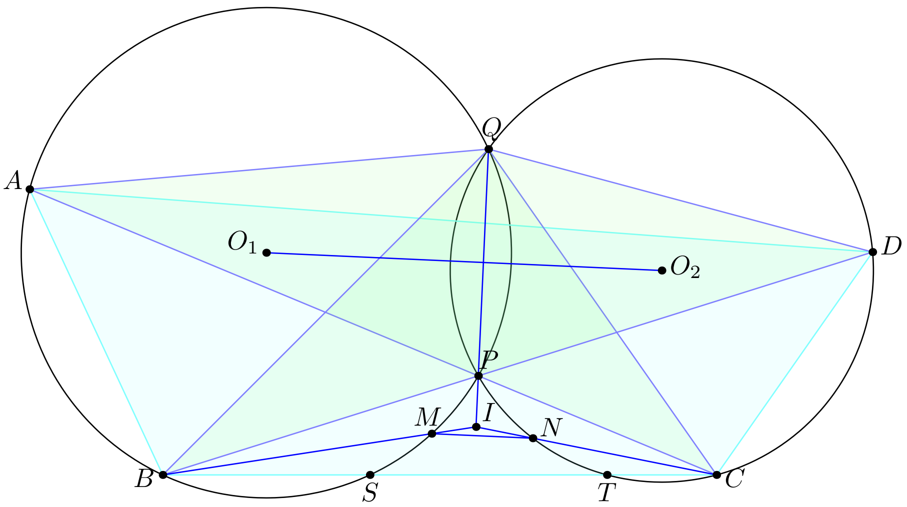
Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

(Available online at https://aops.com/community/p2745851.)

Let Q be the second intersection point of ω_1, ω_2. Suffice to show $\overline{QP} \perp \overline{MN}$. Now Q is the center of a spiral *congruence* which sends $\overline{AC} \mapsto \overline{BD}$. So $\triangle QAB$ and $\triangle QCD$ are similar isosceles. Now,

$$\measuredangle QPA = \measuredangle QBA = \measuredangle DCQ = \measuredangle DPQ$$

and so $\overline{QP}$ is bisects $\angle BPC$.



Now, let $I = \overline{BM} \cap \overline{CN} \cap \overline{PQ}$ be the incenter of $\triangle PBC$. Then $IM \cdot IB = IP \cdot IQ = IN \cdot IC$, so $BMNC$ is cyclic, meaning $\overline{MN}$ is antiparallel to $\overline{BC}$ through $\angle BIC$. Since $\overline{QPI}$ passes through the circumcenter of $\triangle BIC$, it follows now $\overline{QPI} \perp \overline{MN}$ as desired.

§7q IMO 2004/5

In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

(Available online at https://aops.com/community/p99759.)

We show two solutions. We note that the first hypothesis cannot be dropped, because $ABCD$ being a kite with $BA = BC$ and $DA = DC$ with center P is a counterexample (where $AP = CP$ but $ABCD$ is not cyclic). However, the condition that BD bisects neither angle $\angle B$ nor $\angle D$ is equivalent to requiring that the point P does not lie on line BD and we assume this henceforth.

¶ Barycentric solution (from teenager Evan). Apply barycentric coordinates to $\triangle PBD$ (nondegenerate by the non-bisect condition) with $P = (1, 0, 0)$, $B = (0, 1, 0)$ and $D = (0, 0, 1)$. Define $a = BD$, $b = DP$ and $c = PB$.

Since A and C are isogonal conjugates with respect to $\triangle PBD$, we set

$$A = (au : bv : cw) \qquad \text{and} \qquad C = \left(\frac{a}{u} : \frac{b}{v} : \frac{c}{w}\right).$$

For brevity define $M = au + bv + cw$ and $N = avw + bwu + cuv$.

We now compute each condition.

Claim — Quadrilateral $ABCD$ is cyclic if and only if $N^2 = u^2M^2$.

Proof. W know a circle through B and D is a locus of points with

$$\frac{a^2yz + b^2zx + c^2xy}{x(x + y + z)}$$

is equal to some constant. Therefore quadrilateral $ABCD$ is cyclic if and only if $\frac{abc \cdot N}{au \cdot M}$ is equal to $\frac{abc \cdot uvw \cdot M}{avw \cdot N}$ which rearranges to $N^2 = u^2 M^2$. □

Claim — We have $PA = PC$ if and only if $N^2 = u^2 M^2$.

Proof. We have the displacement vector $\overrightarrow{PA} = \frac{1}{M}(bv + cw, -bv, -cw)$. Therefore,

$$\begin{aligned} M^2 \cdot |PA|^2 &= -a^2(bv)(cw) + b^2(cw)(bv + cw) + c^2(bv)(bv + cw) \\ &= bc(-a^2 vw + (bw + cv)(bv + cw)). \end{aligned}$$

In a similar way (by replacing u, v, w with their inverses) we have

$$\begin{aligned} \left(\frac{N}{uvw}\right)^2 \cdot |PC|^2 &= (vw)^{-2} \cdot bc(-a^2 vw + (bv + cw)(bw + cv)) \\ \iff N^2 \cdot |PC|^2 &= u^2 bc(-a^2 vw + (bw + cv)(bv + cw)) \end{aligned}$$

These are equal if and only if $N^2 = u^2 M^2$, as desired. □

¶ Angle chasing solution (Evan Chen and Petko Lazarov). The solution consists of two parts. The first part is that by angle chasing, we will prove that

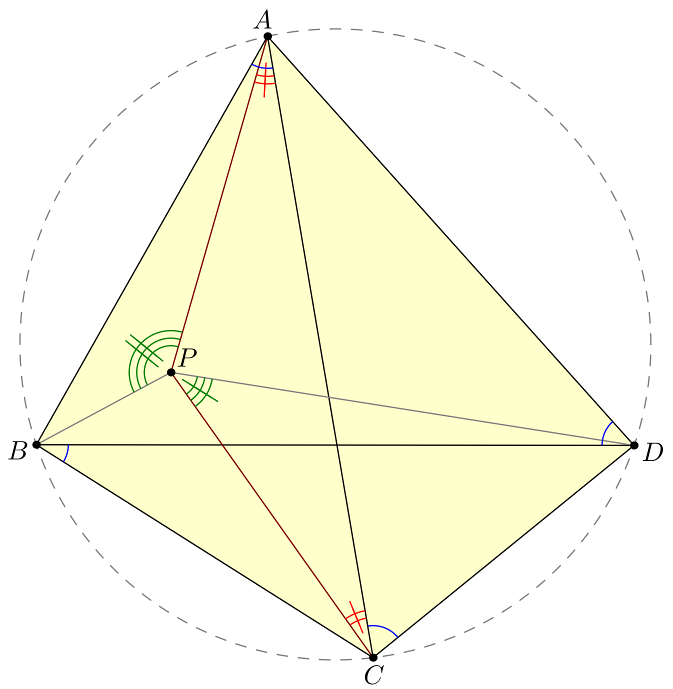
$$\measuredangle PAC = \measuredangle ACP \iff \measuredangle BAC + \measuredangle CBD + \measuredangle DCA + \measuredangle ADB = 0. \qquad (\spadesuit).$$

A careless reader would be forgiven for thinking that ($\spadesuit$) implies the problem or at least one direction, but it turns out the situation is more subtle. The second part analyzes the angle conditions more carefully and provides a complete proof.

Proof of the equivalence ($\spadesuit$) by angle chasing. We start with the following unconditional claim, valid for any quadrilateral.

Claim (Isogonal conjugation) — Let $ABCD$ and P be as in the problem statement. Then $\measuredangle APB + \measuredangle CPD = 180°$.

Proof. The angles in the statement imply that A and C are isogonal conjugates with respect to $\triangle PBC$. Thus, lines PA and PC are isogonal with respect to $\angle BPC$, as needed. □



Next we rewrite the two angles $\measuredangle APB$ and $\measuredangle CPD$ in the claim (colored green with three rings) so that their only dependence on P is through the angles $\measuredangle PAC$ and $\measuredangle CAP$ (colored red with two rings), as follows:

$$\begin{aligned} -\measuredangle APB &= \measuredangle PBA + \measuredangle BAP = \measuredangle PBA + (\measuredangle BAC - \measuredangle PAC) \\ &= \measuredangle CBD + \measuredangle BAC - \measuredangle PAC \\ -\measuredangle CPD &= \measuredangle PDC + \measuredangle DCP = \measuredangle PDC + (\measuredangle DCA + \measuredangle ACP) \\ &= \measuredangle ADB + \measuredangle DCA + \measuredangle ACP. \end{aligned}$$

Since the claim says $\measuredangle APB + \measuredangle CPD = 0$, summing lets us finally rewrite $\measuredangle PAC - \measuredangle APC$ in terms of only A, B, C, D:

$$\begin{aligned} 0 &= (\measuredangle ACP - \measuredangle PAC) + \measuredangle ADB + \measuredangle DCA + \measuredangle CBD + \measuredangle BAC \\ \implies \measuredangle PAC - \measuredangle ACP &= \measuredangle ADB + \measuredangle DCA + \measuredangle CBD + \measuredangle BAC. \end{aligned}$$

These four latter angles are colored blue with one ring in the figure. This proves (♠).

Quasi-harmonic quadrilaterals. To interpret the condition (♠), we define a new term: a quadrilateral $ABCD$ is *quasi-harmonic* if $AB \cdot CD = BC \cdot DA$. (See IMO 2018/6 for another problem involving quasi-harmonic quadrilaterals.) The following two lemmas show why this condition is relevant:

Lemma

The condition
$$\measuredangle BAC + \measuredangle CBD + \measuredangle DCA + \measuredangle ADB = 0$$
is equivalent to $ABCD$ being *either* cyclic or quasi-harmonic or both.

Proof. A clean approach with complex numbers (posted at https://aops.com/community/p15327734) goes as follows: define complex numbers $\mu = (a - b)(c - d) \neq 0$ and

$\nu = (b-c)(d-a) \neq 0$. Then

$$\mu - \nu = (a-c)(b-d) \neq 0.$$

Now the angle condition is equivalent to

$$\begin{aligned} \mathbb{R} &\ni \frac{a-b}{d-b} \cdot \frac{b-c}{a-c} \cdot \frac{c-d}{b-d} \cdot \frac{d-a}{c-a} \\ &= \frac{\mu\nu}{(\mu-\nu)^2} \\ &= \frac{1}{\frac{\mu}{\nu} + \frac{\nu}{\mu} - 2} \\ \iff \mathbb{R} &\ni \frac{\mu}{\nu} + \frac{\nu}{\mu}. \end{aligned}$$

In general, though $z + 1/z$ is real exactly when either $|z| = 1$ or $z \in \mathbb{R}$. Note $\mu/\nu \in \mathbb{R}$ is equivalent to $ABCD$ cyclic and $|\mu| = |\nu|$ is equivalent to $ABCD$ being quasiharmonic; the proof follows.

This can be also proved by inversion at A, though we don't write the details here. (See also https://problems.ru/view_problem_details_new.php?id=116602). □

Lemma

Quadrilateral $ABCD$ is quasi-harmonic if and only if P lies on line AC.

Proof. Let $X = \overline{BD} \cap \overline{AC}$. If the isogonal of line BD with respect to $\angle B$ meets line AC at Y, then $\frac{AX}{CX}\frac{AY}{CY} = \left(\frac{BA}{BC}\right)^2$. Similarly if the isogonal to $\angle D$ meets line AC at Y', then $\frac{AX}{CX}\frac{AY'}{CY'} = \left(\frac{DA}{DC}\right)^2$. Hence $ABCD$ is quasi-harmonic if and only if $Y = Y'$ (that is, $Y = Y' = P$). □

Wrap-up. We now show that $PA = PC$ if and only if $ABCD$ is cyclic by cases on whether P lies on $\overline{AC}$.

- If $ABCD$ is *not* quasi-harmonic, then (♠) implies the problem statement immediately. Indeed, $\measuredangle PAC = \measuredangle ACP$ if and only if $PA = PC$ (as $\triangle PAC$ is not degenerate) and the second lemma turns our angle condition into $ABCD$ cyclic.
- Now assume $ABCD$ is quasi-harmonic and P lies on line AC. We ignore (♠). Instead, note that if $ABCD$ is also cyclic then $\overline{BD}$ is a symmedian of $\triangle ABC$ and hence P is the midpoint. Conversely, suppose we know $\overline{BD}$ is a symmedian of $\triangle ABC$. Let $D' \neq B$ be the point for which $ABCD'$ is cyclic and harmonic; then B, D, D' are collinear and $\frac{BD}{CD} = \frac{BD'}{CD'} = \frac{BA}{CA}$. So $D' = D$ (the corresponding Apollonian circle only meets line BD twice), as needed.

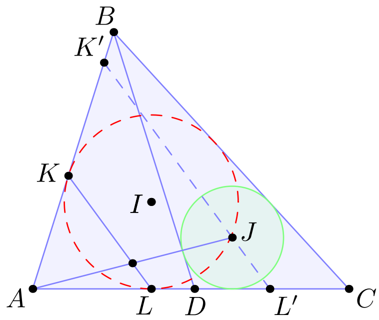
§7r Shortlist 2006 G4

A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

(Available online at https://aops.com/community/p842901.)

Let K' and L' be the reflections of A across K and L.

$$\begin{aligned} K = (s-b : s-a : 0) &\implies K' = (a-b : 2(s-a) : 0) \\ L = (s-c : 0 : s-a) &\implies L' = (a-c : 0 : 2(s-a)). \end{aligned}$$



Now consider the phantom point $J' = (a : b : t-a)$ such that $\overline{CJ'}$ bisects $\angle ACB$ and J' lies on $\overline{K'L'}$. To compute its coordinates, we write

$$0 = \det \begin{bmatrix} a-b & 2(s-a) & 0 \\ a-c & 0 & 2(s-a) \\ a & b & t-a \end{bmatrix} \implies (a-c)(t-a) + b(a-b) = 2a(s-a).$$

So,

$$t = \frac{a(b+c-a) + a(a-c) - b(a-b)}{a-c} = \frac{b^2}{a-c}.$$

In other words $J = \big(a(a-c) : b(a-c) : b^2 - a(a-c)\big)$. So if $E = \overline{BJ} \cap \overline{AC}$ then

$$CE = \frac{a-c}{b^2} \cdot a.$$

Now let F be the foot of $\angle DBC$-bisector on $\overline{BC}$. Since $D = (2S_C - b^2 : 0 : 2S_A)$ (by reflecting the foot of B) the **angle bisector theorem** applied to $BD = c$ and $BC = a$ implies that

$$CF = \frac{CD \cdot a}{a+c} = \frac{\frac{2S_C - b^2}{2S_A + 2S_C - b^2} \cdot a}{a+c} = \frac{a-c}{b^2} \cdot a = CE$$

from which we conclude that $E = F$ as desired.

8 Solutions for Inversion

> Humans are like high templar. They're fragile, weak, and cause storms when they're mad. And they love giving feedback to others despite being unable to receive feedback themselves.

§8a BAMO 2011/4

A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

(Available online at https://aops.com/community/p13035680.)

Inversion at D reduces this to a Ceva picture, which completely destroys the problem.

§8b Shortlist 2003 G4

Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_4, Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

(Available online at https://aops.com/community/p119988.)

Invert arcound P with radius 1.

The conditions in the problem imply that Γ_1^* and Γ_3^* are parallel lines, as are Γ_2^* and Γ_4^*. So $A^*B^*C^*D^*$ is a parallelogram,

$$\begin{aligned} A^*B^* = D^*C^* &\iff \frac{AB}{PA \cdot PB} = \frac{CD}{PC \cdot PD} \\ \text{and } A^*D^* = B^*C^* &\iff \frac{AD}{PA \cdot PD} = \frac{BC}{PB \cdot PD}. \end{aligned}$$

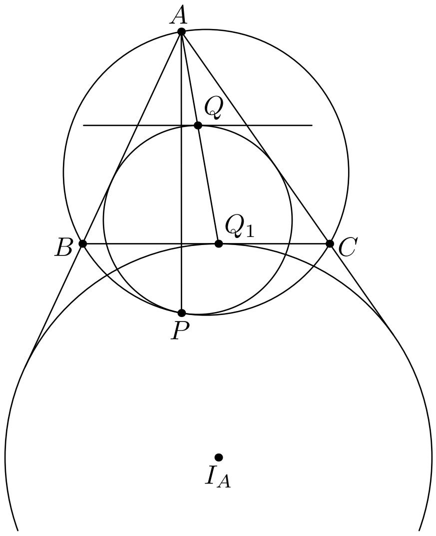
Take the quotient of these two to extract the desired result.

§8c EGMO 2013/5

Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.

(Available online at https://aops.com/community/p3014767.)

First, let us extend $\overline{AQ}$ to meet $\overline{BC}$ at Q_1. By homothety, we see that Q_1 is just the contact point of the A-excircle with $\overline{BC}$.



Now let us perform an inversion around A with radius $\sqrt{AB \cdot AC}$ followed by a reflection around the angle bisector; call this map Ψ. Note that Ψ fixes B and C. Moreover it swaps $\overline{BC}$ and (ABC). Hence, this map swaps the A-excircle with the A-mixtilinear incircle ω. Hence Ψ swaps P and Q_1. It follows that $\overline{AP}$ and $\overline{AQ_1}$ are isogonal with respect to $\angle BAC$, meaning $\angle BAP = \angle CAQ_1$. Since $\angle CAQ = \angle CAQ_1$ we are done.

§8d Russia 2009/10.2

In triangle ABC with circumcircle Ω, the internal angle bisector of $\angle A$ intersects $\overline{BC}$ at D and Ω again at E. The circle with diameter $\overline{DE}$ meets Ω again at F. Prove that $\overline{AF}$ is a symmedian of triangle ABC.

(Available online at https://aops.com/community/p1493622.)

A $\sqrt{bc}$ inversion fixes the circle with diameter $\overline{DE}$. Hence it maps F to the midpoint of $\overline{BC}$. This implies the result.

§8e Shortlist 1997/9

Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let Γ_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices taken mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of Γ_{i+1} and Γ_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear.

(Available online at https://aops.com/community/p1219054.)

It suffices to prove the circles are coaxial. Let DEF be the intouch triangle. Note thatof Γ_1^* is exactly the circle with diameter $\overline{ID}$, etc.

We proceed by inversion around I.

Claim — The triangle $A_1^* A_2^* A_3^*$ is the medial triangle of DEF.

Proof. Circles Γ_2 and Γ_3 are mapped to the circles with diameter $\overline{IE}$ and $\overline{IF}$, hence their second intersection A_1^* is exactly the midpoint of $\overline{EF}$. □

Claim — The triangle $B_1^* B_2^* B_3^*$ is homothetic to triangle DEF.

Proof. This is the triangle determined by the lines Γ_1^*, Γ_2^*, Γ_3^*. Since Γ_1^* is clearly perpendicular to $\overline{A_1 I}$, it is parallel to $\overline{EF}$, and similarly. □

This means $A_1^* B_1^*$, $A_2^* B_2^*$, $A_3^* B_3^*$ are indeed concurrent as needed.

§8f IMO 1993/2

Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio $\frac{AB \cdot CD}{AC \cdot BD}$, and prove that the circumcircles of the triangles ACD and BCD are orthogonal.

(Available online at https://aops.com/community/p99766.)

Answer: $\sqrt{2}$.
The conditions should translate to $\angle D^* B^* C^* = 90^\circ$ and $B^* D^* = B^* C^*$.

§8g IMO 1996/2

Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE concur.

(Available online at https://aops.com/community/p3459.)

Invert around A to eliminate the angle condition. One should find that $\angle C^* B^* P^* = \angle B^* C^* P^*$.
How to handle the incenters? Why does $\angle AD^* B^* = \frac{1}{2} \angle AP^* B^*$?

§8h IMO 2015/3

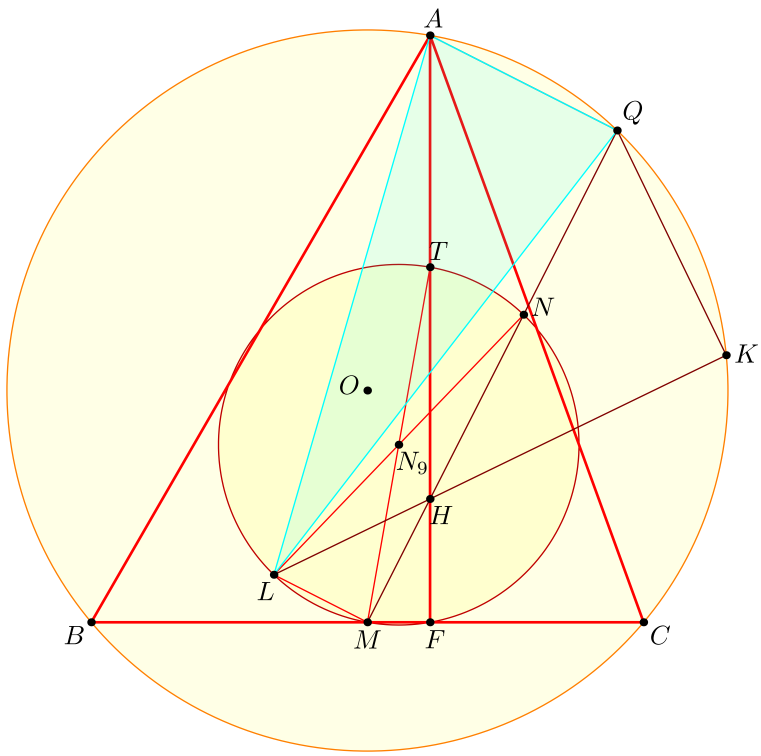
Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

(Available online at https://aops.com/community/p5079655.)

Let L be on the nine-point circle with $\angle HML = 90^\circ$. The negative inversion at H swapping Γ and nine-point circle maps

$$A \longleftrightarrow F, \quad Q \longleftrightarrow M, \quad K \longleftrightarrow L.$$

In the inverted statement, we want line ML to be tangent to (AQL).



Claim — $\overline{LM} \parallel \overline{AQ}$.

Proof. Both are perpendicular to $\overline{MHQ}$. □

Claim — $LA = LQ$.

Proof. Let N and T be the midpoints of $\overline{HQ}$ and $\overline{AH}$, and O the circumcenter. As $\overline{MT}$ is a diameter, we know $LTNM$ is a rectangle, so $\overline{LT}$ passes through O. Since $\overline{LOT} \perp \overline{AQ}$ and $OA = OQ$, the proof is complete. □

Together these two claims solve the problem.

9 Solutions for Projective Geometry

> I don't think Jane Street would appreciate all their thousands of dollars going to fruit snacks.
>
> Debbie Lee, at MOP 2022

§9a TSTST 2012/4

In scalene triangle ABC, let the feet of the perpendiculars from A to $\overline{BC}$, B to $\overline{CA}$, C to $\overline{AB}$ be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Show that the perpendiculars from D to $\overline{AA_2}$, E to $\overline{BB_2}$ and F to $\overline{CC_2}$ are concurrent.

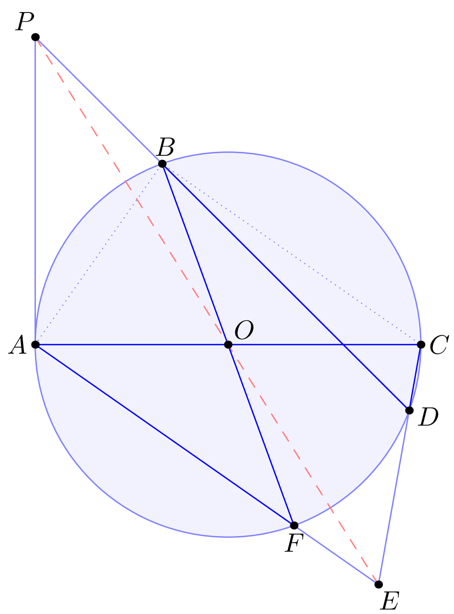
(Available online at https://aops.com/community/p2745854.)

We claim that they pass through the orthocenter H. Indeed, consider the circle with diameter $\overline{BC}$, which circumscribes quadrilateral BCB_1C_1 and has center D. Then by Brocard theorem, $\overline{AA_2}$ is the polar of line H. Thus $\overline{DH} \perp \overline{AA_2}$.

§9b Singapore TST

Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90°$. Let P be any point on the tangent to ω at A other than A, and suppose ray PB intersects ω again at D. Point E lies on line CD such that $\overline{AE} \parallel \overline{BC}$. Prove that P, O, and E are collinear.

Let F be the point diametrically opposite B, and apply Pascal theorem to $AAFBDC$.



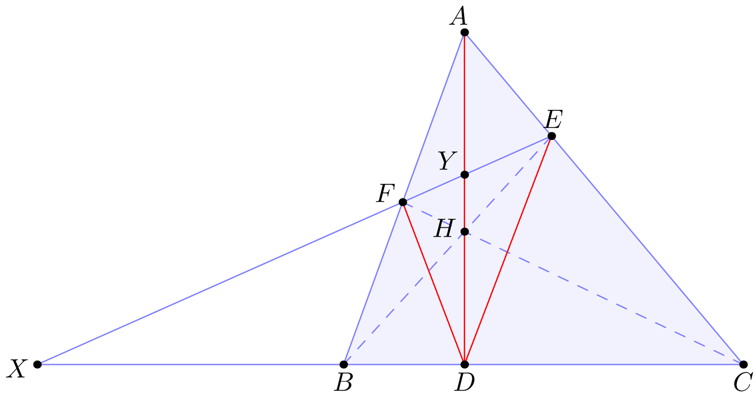
§9c Canada 1994/5

Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively.

Prove that $\angle EDH = \angle FDH$.

(Available online at https://aops.com/community/p2268953.)

Let line EF meet BC again at X. Moreover, let line AH meet line EF at Y.



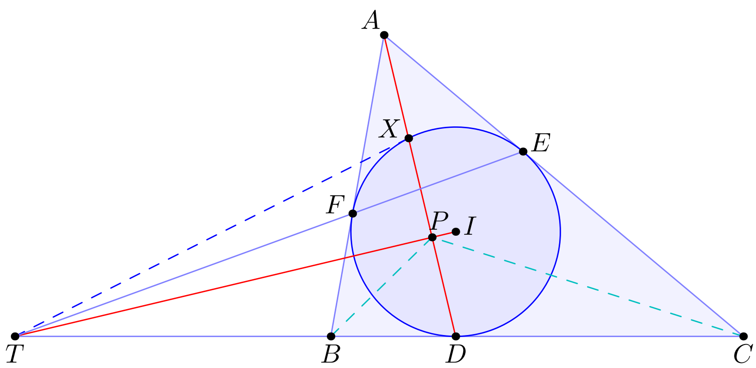
Note derive that $(X,D;B,C) = -1$; perspectivity at A gives $(X,Y;E,F) = -1$. In any case, since we know $\angle XDY = 90^\circ$, the harmonic bundle tells us $\overline{DH}$ bisects $\angle FDE$.

§9d ELMO SL 2012 G3

Let ABC be a triangle with incenter I. The foot of the perpendicular from I to $\overline{BC}$ is D, and the foot of the perpendicular from I to $\overline{AD}$ is P. Prove that $\angle BPD = \angle DPC$.

(Available online at https://aops.com/community/p2728462.)

Let $\triangle DEF$ be the contact triangle, and X be the second intersection of $\overline{AD}$ with the incircle.



Note that $XFED$ is harmonic due to the tangents at A, and thus the tangents to D and X meet on $\overline{EF}$, say at T. In that case $\overline{AXD}$ is the polar of point T, hence $\overline{IT} \perp \overline{AD}$, hence $P = \overline{IT} \cap \overline{AD}$.

Now $(TD;BC) = -1$ since $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ concur at the Gergonne point. Since $\angle TPD = 90^\circ$ this gives the desired angle bisection.

Remark. After showing T lies on line EF, Anka Hu points out that one can avoid appealing to the Gergonne point as follows: one has

$$(TD;BC) \overset{E}{=} (FD;YE) = -1$$

where Y is the second intersection of $\overline{BE}$ with the incircle. (The quadrilateral $YFED$ is harmonic due to the tangents from B.)

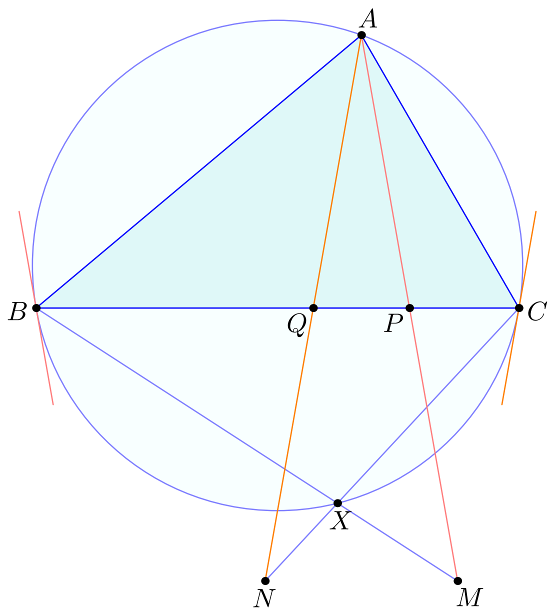
§9e IMO 2014/4

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

(Available online at https://aops.com/community/p3543136.)

We give three solutions.

¶ First solution by harmonic bundles. Let $\overline{BM}$ intersect the circumcircle again at X.



The angle conditions imply that the tangent to (ABC) at B is parallel to $\overline{AP}$. Let ∞ be the point at infinity along line AP. Then

$$-1 = (AM;P\infty) \overset{B}{=} (AX;BC).$$

Similarly, if $\overline{CN}$ meets the circumcircle at Y then $(AY;BC) = -1$ as well. Hence $X = Y$, which implies the problem.

¶ Second solution by similar triangles. Once one observes $\triangle CAQ \sim \triangle CBA$, one can construct D the reflection of B across A, so that $\triangle CAN \sim \triangle CBD$. Similarly, letting E be the reflection of C across A, we get $\triangle BAP \sim \triangle BCA \implies \triangle BAM \sim \triangle BCE$. Now to show $\angle ABM + \angle ACN = 180^\circ$ it suffices to show $\angle EBC + \angle BCD = 180^\circ$, which follows since $BCDE$ is a parallelogram.

¶ Third solution by barycentric coordinates. Since $PB = c^2/a$ we have

$$P = (0 : a^2 - c^2 : c^2)$$

so the reflection $\vec{M} = 2\vec{P} - \vec{A}$ has coordinates

$$M = (-a^2 : 2(a^2 - c^2) : 2c^2).$$

Similarly $N = (-a^2 : 2b^2 : 2(b^2 - a^2))$. Thus

$$\overline{BM} \cap \overline{CN} = (-a^2 : 2b^2 : 2c^2)$$

which clearly lies on the circumcircle, and is in fact the point identified in the first solution.

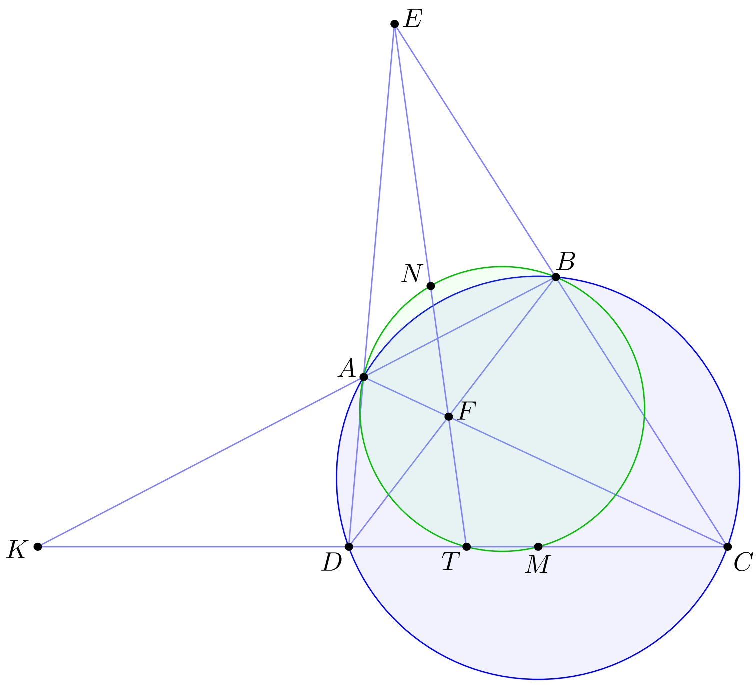
§9f Shortlist 2004 G8

Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = \overline{AD} \cap \overline{BC}$ and $F = \overline{AC} \cap \overline{BD}$.

(Available online at https://aops.com/community/p243438.)

We present two solutions.

¶ First solution by projective geometry. Let $T = \overline{EF} \cap \overline{CD}$, and $K = \overline{AB} \cap \overline{CD}$. Then $KT \cdot KM = KC \cdot KD$ (the latter since $(KM;CD) = -1$), since $ABTM$ is cyclic.



Now that we know $ABTM$ is cyclic, we obtain

$$-1 = (DC;TK) \overset{F}{=} (AB;XK) \overset{T}{=} (AB;NM)$$

where $X = \overline{AB} \cap \overline{FT}$. This completes the proof.

¶ Second solution by complex numbers (Anant Mudgal). By Brocard theorem it's enough to check that N lies on the polar of $K = \overline{AB} \cap \overline{CD}$. We use complex numbers with $ABCD$ the unit circle. First, from the condition, we ought to have

$$-1 = (AB; MN) = \frac{m-a}{m-b} \div \frac{n-a}{n-b}$$

and so solving gives

$$n = \frac{2ab - m(a+b)}{a+b-2m}.$$

To deal with the polar, we use the following lemma (which seems fundamental yet not so well-known).

Lemma

N lies on the polar of K if and only if

$$n\overline{k} + k\overline{n} = 2.$$

Proof. If KX and KY are tangents, we have $\frac{2xy}{x+y} = k$ and $\frac{2}{x+y} = \overline{k}$, and we want $n + xy\overline{n} = x + y$, which rearranges to the lemma. □

To finish, we have $k = \frac{cd(a+b)-ab(c+d)}{cd-ab}$; then a computation shows that

$$n\overline{k} + \overline{k}n = \frac{(a+b)(c+d) - 4ab}{2(cd-ab)} + \frac{4cd - (a+b)(c+d)}{2(cd-ab)} = 2$$

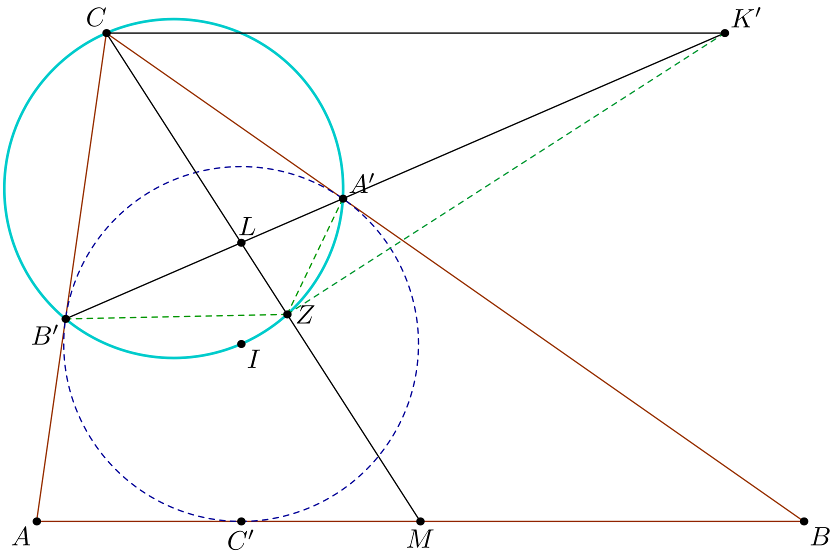
as desired.

> **Remark.** Times change. Rumor has it that in 2005 when this problem was given at MOP, no contestants solved it. (I even heard this was an example of "why you should learn complex numbers".) Even in 2010 ago the use of cross ratios in olympiad geometry was not canon; it was an advanced technique that you only learned your second or third time at MOP. These days, it seems even the middle schoolers know what a harmonic bundle is.

§9g Sharygin 2013/16

The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points A', B' and C' respectively. The perpendicular from the incenter I to the C-median meets the line $A'B'$ in point K. Prove that $\overline{CK} \parallel \overline{AB}$.

Let ω be the circumcircle of $\triangle A'B'C$ and let K' be the intersection of line $A'B'$ with the line through C parallel to AB. Furthermore, let Z be the foot of the perpendicular from I to CM and observe that $Z \in \omega$. It suffices to prove that $\angle K'ZL$ is right, because this will imply $K' = K$.



Let P_∞ be the point at infinity on line AB. Then the quadruple $(A, B; M, P_\infty)$ is clearly harmonic. Taking perspectivity from C onto line $A'B'$ we observe that $(B', A'; L, K')$ is harmonic.

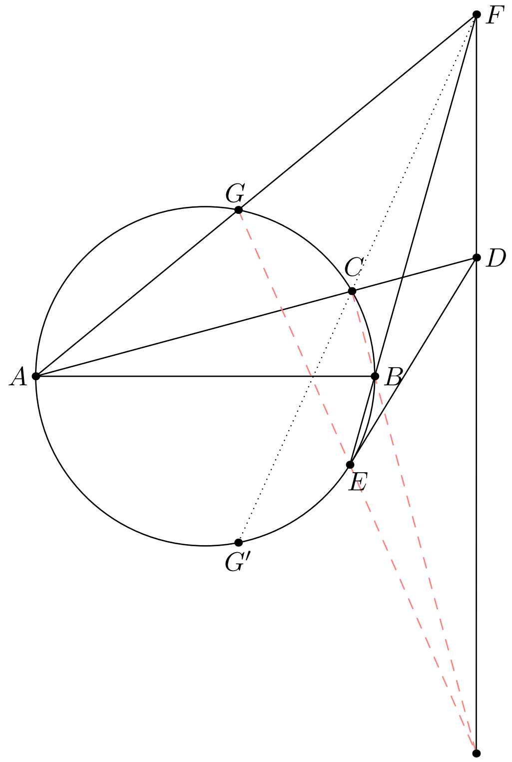
Now consider point Z. Observe that ZL is an angle bisector of $\angle BZA'$, since $B'C = A'C$ implies the arcs $B'C$ and $A'C$ are equal. Since we have a harmonic bundle, we conclude that $LZ \perp K'Z$ as desired.

§9h Shortlist 2004 G2

Circle Γ has diameter $\overline{AB}$, and line d is perpendicular to $\overline{AB}$. Assume d does not intersect Γ and is closer to B than A. Let C be an arbitrary point on Γ, different from the points A and B. Line AC meets d at D. One of the two tangents from the point D to the circle Γ touches Γ at a point E on the same side of $\overline{AC}$ as B. Line BE meets d at F. Line AF meets Γ at a point G different from A. Prove that the reflection of the point G in the line AB lies on the line CF.

(Available online at https://aops.com/community/p152744.)

Pascal's theorem on $AGEEBC$ shows that $\overline{BC} \cap \overline{GE}$ lies on d.



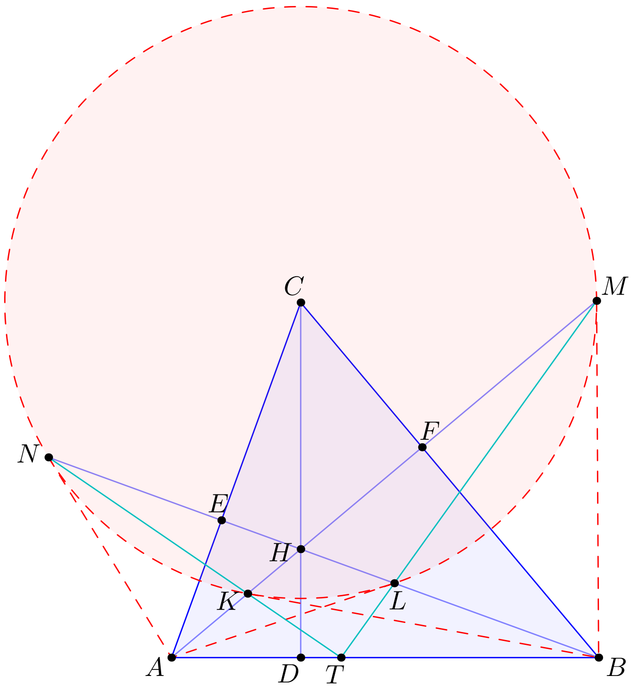
Let G' be the reflection of G over $\overline{AB}$. Then applying Pascal's theorem to $CG'GEBB$ forces $\overline{CG'} \cap \overline{BE}$ to lie on d, so the intersection must be the point F.

§9i January TST 2013/2

Let ABC be an acute triangle. Circle ω_1, with diameter $\overline{AC}$, intersects side $\overline{BC}$ at F (other than C). Circle ω_2, with diameter $\overline{BC}$, intersects side $\overline{AC}$ at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

(Available online at https://aops.com/community/p3161948.)

Let $\overline{CD}$ be the third altitude. Quadrilateral $KLMN$ is cyclic, by power of a point; after all we have $NH \cdot LH = CH \cdot DH = KH \cdot MH$ (since $CNADL$ and $CMBDK$ are cyclic). Denote its circumcircle by γ. Then its center must be C, since it lies on the perpendicular bisectors of $\overline{KM}$, $\overline{LN}$.



Now $\overline{AN}$ and $\overline{AL}$ are tangents to γ, since $\angle ANC = \angle ALC = 90^\circ$. Similarly, so are $\overline{BK}$ and $\overline{BM}$. So by Brocard theorem it follows H is the pole of $\overline{AB}$. Also by Brocard theorem, $\overline{NK} \cap \overline{LM}$ lies on the polar of H, which was what we wanted to prove.

§9j Brazil 2011/5

Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

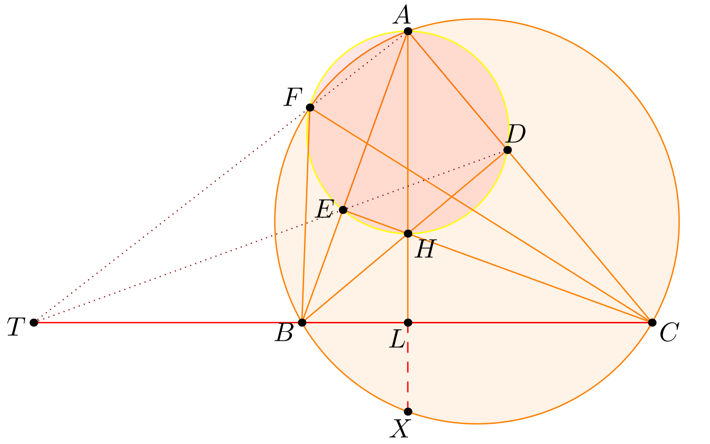
(Available online at https://aops.com/community/p2477427.)

¶ First solution (harmonic). First, notice that lines AF, ED and BC concur at a point T, which is the radical center of the circumcircle, the circle with diameter $\overline{AH}$ (of course H is the orthocenter of ABC), and the circle with diameter $\overline{BC}$.

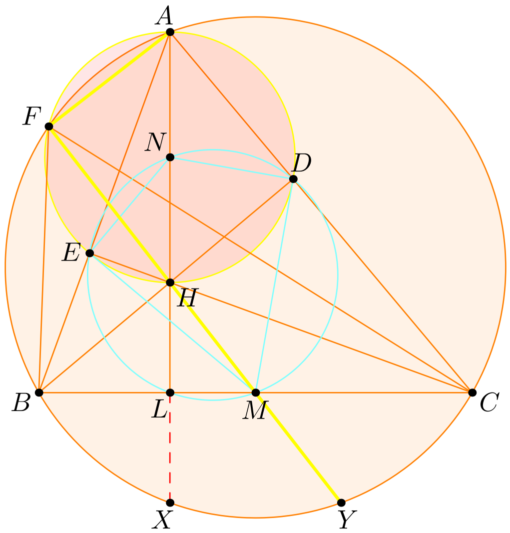
Now let L be the foot of A on $\overline{BC}$ and X the reflection of H over L (which lies on the circumcircle). In light of angle bisector theorem, it suffices to show $BFCX$ is harmonic. But now

$$-1 = (TL; BC) \stackrel{A}{=} (FX; BC)$$

since $\overline{AL}$, $\overline{BD}$, $\overline{CE}$ meet at the orthocenter H. (We are given $F \neq A$, thus $AB \neq AC$, so $\overline{DE} \nparallel \overline{BC}$.)



¶ Second solution (variant by David Hu). As before it suffices to show $FBXC$ is harmonic, where X is the reflection of H. Projecting from A onto (AH), it's equivalent to show $FEHD$ is a harmonic quadrilateral.



Let M be the midpoint of $\overline{BC}$. Then

- It's known that $\overline{ME}$ and $\overline{MD}$ are tangents (for example, by noting that $\overline{NM}$ is a diameter of the nine-point circle for N the midpoint of $\overline{AH}$).
- Moreover, $\overline{MHF}$ are collinear by considering the antipode Y of A on $\overline{MH}$.

These two results together imply $FEHD$ is harmonic.

¶ Third solution (spiral similarity). Note that F is Miquel point of complete quadrilateral $BEDC$. Thus $BF/CF = BE/CD$. The fact $BE/CD = BH/CH$ is obvious.

§9k ELMO SL 2013 G3

In non-right triangle ABC, a point D lies on line $\overline{BC}$. The circumcircle of ABD meets $\overline{AC}$ at F (other than A), and the circumcircle of ADC meets $\overline{AB}$ at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to $\overline{BC}$.

(Available online at https://aops.com/community/p3151962.)

After a $\sqrt{bc}$ inversion around A, it suffices to prove that for variable D^* on (ABC), the line through $E^* = \overline{BD^*} \cap \overline{AC}$ and $F^* = \overline{CD^*} \cap \overline{AB}$ passes through a fixed point on the A-symmedian. By Brocard's theorem this is the pole of $\overline{BC}$.

Alternatively, use barycentric coordinates with $A = (1, 0, 0)$, etc. Let $D = (0 : m : n)$ with $m + n = 1$. Then the circle ABD has equation $-a^2yz - b^2zx - c^2xy + (x + y + z)\left(a^2m \cdot z\right)$. To intersect it with side AC, put $y = 0$ to get $(x + z)(a^2mz) = b^2zx \implies \frac{b^2}{a^2m} \cdot x = x + z \implies \left(\frac{b^2}{a^2m} - 1\right) x = z$, so

$$F = (a^2m : 0 : b^2 - a^2m)$$

Similarly,

$$G = (a^2n : c^2 - a^2n : 0).$$

Then, the circle (AFG) has equation

$$-a^2yz - b^2zx - c^2xy + a^2(x + y + z)(my + nz) = 0.$$

Upon picking $y = z = 1$, we easily see there exists a t such that $(t : 1 : 1)$ is on the circle, implying the conclusion.

One can also use trigonometry directly. Let M be the midpoint of BC. By power of a point, $c \cdot BE + b \cdot CF = a \cdot BD + a \cdot CD = a^2$ is constant. Fix a point D_0; and let $P_0 = AM \cap (AE_0F_0)$. For any other point D, we have $\frac{E_0E}{F_0F} = \frac{b}{c} = \frac{\sin \angle BAM}{\sin \angle CAM} = \frac{P_0E_0}{P_0F_0}$ from the extended law of sines, so triangles P_0E_0E and P_0F_0F are directly similar, whence AEP_0F is cyclic, as desired.

§9I APMO 2008/3

Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides $\overline{BC}$ and $\overline{BA}$ at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines to Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(Available online at https://aops.com/community/p1073985.)

¶ First solution. We will ignore the condition that $ACDE$ is cyclic.

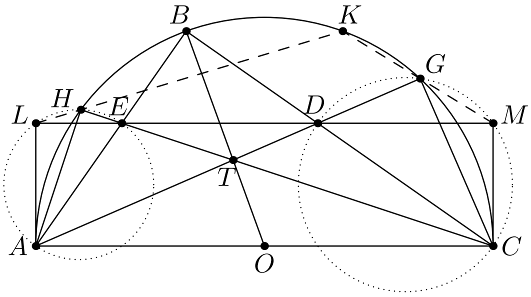
Let $T = \overline{AD} \cap \overline{CE}$ and $O = \overline{BT} \cap \overline{AC}$.

Now we can take a projective transformation that preserves the circumcircle of ABC and sends O to the center of the circle. In that case, $\overline{AC}$ is a diameter, and moreover T lies on the B-median of $\triangle ABC$, meaning that $\overline{DE} \parallel \overline{AC}$.

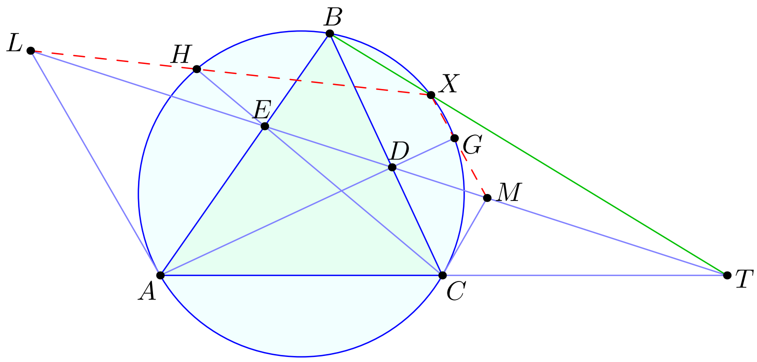
From this we deduce that $ALMC$ is a rectangle. Now we see that $ALHE$ and $DGMC$ are cyclic. From this we can use angle chasing to compute $\measuredangle HKG$ as

$$\begin{aligned} \measuredangle HKG &= \measuredangle LKM = -\measuredangle KML - \measuredangle MLK \\ &= -\measuredangle GMD - \measuredangle ELH \\ &= -\measuredangle GCD - \measuredangle EAH = -\measuredangle GCB - \measuredangle BAH \\ &= -\measuredangle GAB - \measuredangle BAH = -\measuredangle GAH = -\measuredangle GBH \\ &= \measuredangle HBG. \end{aligned}$$

Hence H, B, K, G are concyclic and we are done.



¶ Second solution (Chen Sun). Let lines DE and AC meet at T, and let X be the second intersection of BT with the circumcircle. We claim X is the intersection of lines LH and MG.



Indeed, Pascal's theorem on $XGACCB$ implies that $\overline{XG} \cap \overline{CC}$, $\overline{GA} \cap \overline{CB} = D$, and $\overline{AC} \cap \overline{BX} = T$ are collinear. Since $M = \overline{DT} \cap \overline{CC}$, it follows that M lies on line XG. Similarly, H lies on line XL (by Pascal on $XHCAAB$).

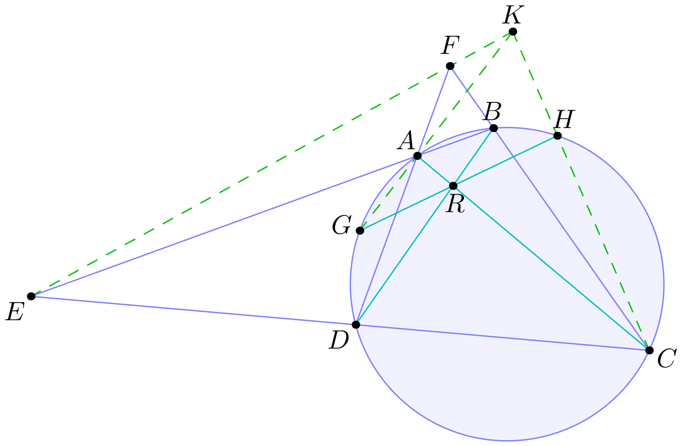
> **Remark.** Colin Tang points out that the condition $AEDC$ cyclic implies that $\overline{ED}$, $\overline{HG}$, $\overline{BB}$ are actually parallel to each other (they're all anti-parallel to $\overline{AC}$). But these three lines are concurrent anyways, by Pascal theorem on $BBAGHC$. So you can think of this as giving a reason to believe the cyclic condition doesn't matter; it's only saying that the concurrency point lies on the infinity line, which isn't special from a projective standpoint.
>
> I have a conjecture that in an problem where up to two conditions are not projective, then those conditions can be deleted.

§9m ELMO SL 2014 G2

Suppose $ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $E = \overline{AB} \cap \overline{CD}$ and $F = \overline{AD} \cap \overline{BC}$. Let ω_1 and ω_2 be the circumcircles of triangles AEF and CEF, respectively. Let G and H be the intersections of ω and ω_1, ω and ω_2, respectively, with $G \neq A$ and $H \neq C$. Show that $\overline{AC}$, $\overline{BD}$, and $\overline{GH}$ are concurrent.

(Available online at https://aops.com/community/p3557483.)

Let K be the radical center of ω, ω_1, ω_2, so that K is the intersection of $\overline{AG}$, $\overline{CH}$, and $\overline{EF}$. Let $R = \overline{AC} \cap \overline{GH}$. The problem is to prove that R lies on $\overline{BD}$. Hence by Brocard's theorem on $ABCD$, it suffices to check that the polar of R is line EF.



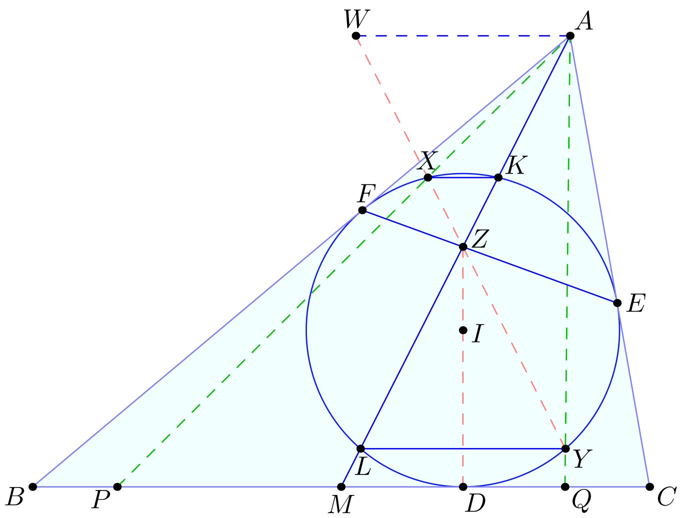
By applying Brocard's theorem on quadrilateral $ACGH$, we find that the polar of R is a line passing through the pole of $\overline{AC}$ and the point $K = \overline{AG} \cap \overline{CH}$. But the pole of $\overline{AC}$ lies on $\overline{EF}$ by Brocard's theorem on $ABCD$. Moreover, so does the point K by construction. Thus the pole of $\overline{AC}$ and the point K both lie on EF. Hence the polar of R really is $\overline{EF}$, and we are done.

§9n Shortlist 2005 G6

Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to $\overline{BC}$, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

(Available online at https://aops.com/community/p463068.)

Recall that $\overline{AKLM}$, $\overline{EF}$, and $\overline{DI}$ are concurrent at a point Z, say. Since $\overline{XY}$ and $\overline{KL}$ are reflections about $\overline{DI}$, it now follows that Z lies on $\overline{XY}$ as well.



From harmonic quadrilaterals, we have $(AZ;KL) = -1$. Let ∞ be the point at infinity along $\overline{BC}$ and set $W = \overline{A\infty} \cap \overline{XY}$. Now

$$-1 = (AZ;KL) \overset{\infty}{=} (WZ;XY) \overset{A}{=} (PQ;M\infty)$$

as desired.

10 Solutions for Complete Quadrilaterals

하늘을 봐 내 맘을 담은 조각을
저 자리에 둘 테니까 날 불러줘 그 언젠가
Look at the sky, I'll leave a piece containing my heart there
So, call me when the time comes

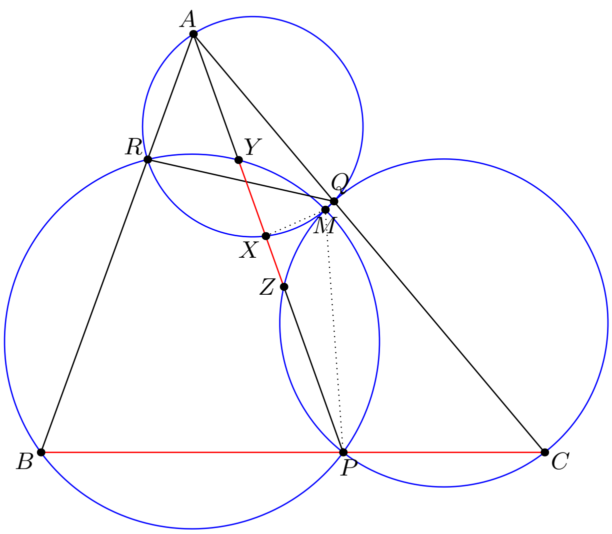
PLEASE PLEASE, by EVERGLOW

§10a USAMO 2013/1

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

(Available online at https://aops.com/community/p3041822.)

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

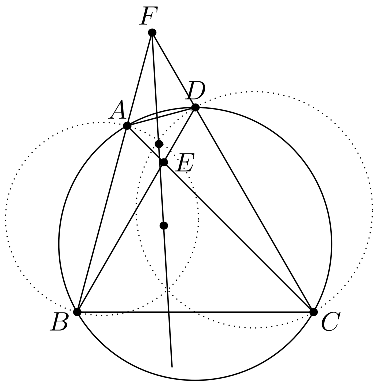
so this follows.

§10b Shortlist 1995 G8

Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{AB} \cap \overline{CD}$. Prove that F lies on the line joining the orthocenters of triangles EAD and EBC.

(Available online at https://aops.com/community/p185022.)

Consider the circle ω_1 with diameter $\overline{AB}$ and the circle ω_2 with diameter $\overline{CD}$. Moreover, let ω be the circumcircle of $ABCD$.



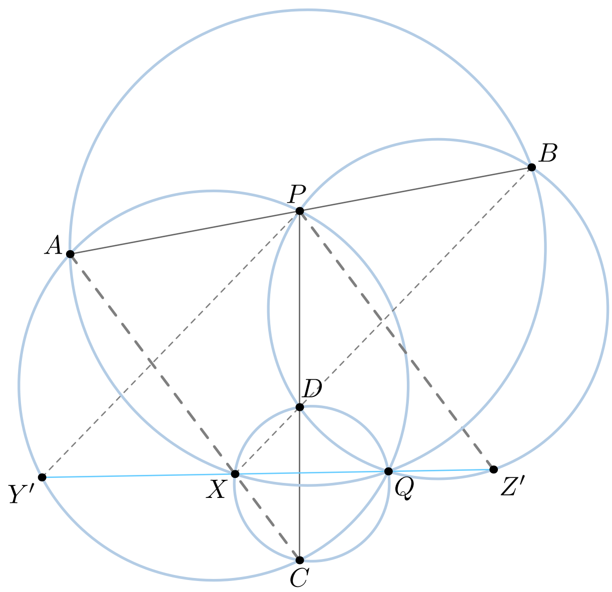
We saw already in the proof of the Gauss line that the two orthocenters lie on the radical axis of ω_1 and ω_2 (i.e., the Steiner line of $ADBC$). Hence the problem is solved if we can prove that F also lies on this radical axis. But this follows from the fact that F is actually the radical center of circles ω_1, ω_2 and ω.

§10c USA TST 2007/1

Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $\overline{PY} \parallel \overline{BD}$. Point Z lies on ω_2 such that $\overline{PZ} \parallel \overline{AC}$. Prove that points Q, X, Y, Z are collinear.

(Available online at https://aops.com/community/p982011.)

Let Y' be the second intersection of ray QX with ω_1. We prove that $\overline{PY'} \parallel \overline{BD}$, which implies that Q, X, Y are collinear. (The point Z is handled similarly.)



The given conditions imply that Q is the Miquel point of complete quadrilateral $DXAP$. Hence quadrilaterals $CQDX$ and $BQXA$ are cyclic. Therefore,

$$\measuredangle QY'P = \measuredangle QCP = \measuredangle QCD = \measuredangle QXD = \measuredangle QXB$$

which implies $\overline{PY'} \parallel \overline{BX}$.

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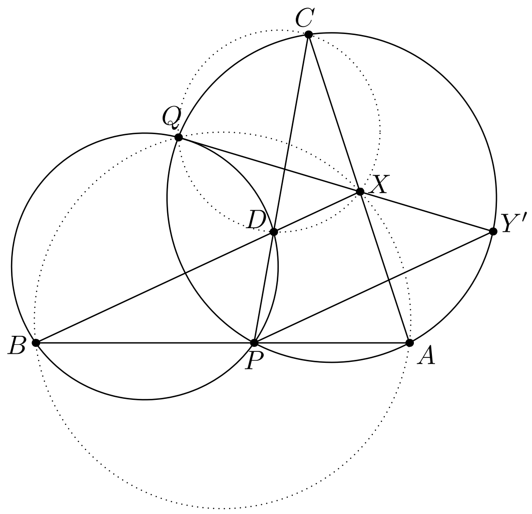


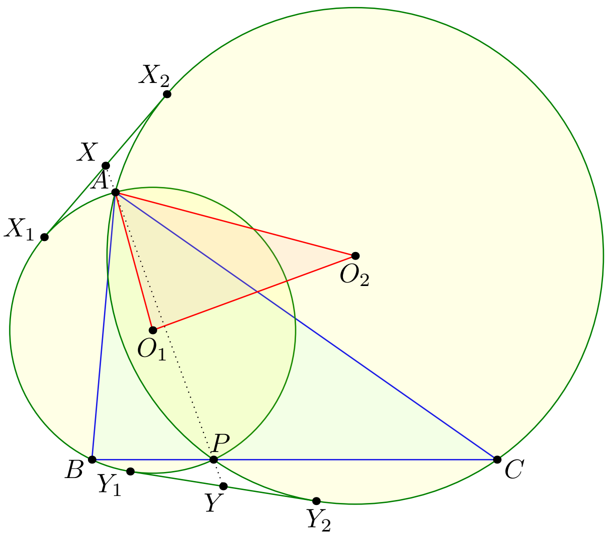
§10d USAMO 2013/6

Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

(Available online at https://aops.com/community/p3043749.)

Let O_1 and O_2 denote the circumcenters of PAB and PAC. The main idea is to notice that $\triangle ABC$ and $\triangle AO_1O_2$ are spirally similar.



Claim (Salmon theorem) — We have $\triangle ABC \overset{+}{\sim} \triangle AO_1O_2$.

Proof. We first claim $\triangle AO_1B \overset{+}{\sim} \triangle AO_2C$. Assume without loss of generality that $\angle APB \le 90°$. Then

$$\angle AO_1B = 2\angle APB$$

but

$$\angle AO_2C = 2\,(180 - \angle APC) = 2\angle ABP.$$

Hence $\angle AO_1B = \angle AO_2C$. Moreover, both triangles are isosceles, establishing the first similarity. The second part follows from spiral similarities coming in pairs. □

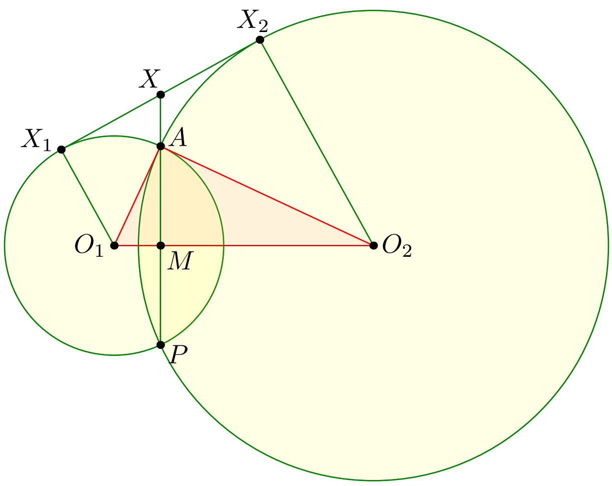
Claim — We always have

$$\left(\frac{PA}{XY}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2.$$

(In particular, this does not depend on P.)

Proof. We now delete the points B and C and remember only the fact that $\triangle AO_1O_2$ has angles A, B, C. The rest is a computation and several approaches are possible.

Without loss of generality A is closer to X than Y, and let the common tangents be $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$. We'll perform the main calculation with the convenient scaling $O_BO_C = a$, $AO_C = b$, and $AO_B = c$. Let B_1 and C_1 be the tangency points of X, and let $h = AM$ be the height of $\triangle AO_BO_C$.



Note that by Power of a Point, we have $XX_1^2 = XX_2^2 = XM^2 - h^2$. Also, by Pythagorean theorem we easily obtain $X_1X_2 = a^2 - (b-c)^2$. So putting these together gives

$$XM^2 - h^2 = \frac{a^2-(b-c)^2}{4} = \frac{(a+b-c)(a-b+c)}{4} = (s-b)(s-c).$$

Therefore, we have

Then

$$\frac{XM^2}{h^2} = 1 + \frac{(s-b)(s-c)}{h^2} = 1 + \frac{a^2(s-b)(s-c)}{a^2h^2}$$

$$= 1 + \frac{a^2(s-b)(s-c)}{4s(s-a)(s-b)(s-c)} = 1 + \frac{a^2}{4s(s-a)}$$
$$= 1 + \frac{a^2}{(b+c)^2 - a^2} = \frac{(b+c)^2}{(b+c)^2 - a^2}.$$

Thus

$$\left(\frac{PA}{XY}\right)^2 = \left(\frac{h}{XM}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2. \qquad \square$$

To finish, note that when P is the foot of the $\angle A$-bisector, we necessarily have

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{\left(\frac{b}{b+c}a\right)\left(\frac{c}{b+c}a\right)}{bc} = \left(\frac{a}{b+c}\right)^2.$$

Since there are clearly at most two solutions as $\frac{PA}{XY}$ is fixed, these are the only two solutions.

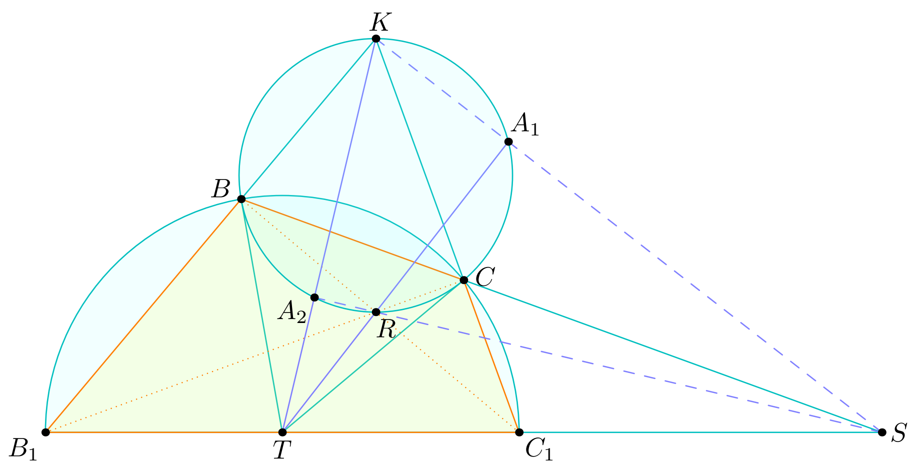
§10e USA TST 2007/5

Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $\overline{AS} \perp \overline{AT}$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar.

(Available online at https://aops.com/community/p982020.)

We ignore for now the point A, and think about the problem in terms of B_1BCC_1.

Let $K = \overline{BB_1} \cap \overline{CC_1}$ and $R = \overline{B_1C} \cap \overline{C_1B}$. Hence R is the orthocenter of $\triangle KB_1C_1$ and C, B are the feet of the altitudes, while T is the midpoint of $\overline{B_1C_1}$. It is known that $\overline{TB}$ and $\overline{TC}$ are tangent to $(KBCR)$, whence this circle actually coincides with ω.



Now, we know that point A satisfies the following two conditions:

- Point A lies on ω.
- We have $\angle TAS = 90°$.

There are two points A with this condition, since the locus is the intersection of two circles.

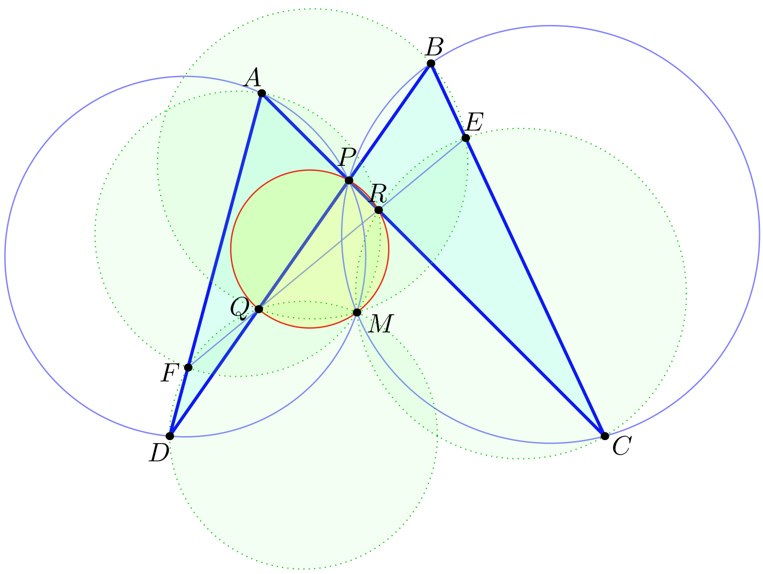
One of these points is the Miquel point of (convex) quadrilateral B_1BCC_1, and we denote it by A_1. It is the inverse of the intersection of the diagonals R. The other is the Miquel point of quadrilateral B_1CBC_1 (which is self-intersecting), which we denote by A_2; indeed that point also lies on ω, and satisfies $\measuredangle TA_2R = \measuredangle TA_2S = 90°$. In the first case we get that $\triangle ABC \sim \triangle AB_1C_1$ directly and in the other case we get $\triangle ABC \sim \triangle AC_1B_1$ instead.

§10f IMO 2005/5

Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $\overline{BC} \nparallel \overline{DA}$. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

(Available online at https://aops.com/community/p282140.)

Let M be the Miquel point of complete quadrilateral $ADBC$; in other words, let M be the second intersection point of the circumcircles of $\triangle APD$ and $\triangle BPC$. (A good diagram should betray this secret; all the points are given in the picture.) This makes lots of sense since we know E and F will be sent to each other under the spiral similarity too.



Thus M is the Miquel point of complete quadrilateral $FACE$. As $R = \overline{FE} \cap \overline{AC}$ we deduce $FARM$ is a cyclic quadrilateral (among many others, but we'll only need one).

Now look at complete quadrilateral $AFQP$. Since M lies on (DFQ) and (RAF), it follows that M is in fact the Miquel point of $AFQP$ as well. So M lies on (PQR).

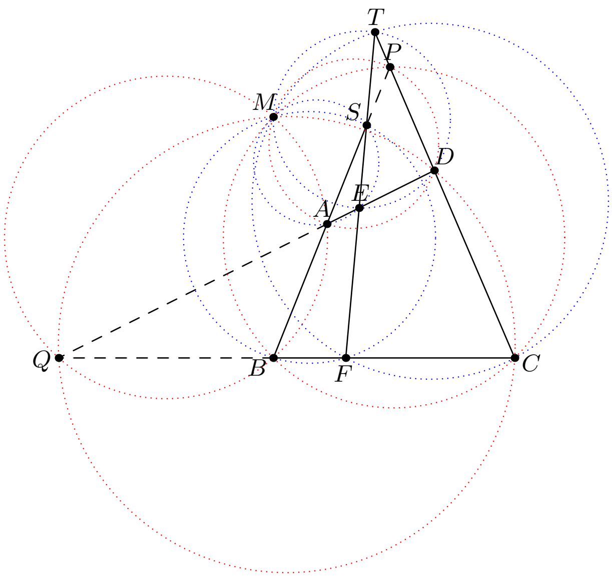
Thus M is the fixed point that we wanted.

> **Remark.** Naturally, the congruent length condition can be relaxed to $DF/DA = BE/BC$.

§10g USAMO 2006/6

Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

(Available online at https://aops.com/community/p490691.)



Let M be the Miquel point of $ABCD$. Then M is the center of a spiral similarity taking AD to BC. The condition guarantees that it also takes E to F. Hence, we see that M is the center of a spiral similarity taking $\overline{AB}$ to $\overline{EF}$, and consequently the circumcircles of QAB, QEF, SAE, SBF concur at point M.

In other words, the Miquel point of $ABCD$ is also the Miquel point of $ABFE$. Similarly, M is also the Miquel point of $EDCF$, so all four circles concur at M.

§10h Balkan 2009/2

Let $\overline{MN}$ be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines $\overline{BN}$ and $\overline{CM}$ meet at point P. The circumcircles of triangles BMP and CNP intersect at a point $Q \neq P$. Prove that $\angle BAQ = \angle CAP$.

(Available online at https://aops.com/community/p1484879.)

By Ceva, $\overline{AP}$ is a median, so we wish to show $\overline{AQ}$ is a symmedian. But Q is the center of the spiral similarity

$$\triangle QBM \sim \triangle QNC$$

so the ratio of distance from Q to sides $\overline{BM}$ and $\overline{CN}$ is equal to $BM : NC = AB : AC$, hence the result.

§10i TSTST 2012/7

Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

(Available online at https://aops.com/community/p2745857.)

By angle chasing, equivalent to show $\overline{MN} \parallel \overline{AD}$, so discard the point H. We now present a three solutions.

¶ First solution using vectors. We first contend that:

Claim — We have $QB = PC$.

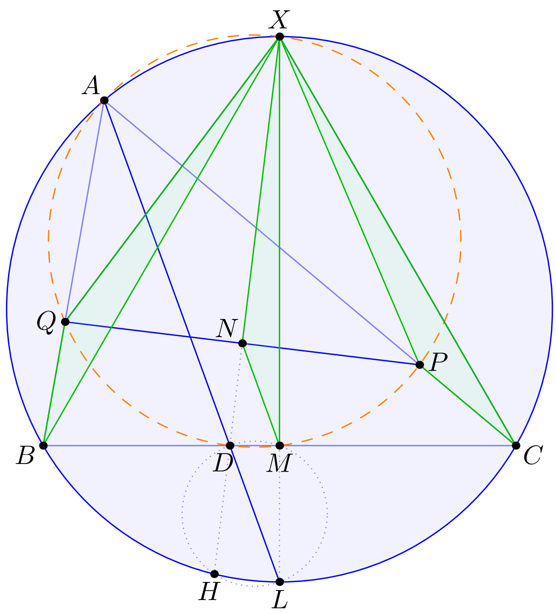
Proof. Power of a Point gives $BM \cdot BD = AB \cdot QB$. Then use the angle bisector theorem. □

Now notice that the vector

$$\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{BQ} + \overrightarrow{CP}\right)$$

which must be parallel to the angle bisector since $\overrightarrow{BQ}$ and $\overrightarrow{CP}$ have the same magnitude.

¶ Second solution using spiral similarity. let X be the arc midpoint of BAC. Then $ADMX$ is cyclic with diameter $\overline{XD}$, and hence X is the Miquel point of $QBPC$ and thus the midpoint of arc BAC. Moreover $\overline{XND}$ collinear (as $XP = XQ$, $DP = DQ$) on (APQ).



Then $\triangle XNM \sim \triangle XPC$ spirally, and

$$\measuredangle XMN = \measuredangle XCP = \measuredangle XCA = \measuredangle XLA$$

thus done.

¶ Third solution using barycentrics (mine). Once reduced to $\overline{MN} \parallel \overline{AB}$, straight bary will also work. By power of a point one obtains

$$\begin{aligned} P &= \left(a^2 : 0 : 2b(b+c) - a^2\right) \\ Q &= \left(a^2 : 2c(b+c) - a^2 : 0\right) \\ \implies N &= \left(a^2(b+c) : 2bc(b+c) - ba^2 : 2bc(b+c) - ca^2\right). \end{aligned}$$

Now the point at infinity along $\overline{AD}$ is $(-(b+c) : b : c)$ and so we need only verify

$$\det \begin{bmatrix} a^2(b+c) & 2bc(b+c) - ba^2 & 2bc(b+c) - ca^2 \\ 0 & 1 & 1 \\ -(b+c) & b & c \end{bmatrix} = 0$$

which follows since the first row is $-a^2$ times the third row plus $2bc(b+c)$ times the second row.

§10j TSTST 2012/2

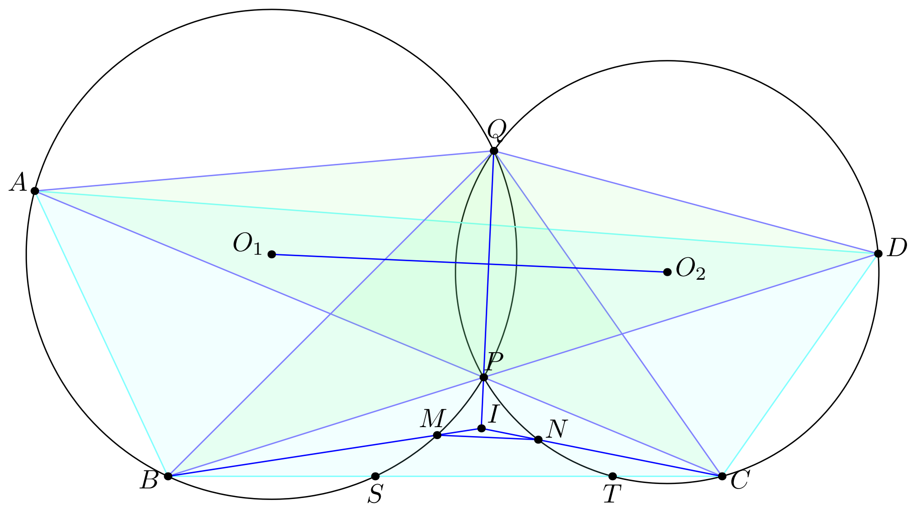
Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $\overline{MN} \parallel \overline{O_1O_2}$.

(Available online at https://aops.com/community/p2745851.)

Let Q be the second intersection point of ω_1, ω_2. Suffice to show $\overline{QP} \perp \overline{MN}$. Now Q is the center of a spiral *congruence* which sends $\overline{AC} \mapsto \overline{BD}$. So $\triangle QAB$ and $\triangle QCD$ are similar isosceles. Now,

$$\measuredangle QPA = \measuredangle QBA = \measuredangle DCQ = \measuredangle DPQ$$

and so $\overline{QP}$ is bisects $\angle BPC$.



Now, let $I = \overline{BM} \cap \overline{CN} \cap \overline{PQ}$ be the incenter of $\triangle PBC$. Then $IM \cdot IB = IP \cdot IQ = IN \cdot IC$, so $BMNC$ is cyclic, meaning $\overline{MN}$ is antiparallel to $\overline{BC}$ through $\angle BIC$. Since $\overline{QPI}$ passes through the circumcenter of $\triangle BIC$, it follows now $\overline{QPI} \perp \overline{MN}$ as desired.

§10k USA TST 2009/2

Let ABC be an acute triangle. Point D lies on side BC. Let O_B, O_C be the circumcenters of triangles ABD and ACD, respectively. Suppose that the points B, C, O_B, O_C lie on a circle centered at X. Let H be the orthocenter of triangle ABC. Prove that $\angle DAX = \angle DAH$.

(Available online at https://aops.com/community/p1566047.)

Without loss of generality $AC > AB$. It is easy to verify via angle chasing that $\angle AO_BB = \angle AO_CC$. Since O_BO_CCB is cyclic, it follows that A is the Miquel point of O_BO_CCB. Therefore, AO_CXB is cyclic.

Set $x = \angle BAD$, $y = \angle CAD$. Then

$$\begin{aligned} \angle BO_BO_C &= \angle BO_BD + \angle DO_BC = 2x + B \\ \implies \angle BXC &= 360 - 4x - 2B \\ \implies \angle BAX &= \angle BO_CX = 2x + B - 90. \end{aligned}$$

On the other hand, $\angle BAH = 90 - B$. From here it is easy to derive that $\angle HAD = x + B - 90 = \angle XAD$, as desired.

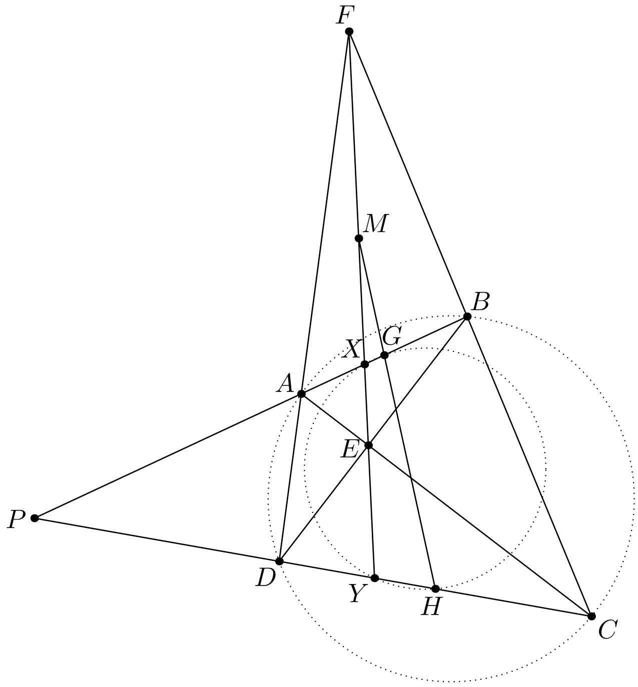
§10l Shortlist 2009 G4

Given a cyclic quadrilateral $ABCD$, let $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AD} \cap \overline{BC}$. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G, and H.

(Available online at https://aops.com/community/p1932936.)

We present two approaches.

¶ First solution with harmonic bundles. Let M be the midpoint of $\overline{EF}$. Then M, G, H lie on the Gauss line of complete quadrilateral $ADBC$. Let $P = \overline{AB} \cap \overline{CD}$ and let line EF meet $\overline{AB}$ and $\overline{CD}$ at X and Y, respectively.



Note that we have harmonic bundles

$$(XY;EF) = (PX;AB) = (PY;DC) = -1.$$

We thus obtain $XYGH$ cyclic from

$$PX \cdot PG = PA \cdot PB = PD \cdot PC = PY \cdot PH.$$

Now, from $(ME;XY) = -1$ we have

$$ME^2 = MX \cdot MY = MG \cdot MH$$

which gives the desired conclusion.

¶ Second solution using complex numbers (Sanjana Das). As before let $P = \overline{AB} \cap \overline{CD}$. We are supposed to verify that

$$\frac{e-f}{e-g} \div \frac{e-h}{g-h} \in \mathbb{R}$$

to get the desired equality of directed angles. To avoid involving the point E at all, we use the following two ideas:

- By Brocard's theorem, the direction of $e-f$ is perpendicular to $p = \frac{ab(c+d)-cd(a+b)}{ab-cd}$.
- Since $\triangle EBA \overset{-}{\sim} \triangle ECD$ we also have $\triangle EBG \overset{-}{\sim} \triangle ECH$. Consequently, the complex number $(e-g)(e-h)$ has the same direction as $(e-b)(e-c)$, and hence the same direction as $(d-b)(a-c)$.

On the other hand, $g - h = \frac{a+b-c-d}{2}$. So putting this all together, we need to verify

$$\frac{i \cdot \frac{ab(c+d)-cd(a+b)}{ab-cd} \cdot \frac{a+b-c-d}{2}}{(d-b)(a-c)} \in \mathbb{R}$$

which is immediate.

§10m Shortlist 2006 G9

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

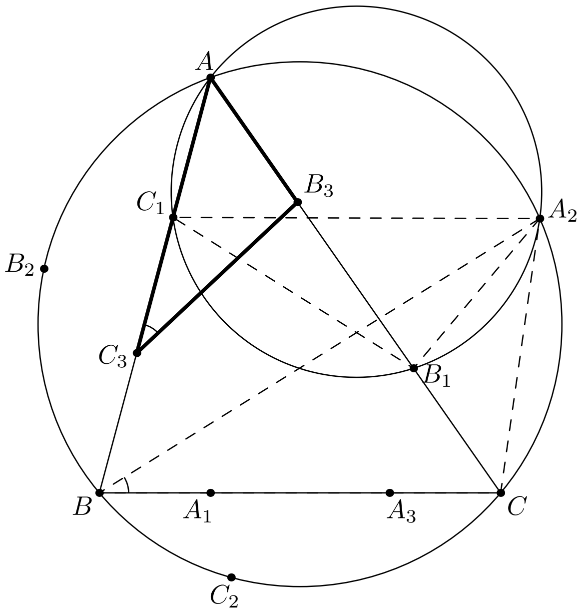
(Available online at https://aops.com/community/p875036.)

We will prove the following claim, after which only angle chasing remains.

Claim — We have $\measuredangle AC_3B_3 = \measuredangle A_2BC$.

Proof. By spiral similarity at A_2, we deduce that $\triangle A_2C_1B \sim \triangle A_2B_1C$, hence

$$\frac{A_2B}{A_2C} = \frac{C_1B}{B_1C} = \frac{AC_3}{AB_3}.$$



It follows that

$$\triangle A_2BC \sim \triangle AC_3B_3$$

since we also have $\measuredangle BA_2C = \measuredangle BAC = \measuredangle C_3AB_3$. (Configuration issues: we can check that A_2 lies on the same side of A as $\overline{BC}$ since B_1 and C_1 are constrained to lie on the sides of the triangle. So we can deduce $\angle C_3AB_3 = \angle BA_2C$.)

Thus $\measuredangle AC_3B_3 = \measuredangle A_2BC$, completing the proof. $\square$

Similarly, $\measuredangle BC_3A_3 = \measuredangle B_2AC$

The rest is angle chasing; we have

$$\begin{aligned}
\measuredangle A_3C_3B_3 &= \measuredangle A_3C_3A + \measuredangle AC_3B_3 \\
&= \measuredangle A_3C_3B + \measuredangle AC_3B_3 \\
&= \measuredangle CAB_2 + \measuredangle A_2BC \\
&= \measuredangle A_2C_2C + \measuredangle CC_2B_2 \\
&= \measuredangle A_2C_2B_2.
\end{aligned}$$

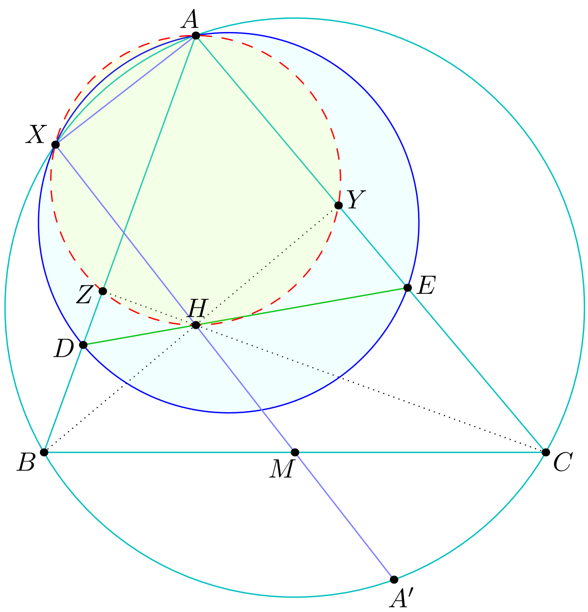
§10n Shortlist 2005 G5

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the radical axis of the circumcircles of $\triangle ADE$ and $\triangle ABC$.

(Available online at https://aops.com/community/p519896.)

Let X be the second intersection of the circumcircles of ADE and ABC (in other words, the Miquel point of complete quadrilateral $DECB$). We will in fact prove that $\angle MXA = 90^\circ$. This will establish the problem.

(Note that one could have "guessed" this was the case by reflecting H over M to A', and then realizing that the foot of the altitude from A to $\overline{HM}$ must in fact lie on the circumcircle of ABC.)



Let Y and Z be the feet of the altitudes from B and C to $\overline{AC}$ and $\overline{AB}$. It suffices to prove that X lies on the circle with diameter $\overline{AH}$. Since X is already the center of a spiral similarity mapping $\overline{BD}$ to $\overline{CE}$, we just need it to also map Z to Y. In other words, we want

$$\frac{BD}{ZD} = \frac{CE}{YE}.$$

This can be done easily enough with explicit calculation. However, here is a more elegant solution. Notice that

$$\angle ZHB = 90^\circ - \angle ZBH = \angle A.$$

On the other hand,

$$\angle DHZ = 90^\circ - \angle ADE = 90^\circ - \left(90^\circ - \frac{1}{2}\angle A\right) = \frac{1}{2}\angle A.$$

Therefore, $\overline{HD}$ bisects $\angle ZHB$. Similarly, $\overline{EH}$ bisects $\angle YHC$. Finally, $ZH \cdot HC = YH \cdot HB$ since the points Z, Y, B, C are concyclic. Tying these all together, we have

$$\frac{BD}{ZD} = \frac{ZH}{BH} = \frac{YH}{CH} = \frac{CE}{YE}$$

as required.

> **Remark.** One can phrase this solution using the forgotten coaxiality lemma, see `https://aops.com/community/p27873074`.

11 Solutions for Personal Favorites

How do you *accidentally* rob a bank??

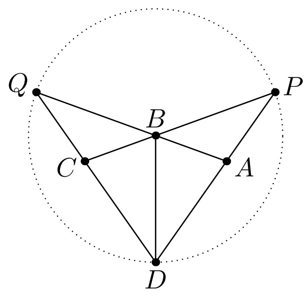
RWBY Chibi, Season 3, Episode 1

§11a Canada 2000/4

Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$. Prove that $AD = CD$.

(Available online at https://aops.com/community/p445434.)

Let $P = \overline{AD} \cap \overline{BC}$, $Q = \overline{AB} \cap \overline{CD}$. Now $2\angle ADB = \angle CBD = \angle BPD + \angle PDB$, meaning $\angle BPD = \angle BDP$ and $BP = BD$. Similarly, $BQ = BD$.



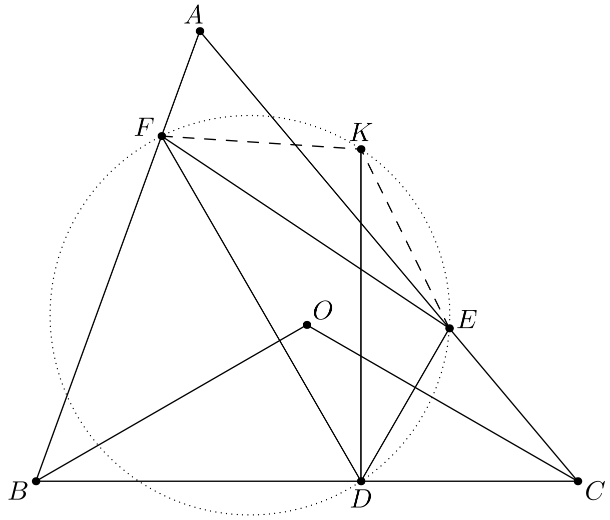
Now $BP = BQ$ and $BC = BA$ give $\triangle QBC \cong \triangle PBA$; from here the solution follows readily.

§11b EGMO 2012/1

Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that $\overline{DE} \perp \overline{CO}$ and $\overline{DF} \perp \overline{BO}$. Let K be the circumcenter of triangle AFE. Prove that the lines $\overline{DK}$ and $\overline{BC}$ are perpendicular.

(Available online at https://aops.com/community/p2658992.)

First, note $\measuredangle EDF = 180^\circ - \measuredangle BOC = 180^\circ - 2A$, so $\measuredangle FDE = 2A$.



Observe that $\measuredangle FKE = 2A$ as well; hence $KFDE$ is cyclic. Hence

$$\begin{aligned} \measuredangle KDB &= \measuredangle KDF + \measuredangle FDB \\ &= \measuredangle KEF + (90^\circ - \measuredangle DBO) \\ &= (90^\circ - A) + (90^\circ - (90^\circ - A)) \\ &= 90^\circ. \end{aligned}$$

and the proof ends here.

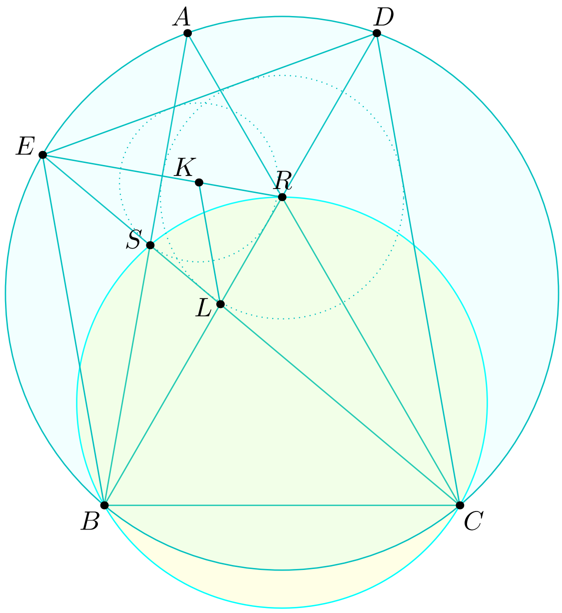
§11c ELMO 2013/4

Triangle ABC is inscribed in circle ω. A circle through BC intersects segments AB and AC at S and R, respectively. Lines BR and CS meet at L, and intersect ω at D and E, respectively. The angle bisector of $\angle BDE$ meets ER at K.

Prove that if $BE = BR$, then $\angle ELK = \frac{1}{2}\angle BCD$.

(Available online at https://aops.com/community/p3104305.)

First, we claim that $BE = BR = BC$. Indeed, construct a circle with radius $BE = BR$ centered at B, and notice that $\angle ECR = \frac{1}{2}\angle EBR$, implying that it lies on the circle.



Now, CA bisects $\angle ECD$ and DB bisects $\angle EDC$, so R is the incenter of $\triangle CDE$. Then, K is the incenter of $\triangle LED$, so

$$\angle ELK = \frac{1}{2}\angle ELD = \frac{1}{2}\left(\frac{\widehat{ED}+\widehat{BC}}{2}\right) = \frac{1}{2}\frac{\widehat{BED}}{2} = \frac{1}{2}\angle BCD.$$

¶ Authorship comments. This problem was actually written backwards; the idea is a phantom circle with center B and radius BE. This causes a certain isosceles triangle to appear, and I wanted to see what I could do with it.

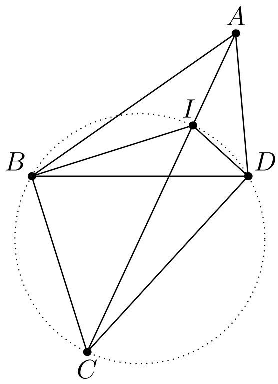
After some messing around I eventually found that making the cyclic quadrilateral through BC created the right setup for the angles I wanted. (Originally the problem was phrased in terms of the cyclic quadrilateral $BCSR$, which was then named $ABCD$.) I started drawing lines to see where I could take the hidden isosceles triangle. Four hours later, I got something sort of contrived which I showed Aaron Lin.

He liked it, but then pointed out that R was the incenter of $\triangle DEC$, something I hadn't noticed earlier. So I decided to make another incenter K and put in a random angle condition. I was somewhat satisfied with the result.

§11d USAMTS 3/3/24

In quadrilateral $ABCD$, $\angle DAB = \angle ABC = 110°$, $\angle BCD = 35°$, $\angle CDA = 105°$, and $\overline{AC}$ bisects $\angle DAB$. Find $\angle ABD$.

The following diagram is not drawn to scale.



Let I denote the incenter of $\triangle ABD$. Then quadrilateral $IBCD$ is cyclic since $\angle DIB = 90^\circ + \frac{1}{2}\angle DAB = 145^\circ$. Hence we obtain $\angle IBD = \angle ICD = 180^\circ - (55^\circ + 105^\circ) = 20^\circ$ and so $\angle ABD = 40^\circ$.

§11e Sharygin 2013/21

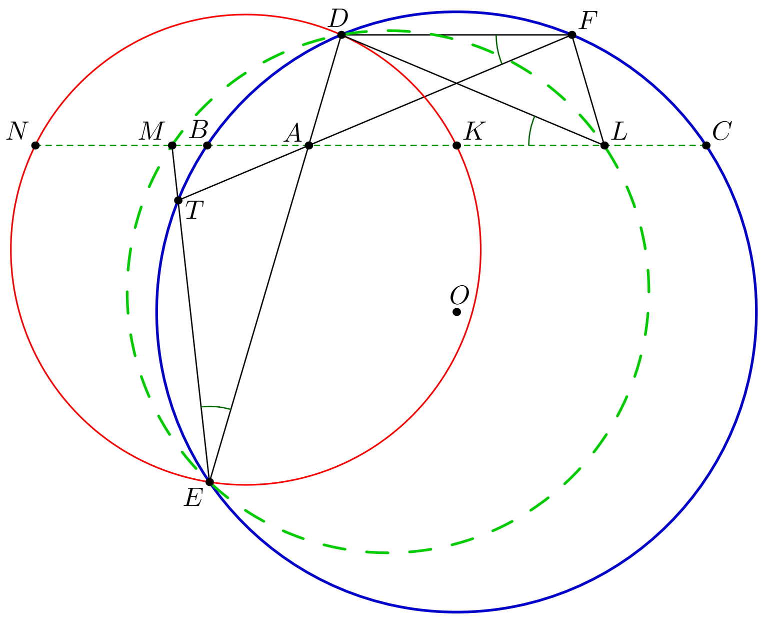
Chords $\overline{BC}$ and $\overline{DE}$ of circle ω meet at point A. The line through D parallel to BC meets ω again at F, and FA meets ω again at T. Let $M = \overline{ET} \cap \overline{BC}$ and let N be the reflection of A over M. Show that (DEN) passes through the midpoint of BC.

(Available online at https://aops.com/community/p3008129.)

Let K be the midpoint of BC, and let L be the reflection of A over K. Note that F is the reflection of D over OK, so we find that $DFLA$ is an isosceles trapezoid. Then,

$$\angle MED = \angle TED = \angle TFD = \angle AFD = \angle ALD = \angle MLD.$$

Therefore, $MELD$ is cyclic.



Now, by Power of a Point, we see that

$$AD \cdot AE = AM \cdot AL$$

$$= AM \cdot 2AK$$
$$= 2AM \cdot AK$$
$$= NA \cdot AK$$

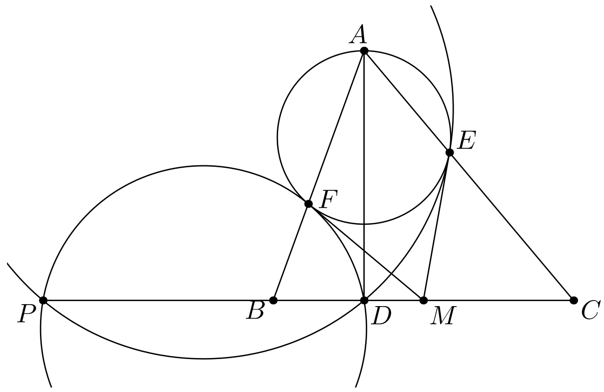
Therefore, $DKEN$ is cyclic, as desired.

§11f ELMO 2012/1

In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on line BC other than D.

(Available online at https://aops.com/community/p2728459.)

Let M denote the midpoint of $\overline{BC}$.



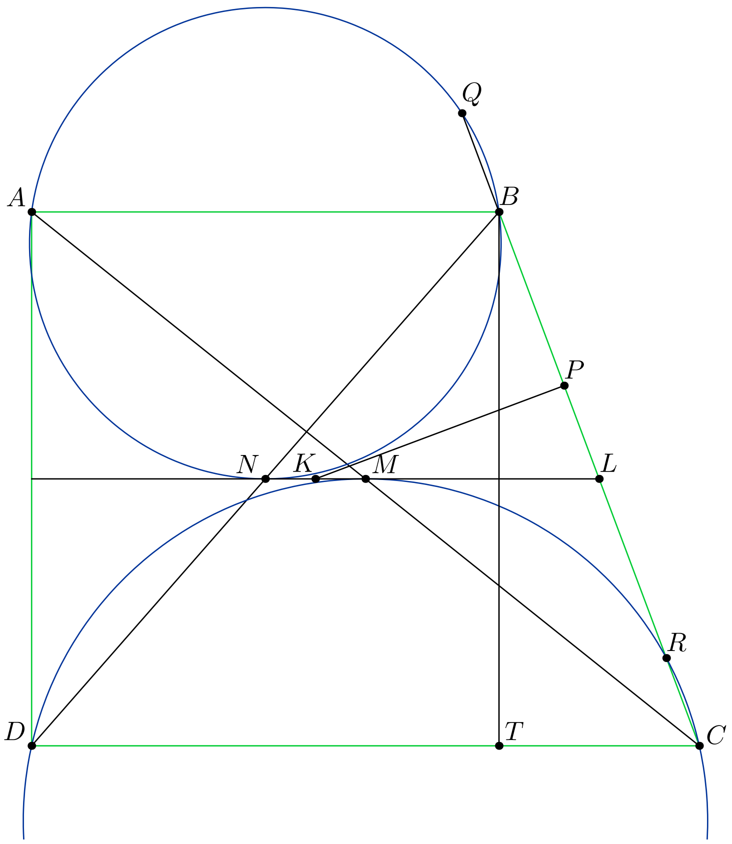
It's known that $\overline{ME}$ and $\overline{MF}$ are tangents to ω (and hence to ω_1, ω_2), so M is the radical center of ω, ω_1, ω_2. Now consider the radical axis of ω_1 and ω_2. It passes through D and M, so it is line BC, and we are done.

(Thus the problem is still true if D is replaced by any point on $\overline{BC}$.)

§11g Sharygin 2013/14

In trapezoid $ABCD$, $\angle A = \angle D = 90^\circ$. Let M and N be the midpoints of diagonals AC and BD, respectively. Let $Q = (ABN) \cap BC$ and $R = (CDM) \cap BC$. If K is the midpoint of MN, show that $KQ = KR$.

Let $AB = 2x$, $CD = 2y$, and assume without loss of generality that $x < y$. Let L be the midpoint of BC and denote $BC = 2\ell$. Let P be the midpoint of QR. Let T be the foot of B on DC.



Since N is the midpoint of the hypotenuse of $\triangle ABD$, it follows that $AN = BN$. Since $MN \parallel AB$, we see that MN is tangent to (ABN). Similarly, it is tangent to (BCM).

Noting that $LM = \frac{1}{2}AB$ via $\triangle ABC$, we obtain

$$LR \cdot LC = LM^2 = \left(\frac{1}{2}AB\right)^2 = x^2 \implies LR = \frac{x^2}{\ell}$$

Similarly, $LQ = \frac{y^2}{\ell}$. Then,

$$PL = \frac{LQ - LR}{2} = \frac{y^2 - x^2}{2\ell} - \text{ and } KL = \frac{ML + NL}{2} = x + y.$$

But then, we find that

$$\frac{KL}{PL} = \frac{\frac{y^2-x^2}{2\ell}}{x+y} = \frac{y-x}{2\ell} = \frac{TC}{BC}$$

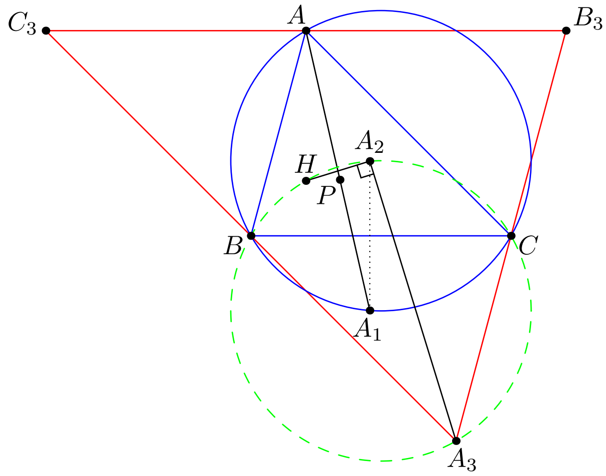
Combined with $\angle KLP = \angle BCT$, we find that $\triangle KLP \sim \triangle BCT$. Therefore, $\angle KPL = \angle BTC = 90^\circ$. But P is the midpoint of QR, so $KQ = KR$.

§11h Bulgaria 2012

Let ABC be a fixed triangle with circumcircle γ, and let P be any point in its interior. Ray AP meets γ again at A_1. We reflect A_1 across $\overline{BC}$ to obtain a point A_2. Define B_1, B_2, C_1 and C_2 similarly. Prove that the circumcircle of $A_2B_2C_2$ passes through a fixed point independent of P.

We claim the fixed point is the orthocenter H. (One might guess this by considering degenerate cases like $P = H$.) We present two solutions. (It is also possible to solve the problem using complex numbers with ABC as the unit circle.)

¶ First elementary solution (Evan Chen). Reflect A through the midpoint of $\overline{BC}$ to a point A_3. Define B_3 and C_3 similarly Notice that B, H, A_2, C, A_3 are concyclic, namely on the reflection of the circumcircle through $\overline{BC}$. Moreover, we have $\angle HA_2A_3 = 90^\circ$.



Notice that

$$\angle BAA_1 = \frac{1}{2}\widehat{BA_1} = \frac{1}{2}\widehat{BA_2} = \angle BA_3A_2.$$

Hence we see, say by Trig Ceva, that the concurrence of lines AA_1, BB_1, CC_1 also implies the lines A_3A_2, B_3B_2, C_3C_2 are concurrent, say at Q. (Alternatively, if you don't like trig: under the similarity $\triangle ABC \sim \triangle A_3B_3C_3$ let P_3 be the image of P. Then Q is the isogonal conjugate of P_3 with respect to $\triangle A_3B_3C_3$.) Then A_2 lies on a circle with diameter $\overline{HQ}$. So do B_2 and C_2 and the problem is solved.

¶ Second solution by tethered moving points. We fix A_1 and A_2, and let P vary on line AA_1. Then the maps $B \mapsto \gamma \mapsto (BHC)$ by $P \mapsto B_1 \mapsto B_2$ is projective, and similarly $P \mapsto C_1 \mapsto C_2$ is projective.

Now, we use the "second intersection of circles lemma" to conclude that the map

$$(HAC) \to (HAB) \qquad \text{by} \qquad B_2 \mapsto (HA_2B_2) \cap (HAB) \neq H$$

is a projective map (note that B_2 is the only point which is moving here). We claim this map coincides with the composed map $B_2 \mapsto C_2$, and for this it suffices to verify it for three points:

- If $P = A$, then $A = B_1 = B_2 = C_1 = C_2$ and we are okay.
- If $P = \overline{AA_1} \cap \overline{BC}$ then $B_1 = B_2 = C$, $C_1 = C_2 = B$, and since BHA_2C is an isosceles trapezoid we are okay.
- If $P = A_1$ then in fact $A_2B_2C_2$ is the dilation of the Simson line from P with ratio 2, which is known to pass through the orthocenter.

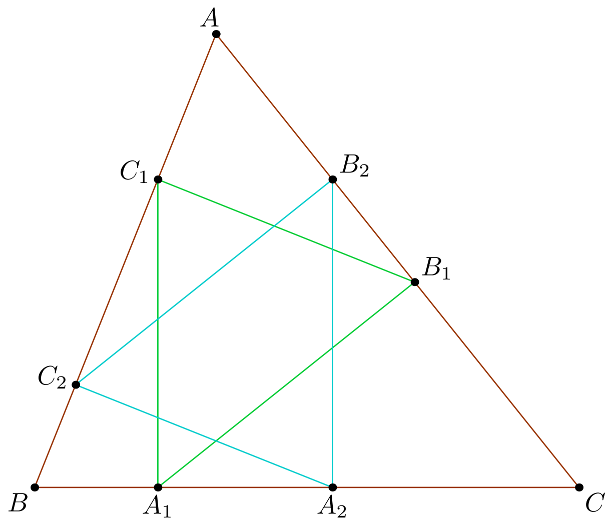
§11i Sharygin 2013/15

Let ABC be a triangle.

(a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are congruent.

(b) Points A_1, B_1, C_1, A_2, B_2, C_2 lie inside a triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2, and the angles BAA_1, CBB_2, ACC_1, CAA_2, ABB_2, BCC_2 are equal. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent.

For part (a), observe that $\angle C_1A_1B_1 = 90^\circ - (90^\circ - \angle B_1CA_1) = \angle C$. Similar calculations yield that $\triangle ABC \sim \triangle C_1A_1B_1 \sim \triangle B_2C_2A_2$.



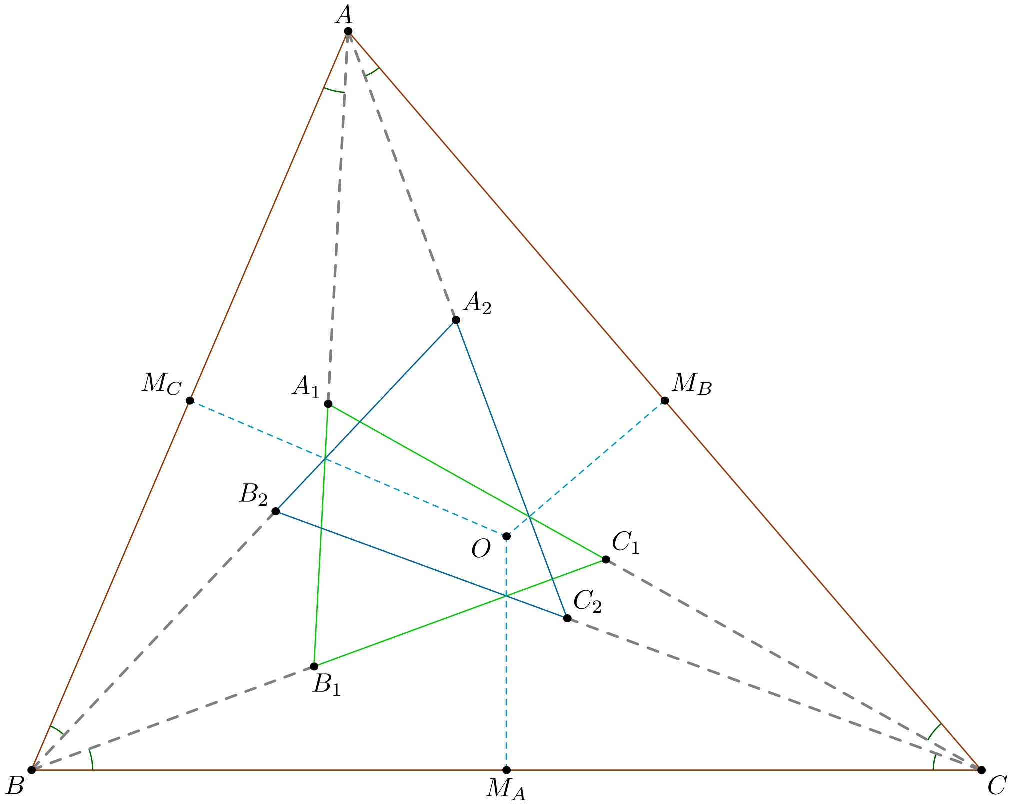
Now, notice that by the Pythagorean Theorem, we have

$$A_1B_2^2 = B_1B_2^2 + A_1B_1^2 = A_1A_2^2 + A_2B_2^2$$
$$B_1C_2^2 = C_1C_2^2 + B_1C_1^2 = B_1B_2^2 + B_2C_2^2$$
$$C_1A_2^2 = A_1A_2^2 + C_1A_1^2 = C_1C_2^2 + C_2A_2^2$$

Summing, we obtain that

$$A_1B_1^2 + B_1C_1^2 + C_1A_1^2 = A_2B_2^2 + B_2C_2^2 + C_2A_2^2.$$

Since $\triangle C_1A_1B_1 \sim \triangle B_2C_2A_2$, and the sums of the square of the sides are equal, it follows that the two triangles must be equal as well.



For part (b), easy angle chasing gives

$$\angle B_2A_2C_2 = \angle ABA_2 + \angle BAA_2 = \angle BAC.$$

Similar calculations yield that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle ABC$.

Now, let O be the circumcenter of $\triangle ABC$. Then O lies on the angle bisector of the angle formed by lines B_2C_2 and B_1C_1; namely, the line through O perpendicular to BC. (Note that $\angle B_1BC = C_2CB$.) Let d_a denote the command distance from O to lines B_2C_2 and B_1C_1. Define d_b and d_c analogously.

Then, since $A_1B_1C_1 \sim A_2B_2C_2$, we observe that O must have the same barycentric coordinates with respect to $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, namely

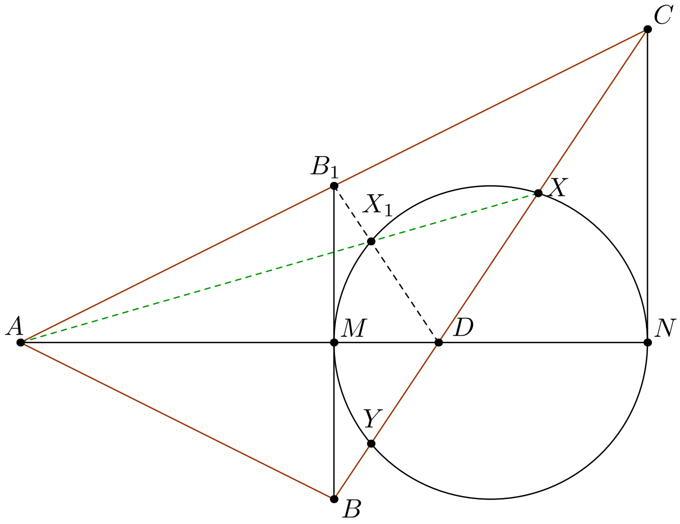
$$(d_a \cdot B_1C_1 : d_b \cdot C_1A_1 : d_c \cdot A_1B_1) = (d_a \cdot B_2C_2 : d_b \cdot C_2A_2 : d_c \cdot A_2B_2).$$

So O corresponds to the same point in both triangles. The congruence of the pedal triangles is then enough to deduce that $\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2$.

§11j Sharygin 2013/18

Let $\overline{AD}$ be a bisector of $\triangle ABC$. Points M and N are the projections of B and C respectively to $\overline{AD}$. The circle with diameter $\overline{MN}$ intersects $\overline{BC}$ at points X and Y. Prove that $\angle BAX = \angle CAY$.

Let B_1 be the reflection of B over M (which is on $\overline{AC}$) and let P_∞ be the point at infinity along $\overline{BM} \parallel \overline{CN}$.



Evidently

$$-1 = (B_1, B; M, P_\infty) \overset{C}{=} (A, D; M, N).$$

But $\angle MYN = \angle MXN = 90^\circ$, so we find that M is the incenter of $\triangle AXY$; hence $\angle XAM = \angle YAM$, and hence $\angle BAX = \angle CAY$ as desired.

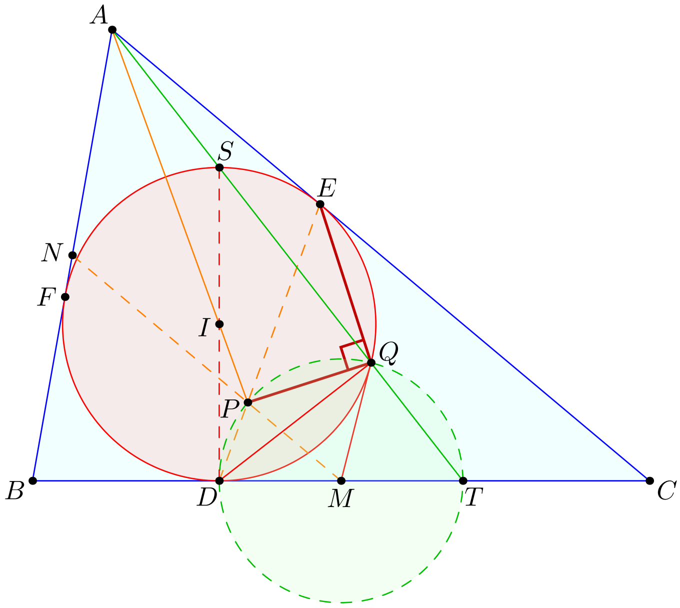
§11k USA TST 2015/1

Let ABC be a scalene triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$ and let P be a point in the interior of $\triangle ABC$ so that $MD = MP$ and $\angle PAB = \angle PAC$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

(Available online at https://aops.com/community/p3683109.)

We present two solutions.

¶ Official solution. Assume without loss of generality that $AB < AC$; we show $\angle PQE = 90^\circ$.



First, we claim that D, P, E are collinear. Let N be the midpoint of $\overline{AB}$. It is well-known that the three lines MN, DE, AI are concurrent at a point (see for example problem 6 of USAJMO 2014). Let P' be this intersection point, noting that P' actually lies on segment DE. Then P' lies inside $\triangle ABC$ and moreover

$$\triangle DP'M \sim \triangle DEC$$

so $MP' = MD$. Hence $P' = P$, proving the claim.

Let S be the point diametrically opposite D on the incircle, which is also the second intersection of $\overline{AQ}$ with the incircle. Let $T = \overline{AQ} \cap \overline{BC}$. Then T is the contact point of the A-excircle; consequently,

$$MD = MP = MT$$

and we obtain a circle with diameter $\overline{DT}$. Since $\angle DQT = \angle DQS = 90^\circ$ we have Q on this circle as well.

As $\overline{SD}$ is tangent to the circle with diameter $\overline{DT}$, we obtain

$$\angle PQD = \angle SDP = \angle SDE = \angle SQE.$$

Since $\angle DQS = 90^\circ$, $\angle PQE = 90^\circ$ too.

¶ Solution using spiral similarity. We will ignore for now the point P. As before define S, T and note $\overline{AQST}$ collinear, as well as $DPQT$ cyclic on circle ω with diameter $\overline{DT}$.

Let τ be the spiral similarity at Q sending ω to the incircle. We have $\tau(T) = D$, $\tau(D) = S$, $\tau(Q) = Q$. Now

$$I = \overline{DD} \cap \overline{QQ} \implies \tau(I) = \overline{SS} \cap \overline{QQ}$$

and hence we conclude $\tau(I)$ is the pole of $\overline{ASQT}$ with respect to the incircle, which lies on line EF.

Then since $\overline{AI} \perp \overline{EF}$ too, we deduce τ sends line AI to line EF, hence $\tau(P)$ must be either E or F as desired.

¶ Authorship comments. Written April 2014. I found this problem while playing with GeoGebra. Specifically, I started by drawing in the points A, B, C, I, D, M, T, common points. I decided to add in the circle with diameter DT, because of the synergy it had with the rest of the picture. After a while of playing around, I intersected ray AI with the circle to get P, and was surprised to find that D, P, E were collinear, which I thought was impossible since the setup should have been symmetric. On further reflection, I realized it was because AI intersected the circle twice, and set about trying to prove this. I noticed the relation $\angle PQE = 90^\circ$ in my attempts to prove the result, even though this ended up being a corollary rather than a useful lemma.

§11I EGMO 2014/2

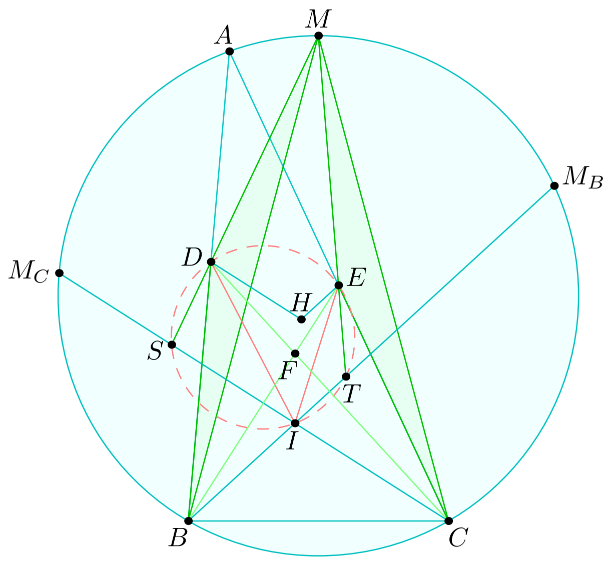
Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incenter I of triangle ABC, the orthocenter H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

(Available online at https://aops.com/community/p3459750.)

¶ First solution (Cynthia Du). Let BI and CI meet the circumcircle again at M_B, M_C. Observe that we have the spiral congruence

$$\triangle MDB \sim \triangle MEC$$

from $\measuredangle MBD = \measuredangle MBA = \measuredangle MCA = \measuredangle MCE$ and $BD = EC$, $BM = CM$. That is, M is the Miquel point of $BDEC$.



Let $T = \overline{ME} \cap \overline{BI}$ and $S = \overline{MD} \cap \overline{CI}$. First, since $\overline{BI}$ is the perpendicular bisector of $\overline{CD}$ we have that

$$\measuredangle DIT = \measuredangle CIT = \measuredangle CIB = 90^\circ - \frac{1}{2}\angle A = \measuredangle MCB = \measuredangle MED = \measuredangle TED$$

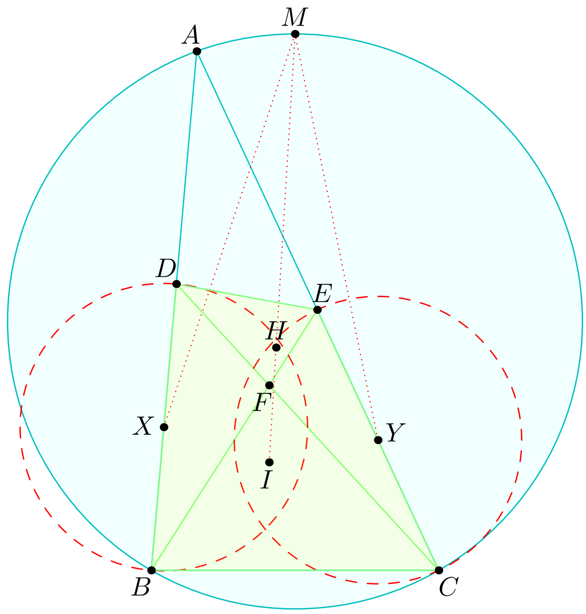
and so D, I, T, E is cyclic. Similarly S lies on this circle too. But $\measuredangle SDE = \measuredangle EDM = \measuredangle MED = \measuredangle TED$ so in fact $\overline{ST} \parallel \overline{DE}$ (isosceles trapezoid).

Then $\triangle IST$ and $\triangle HDE$ are homothetic, so $\overline{IH}$, $\overline{DS}$, and $\overline{ET}$ concur (at M).

¶ Second solution (Evan Chen). Observe that we have the spiral congruence

$$\triangle MDB \sim \triangle MEC$$

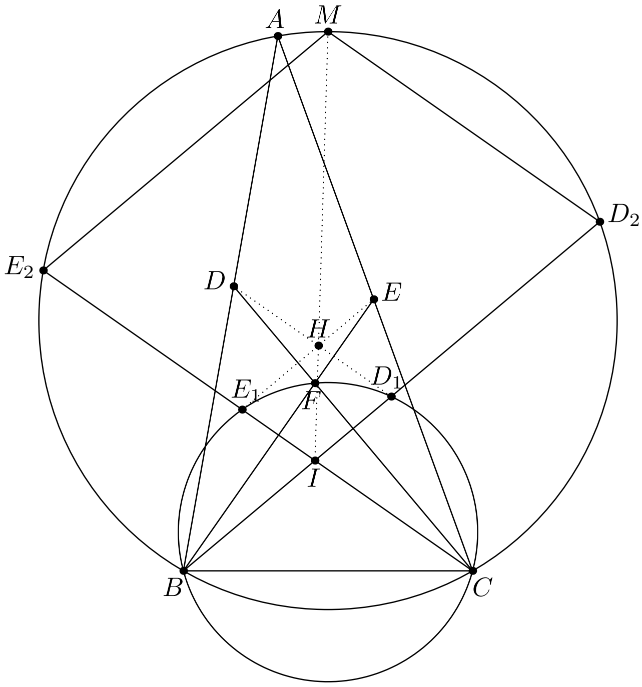
from $\measuredangle MBD = \measuredangle MBA = \measuredangle MCA = \measuredangle MCE$ and $BD = EC$, $BM = CM$. That is, M is the Miquel point of $BDEC$.



Let X and Y be the midpoints of $\overline{BD}$ and $\overline{CE}$. Then $MX = MY$ by our congruence.

Consider now the circles with diameters $\overline{BD}$ and $\overline{CE}$. We now claim that H, I, M all lie on the radical axis of these circles. Note that I is the orthocenter of $\triangle BFC$ and H is the orthocenter of $\triangle DEF$, so this follows from the so-called Steiner line of $BCDE$ (perpendicular to Gauss line $\overline{XY}$). For M, we observe $MX^2 - XB^2 = MY^2 - YC^2$ thus completing the proof.

¶ Third solution (homothety, official solution). Extend DH and EH to meet BI and CI at D_1 and E_1. Note $DD_1 \perp BE$, $CI \perp BE$, so $DD_1 \parallel CI$. Similarly $EE_1 \parallel BI$. So HE_1ID_1.



Angle chase to show that B, E_1, F, C are cyclic – $\angle DCE_1 = \angle DCI$ is computable in terms of ABC and

$$\angle E_1BF = \angle E_1BE = \angle E_1EB = \angle HEF = \angle HDF = \angle HDC = \angle DCE_1 = \angle FCE_1.$$

Thus B, D_1, F, C are also cyclic. So B, D_1, E_1, C are cyclic.

Extend BI and CI to meet the circumcircle again at D_2 and E_2. Direct computation gives that ME_2ID_2 is also a parallelogram. We also get E_1D_1 is parallel to E_2D_2 (both are antiparallel to BC through $\angle BIC$). So we have homothetic paralellograms and that finishes the problem.

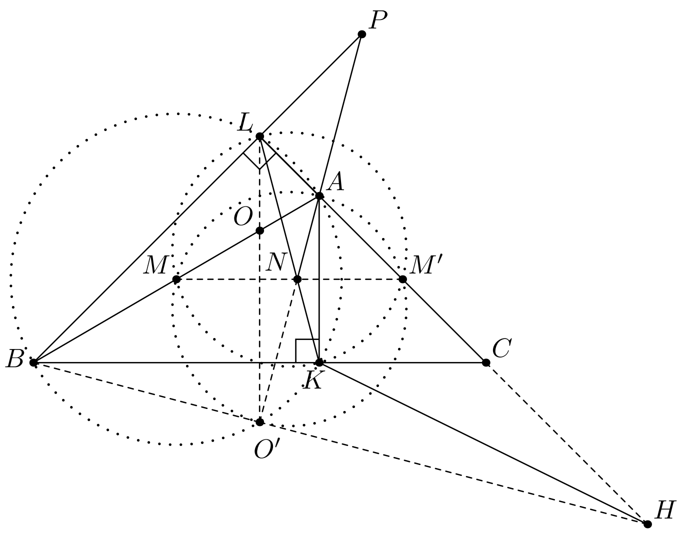
§11m OMO 2013 W49

In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45^\circ$. Let K, L, M lie on lines BC, CA, and AB such that $\overline{AK} \perp \overline{BC}$, $\overline{BL} \perp \overline{CA}$, and $AM = BM$. Let N, O, P lie on $\overline{KL}$, $\overline{BA}$, and $\overline{BL}$ such that $AN = KN$, $BO = CO$, and A lies on line NP.

If H is the orthocenter of $\triangle MOP$, compute HK^2.

(Available online at https://aops.com/community/p2906138.)

Let M' be the midpoint of $\overline{AC}$ and let O' be the circumcenter of $\triangle ABC$. Then $KMLM'$ is cyclic (nine-point circle), as is $AMO'M'$ (since $\angle MOA = \angle MM'A = 45^\circ$). Also, $\angle BO'A = 90^\circ$, so O' lies on the circle with diameter $\overline{AB}$. Then N is the radical center of these three circles; hence A, N, O' are collinear.



Now applying Brocard's theorem to quadrilateral $BLAO'$, we find that M is the orthocenter of the OPH', where $H' = \overline{LA} \cap \overline{BO'}$. Hence H' is the orthocenter of $\triangle MOP$, whence $H = H' = \overline{AC} \cap \overline{BO'}$.

Now we know that

$$\frac{AH}{HC} = \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)}$$

where the ratio is directed as in Menelaus's theorem. Cancelling a factor of 280^2 we can compute:

$$\frac{AH}{HC} = \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)} = \frac{338(576+98-338)}{576(98+338-576)} = -\frac{169}{120}.$$

Therefore,

$$\frac{AC}{HC} = 1 + \frac{AH}{HC} = -\frac{49}{120}$$
$$\implies |HC| = \frac{120}{49} \cdot 1960\sqrt{2} = 4800\sqrt{2}.$$

Now applying the law of cosines to $\triangle KCH$ with $\angle KCH = 135^\circ$ yields

$$\begin{aligned} HK^2 &= KC^2 + CH^2 - 2KC \cdot CH \cdot \cos 135^\circ \\ &= 1960^2 + \left(4800\sqrt{2}\right)^2 - 2(1960)\left(4800\sqrt{2}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ &= 40^2\left(49^2 + 2\cdot 120^2 + 2\cdot 49\cdot 120\right) \\ &= 1600 \cdot 42961 \\ &= 68737600. \end{aligned}$$

§11n USAMO 2007/6

Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \leq R^3$$

with equality if and only if triangle ABC is equilateral.

(Available online at https://aops.com/community/p825515.)

It turns out we can compute P_AQ_A explicitly. Let us invert around A with radius $s-a$ (hence fixing the incircle) and then compose this with a reflection around the angle bisector of $\angle BAC$. We denote the image of the composed map via

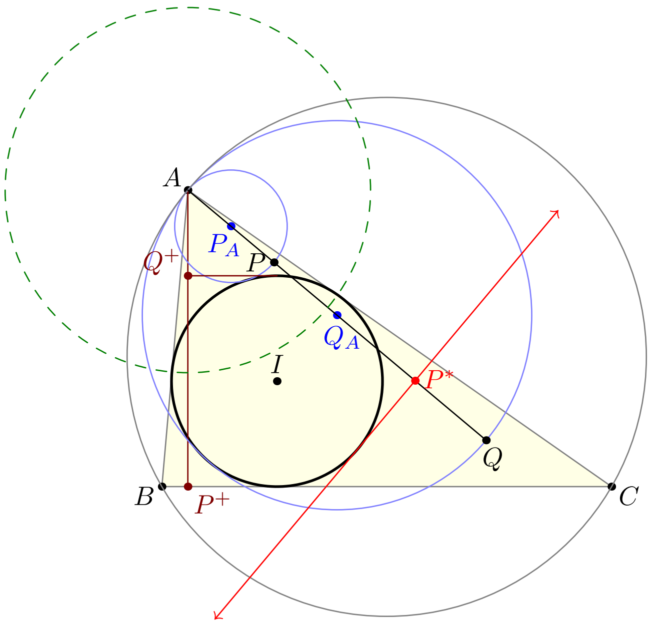
$$\bullet \mapsto \bullet^* \mapsto \bullet^+.$$

We overlay this inversion with the original diagram.

Let P_AQ_A meet ω_A again at P and S_A again at Q. Now observe that ω_A^* is a line parallel to S^*; that is, it is perpendicular to $\overline{PQ}$. Moreover, it is tangent to $\omega^* = \omega$.

Now upon the reflection, we find that $\omega^+ = \omega^* = \omega$, but line $\overline{PQ}$ gets mapped to the altitude from A to $\overline{BC}$, since $\overline{PQ}$ originally contained the circumcenter O (isogonal to the orthocenter). But this means that ω_A^* is none other than the $\overline{BC}$! Hence P^+ is actually the foot of the altitude from A onto $\overline{BC}$.

By similar work, we find that Q^+ is the point on $\overline{AP^+}$ such that $P^+Q^+ = 2r$.



Now we can compute all the lengths directly. We have that

$$AP_A = \frac{1}{2}AP = \frac{(s-a)^2}{2AP^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a}$$

and

$$AQ_A = \frac{1}{2}AQ = \frac{(s-a)^2}{2AQ^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a - 2r}$$

where $h_a = \frac{2K}{a}$ is the length of the A-altitude, with K the area of ABC as usual. Now it follows that

$$P_AQ_A = \frac{1}{2}(s-a)^2 \left(\frac{2r}{h_a(h_a-2r)} \right).$$

This can be simplified, as

$$h_a - 2r = \frac{2K}{a} - \frac{2K}{s} = 2K \cdot \frac{s-a}{as}.$$

Hence

$$P_AQ_A = \frac{a^2rs(s-a)}{4K^2} = \frac{a^2(s-a)}{4K}.$$

Hence, the problem is just asking us to show that

$$a^2b^2c^2(s-a)(s-b)(s-c) \le 8(RK)^3.$$

Using $abc = 4RK$ and $(s-a)(s-b)(s-c) = \frac{1}{s}K^2 = rK$, we find that this becomes

$$2(s-a)(s-b)(s-c) \le RK \iff 2r \le R$$

which follows immediately from $IO^2 = R(R-2r)$. Alternatively, one may rewrite this as Schur's Inequality in the form

$$abc \ge (-a+b+c)(a-b+c)(a+b-c).$$

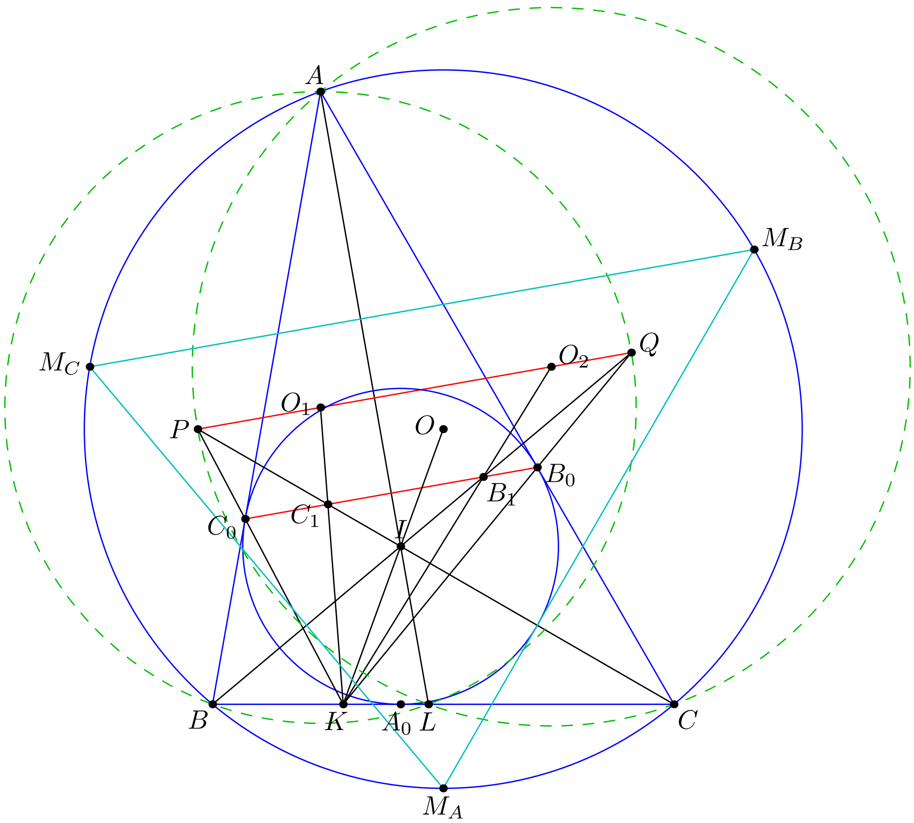
§11o Sharygin 2013/19

Let ABC be a triangle with circumcenter O and incenter I. The incircle is tangent to sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at A_0, B_0, C_0. Point L lies on $\overline{BC}$ so that $\angle BAL = \angle CAL$. The perpendicular bisector of $\overline{AL}$ meets BI and CI at Q and P, respectively. Let C_1 and B_1 denote the projections of B and C onto lines CI and BI. Let O_1 and O_2 denote the circumcenters of triangles ABL and ACL.

Prove that the six lines BC, PC_0, QB_0, C_1O_1, B_1O_2, and OI are concurrent.

(Available online at https://aops.com/community/p3010038.)

First, show that B_0, B_1, C_0, C_1 are collinear. This follows by angle chasing (it's EGMO Lemma 1.45). Moreover, we can check that P is the midpoint of the minor arc AL of the circumcircle of triangle ACL. In particular, A, P, C, L are concyclic. Similarly, A, Q, B, L are concyclic. We also know that P, O_1, O_2, Q are clearly collinear.



By $\angle LPI = \angle LAC$ we observe that $\overline{LP} \perp \overline{BI}$. Similarly $\overline{LQ} \perp \overline{CI}$. This is enough to imply that

$$\triangle A_0B_0C_0 \sim \triangle LPQ$$

are homothetic, with center K. Thus we obtain that BC, PC_0, QB_0 concur at at a point K. Upon noticing that $C_1A_0 = C_1B_0$ and $O_1Q = O_1L$ (as well as $C_1 \in \overline{B_0C_0}$, $O_1 \in \overline{PQ}$) we find that C_1 maps to O_1 under the same homothety, meaning C_1, O_1, K are collinear. Similarly, B_1, O_2, K are collinear.

It remains to show that I, O, K are collinear. Let $M_AM_BM_C$ denote the arc midpoints on the circumcircle of $\triangle ABC$. Note that:

- We had already a positive homothety at K between $\triangle A_0B_0C_0$ and $\triangle PQL$.
- There is evidently a homothety at I mapping $\triangle PQL$ to $\triangle M_cM_aM_b$.
- There is by definition a homothety at X_{56} mapping (I) to (O).

So by Monge's theorem, K, I, X_{56} are collinear, and X_{56} lies on line IO, as desired.

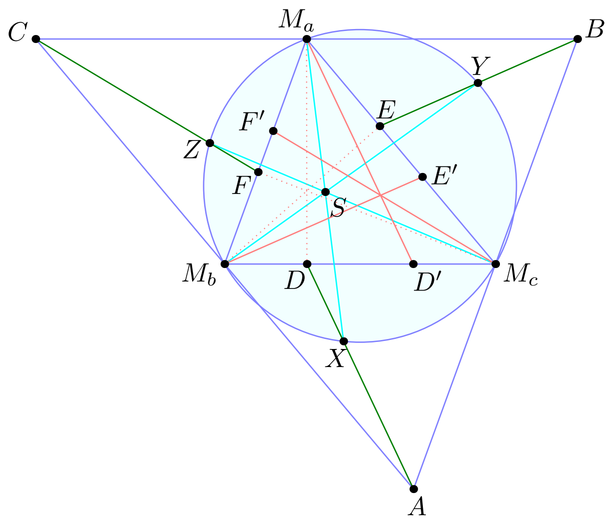
§11p USA TST 2015/6

Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

(Available online at https://aops.com/community/p4628087.)

We assume now and forever that ABC is scalene since the problem follows by symmetry in the isosceles case. We present four solutions.

¶ First solution by barycentric coordinates (Evan Chen). Let AX meet M_bM_c at D, and let X reflected over M_bM_c's midpoint be X'. Let Y', Z', E, F be similarly defined.



By Cevian Nest Theorem it suffices to prove that M_aD, M_bE, M_cF are concurrent. Taking the isotomic conjugate and recalling that $M_aM_bAM_c$ is a parallelogram, we see that it suffices to prove M_aX', M_bY', M_cZ' are concurrent.

We now use barycentric coordinates on $\triangle M_aM_bM_c$. Let

$$S = \left(a^2S_A + t : b^2S_B + t : c^2S_C + t\right)$$

(possibly $t = \infty$ if S is the centroid). Let $v = b^2S_B + t$, $w = c^2S_C + t$. Hence

$$X = \left(-a^2vw : (b^2w + c^2v)v : (b^2w + c^2v)w\right).$$

Consequently,

$$X' = \left(a^2vw : -a^2vw + (b^2w + c^2v)w : -a^2vw + (b^2w + c^2v)v\right)$$

We can compute

$$b^2w + c^2v = (bc)^2(S_B + S_C) + (b^2 + c^2)t = (abc)^2 + (b^2 + c^2)t.$$

Thus

$$-a^2v + b^2w + c^2v = (b^2 + c^2)t + (abc)^2 - (ab)^2S_B - a^2t = S_A((ab)^2 + t).$$

Finally

$$X' = \left(a^2vw : S_A(c^2S_C + t)\left((ab)^2 + 2t\right) : S_A(b^2S_B + t)\left((ac)^2 + 2t\right)\right)$$

and from this it's evident that AX', BY', CZ' are concurrent.

¶ Second solution by moving points (Anant Mudgal). Let H_a, H_b, H_c be feet of altitudes, and let γ denote the nine-point circle. The main claim is that:

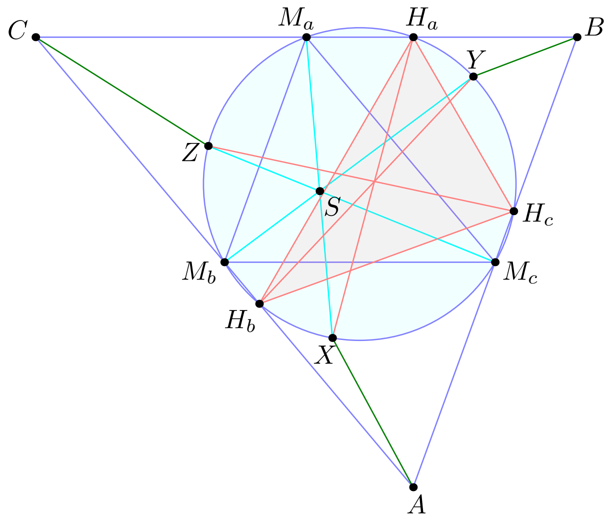
Claim — Lines XH_a, YH_b, ZH_c are concurrent, *Proof.* In fact, we claim that the concurrence point lies on the Euler line ℓ. This gives us a way to apply the moving points method: fix triangle ABC and animate $S \in \ell$; then the map

$$\begin{aligned} \ell &\to \gamma \to \ell \\ S &\mapsto X \mapsto S_a := \ell \cap \overline{H_a X} \end{aligned}$$

is projective, because it consists of two perspectivities. So we want the analogous maps $S \mapsto S_b$, $S \mapsto S_c$ to coincide. For this it suffices to check three positions of S; since you're such a good customer here are four.

- If S is the orthocenter of $\triangle M_aM_bM_c$ (equivalently the circumcenter of $\triangle ABC$) then S_a coincides with the circumcenter of $M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then S_a coincides with the de Longchamps point of $\triangle M_aM_bM_c$ (equivalently orthocenter of $\triangle ABC$). By symmetry S_b and S_c are too.
- If S is either of the intersections of the Euler line with γ, then $S = S_a = S_b = S_c$ (as $S = X = Y = Z$).

This concludes the proof. □



We now use Trig Ceva to carry over the concurrence. By sine law,

$$\frac{\sin \angle M_cAX}{\sin \angle AM_cX} = \frac{M_cX}{AX}$$

and a similar relation for M_b gives that

$$\frac{\sin \angle M_cAX}{\sin \angle M_bAX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{M_cX}{M_bX} = \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \cdot \frac{\sin \angle XM_aM_c}{\sin \angle XM_aM_b}.$$

Thus multiplying cyclically gives

$$\prod_{\text{cyc}} \frac{\sin \angle M_cAX}{\sin \angle M_bAX} = \prod_{\text{cyc}} \frac{\sin \angle AM_cX}{\sin \angle AM_bX} \prod_{\text{cyc}} \frac{\sin \angle XM_aM_c}{\sin \angle XM_aM_b}.$$

The latter product on the right-hand side equals 1 by Trig Ceva on $\triangle M_aM_bM_c$ with cevians $\overline{M_aX}$, $\overline{M_bY}$, $\overline{M_cZ}$. The former product also equals 1 by Trig Ceva for the concurrence in the previous claim (and the fact that $\angle AM_cX = \angle H_cH_aX$). Hence the left-hand side equals 1, implying the result.

¶ Third solution by moving points (Gopal Goel). In this solution, we will instead use barycentric coordinates with resect to $\triangle ABC$ to bound the degrees suitably, and then verify for seven distinct choices of S.

We let R denote the radius of $\triangle ABC$, and N the nine-point center.

First, imagine solving for X in the following way. Suppose $\vec{X} = (1-t_a)\vec{M}_a + t_a\vec{S}$. Then, using the dot product (with $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$ in general)

$$\begin{aligned}\frac{1}{4}R^2 &= \left|\vec{X}-\vec{N}\right|^2 \\ &= \left|t_a(\vec{S}-\vec{M}_a)+\vec{M}_a-\vec{N}\right|^2 \\ &= \left|t_a(\vec{S}-\vec{M}_a)\right|^2 + 2t_a\left(\vec{S}-\vec{M}_a\right)\cdot\left(\vec{M}_a-\vec{N}\right) + \left|\vec{M}_a-\vec{N}\right|^2 \\ &= t_a^2\left|(\vec{S}-\vec{M}_a)\right|^2 + 2t_a\left(\vec{S}-\vec{M}_a\right)\cdot\left(\vec{M}_a-\vec{N}\right) + \frac{1}{4}R^2\end{aligned}$$

Since $t_a \neq 0$ we may solve to obtain

$$t_a = -\frac{2(\vec{M}_a-\vec{N})\cdot(\vec{S}-\vec{M}_a)}{\left|\vec{S}-\vec{M}_a\right|^2}.$$

Now imagine S varies along the Euler line, meaning there should exist linear functions $\alpha,\beta,\gamma\colon \mathbb{R}\to\mathbb{R}$ such that

$$S = (\alpha(s),\beta(s),\gamma(s)) \qquad s\in\mathbb{R}$$

with $\alpha(s)+\beta(s)+\gamma(s)=1$. Thus $t_a = \frac{f_a}{g_a} = \frac{f_a(s)}{g_a(s)}$ is the quotient of a linear function $f_a(s)$ and a quadratic function $g_a(s)$.

So we may write:

$$\begin{aligned}X &= (1-t_a)\left(0,\frac{1}{2},\frac{1}{2}\right) + t_a\left(\alpha,\beta,\gamma\right) \\ &= \left(t_a\alpha, \frac{1}{2}(1-t_a)+t_a\beta, \frac{1}{2}(1-t_a)+t_a\gamma\right) \\ &= \left(2f_a\alpha : g_a - f_a + 2f_a\beta : g_a - f_a + 2f_a\gamma\right).\end{aligned}$$

Thus the coordinates of X are quadratic polynomials in s when written in this way.

In a similar way, the coordinates of Y and Z should be quadratic polynomials in s. The Ceva concurrence condition

$$\prod_{\text{cyc}} \frac{g_a - f_a + 2f_a\beta}{g_a - f_a + 2f_a\gamma} = 1$$

is thus a polynomial in s of degree at most six. Our goal is to verify it is identically zero, thus it suffices to check seven positions of S.

- If S is the circumcenter of $\triangle M_aM_bM_c$ (equivalently the nine-point center of $\triangle ABC$) then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are altitudes of $\triangle ABC$.
- If S is the centroid of $\triangle M_aM_bM_c$ (equivalently the centroid of $\triangle ABC$), then $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are medians of $\triangle ABC$.
- If S is either of the intersections of the Euler line with γ, then $S=X=Y=Z$ and all cevians concur at S.

- If S lies on the $\overline{M_aM_b}$, then $Y = M_a$, $X = M_c$, and thus $\overline{AX} \cap \overline{BY} = C$, which is of course concurrent with $\overline{CZ}$ (regardless of Z). Similarly if S lies on the other sides of $\triangle M_aM_bM_c$.

Thus we are also done.

¶ Fourth solution using Pascal (official one). We give a different proof of the claim that $\overline{XH_a}$, $\overline{YH_b}$, $\overline{ZH_c}$ are concurrent (and then proceed as in the end of the second solution).

Let H denote the orthocenter, N the nine-point center, and moreover let N_a, N_b, N_c denote the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, which also lie on the nine-point circle (and are the antipodes of M_a, M_b, M_c).

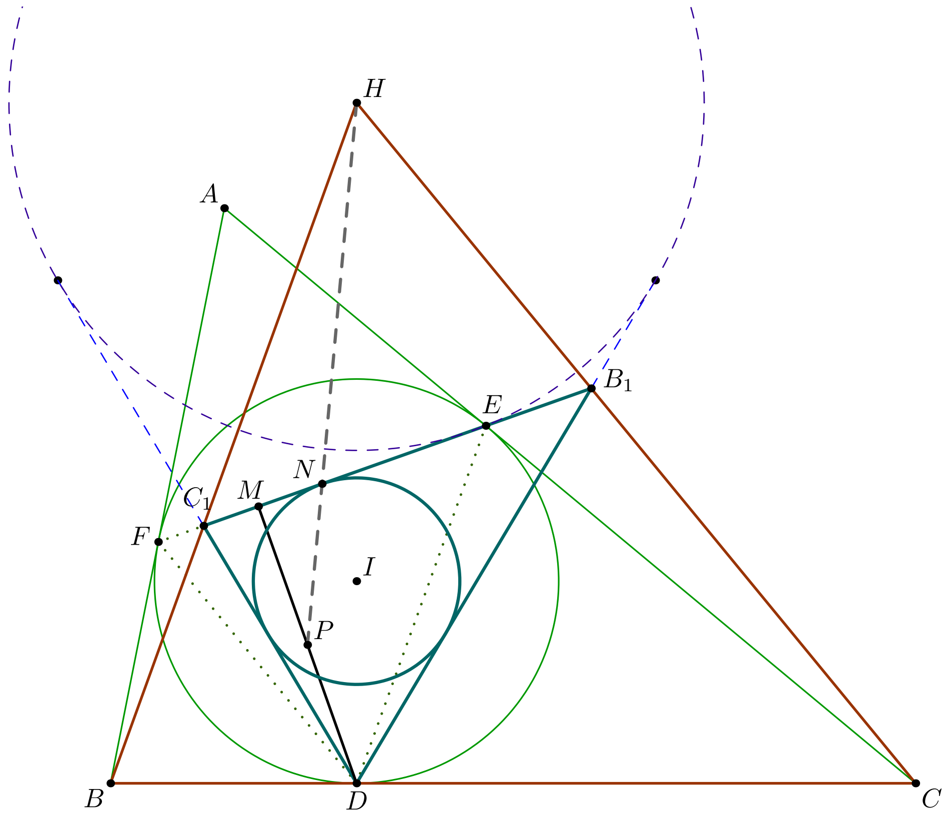
- By Pascal's theorem on $M_bN_bH_bM_cN_cH_c$, the point $P = \overline{M_cH_b} \cap \overline{M_bH_c}$ is collinear with $N = \overline{M_bN_b} \cap \overline{M_cN_c}$, and $H = \overline{N_bH_b} \cap \overline{N_cH_c}$. So P lies on the Euler line.
- By Pascal's theorem on $M_bYH_bM_cZH_c$, the point $\overline{YH_b} \cap \overline{ZH_c}$ is collinear with $S = \overline{M_bY} \cap \overline{M_cZ}$ and $P = \overline{M_bH_c} \cap \overline{M_cH_b}$. Hence YH_b and ZH_c meet on the Euler line, as needed.

§11q Iran TST 2009/9

Let ABC be a triangle with incenter I and intouch triangle DEF. Let M be the foot of the perpendicular from D to $\overline{EF}$ and let P be the midpoint of $\overline{DM}$. If H is the orthocenter of triangle BIC, prove that $\overline{PH}$ bisects $\overline{EF}$.

(Available online at https://aops.com/community/p1499412.)

Let N be the midpoint of $\overline{EF}$, and set $B_1 = \overline{EF} \cap \overline{HC}$, $C_1 = \overline{EF} \cap \overline{HB}$. Focus on triangle DB_1C_1.



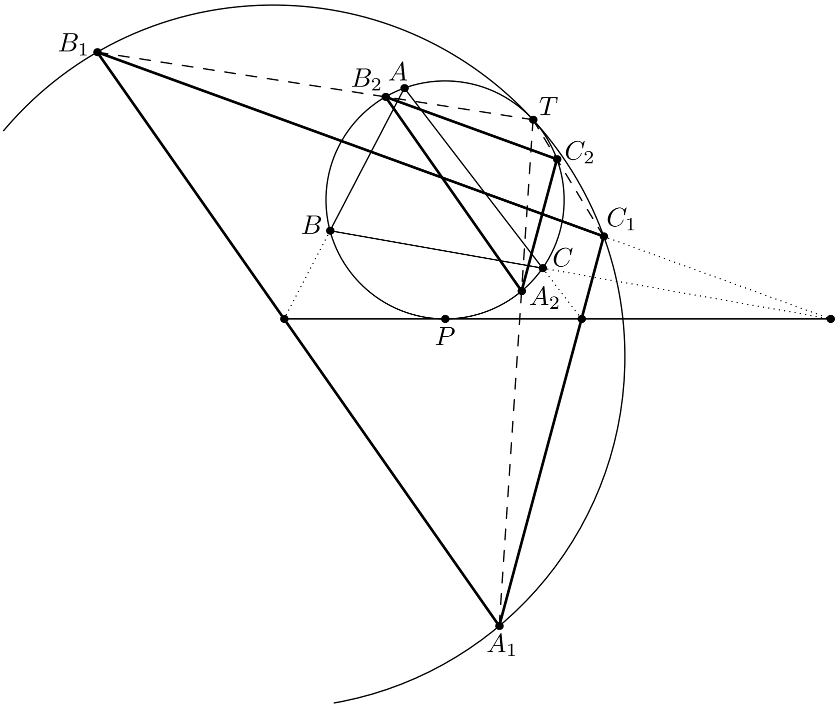
It's known that $\triangle DB_1C_1$ is the orthic triangle of $\triangle HBC$ (by EGMO Lemma 1.45). Moreover, N is the tangency point of its incircle with $\overline{B_1C_1}$. In addition, H is the D-excenter. Finally, because of altitude midpoints, points P, N, and H are collinear.

§11r IMO 2011/6

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

(Available online at https://aops.com/community/p2365045.)

This is a hard problem with many beautiful solutions. The following solution is not very beautiful but not too hard to find during an olympiad, as the only major insight it requires is the construction of A_2, B_2, and C_2.



We apply complex numbers with ω the unit circle and $p = 1$. Let $A_1 = \ell_B \cap \ell_C$, and let $a_2 = a^2$ (in other words, A_2 is the reflection of P across the diameter of ω through A). Define the points B_1, C_1, B_2, C_2 similarly.

We claim that $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ concur at a point on Γ.

We begin by finding A_1. If we reflect the points $1+i$ and $1-i$ over $\overline{AB}$, then we get two points Z_1, Z_2 with

$$\begin{aligned} z_1 &= a + b - ab(1-i) = a + b - ab + abi \\ z_2 &= a + b - ab(1+i) = a + b - ab - abi. \end{aligned}$$

Therefore,

$$\begin{aligned} z_1 - z_2 &= 2abi \\ \overline{z_1} z_2 - \overline{z_2} z_1 &= -2i \left(a + b + \frac{1}{a} + \frac{1}{b} - 2 \right). \end{aligned}$$

Now ℓ_C is the line $\overline{Z_1Z_2}$, so with the analogous equation ℓ_B we obtain:

$$\begin{aligned} a_1 &= \frac{-2i\left(a+b+\frac{1}{a}+\frac{1}{b}-2\right)(2aci) + 2i\left(a+c+\frac{1}{a}+\frac{1}{c}-2\right)(2abi)}{\left(-\frac{2}{ab}i\right)(2aci) - \left(-\frac{2}{ac}i\right)(2abi)} \\ &= \frac{\left[c-b\right]a^2 + \left[\frac{c}{b} - \frac{b}{c} - 2c + 2b\right]a + (c-b)}{\frac{c}{b} - \frac{b}{c}} \\ &= a + \frac{(c-b)\left[a^2 - 2a + 1\right]}{(c-b)(c+b)/bc} \\ &= a + \frac{bc}{b+c}(a-1)^2. \end{aligned}$$

Then the second intersection of $\overline{A_1A_2}$ with ω is given by

$$\frac{a_1 - a_2}{1 - a_2\overline{a_1}} = \frac{a + \frac{bc}{b+c}(a-1)^2 - a^2}{1 - a - a^2 \cdot \frac{(1-1/a)^2}{b+c}}$$

$$= \frac{a + \frac{bc}{b+c}(1-a)}{1 - \frac{1}{b+c}(1-a)}$$
$$= \frac{ab + bc + ca - abc}{a + b + c - 1}.$$

Thus, the claim is proved.

Finally, it suffices to show $\overline{A_1B_1} \parallel \overline{A_2B_2}$. One can also do this with complex numbers; it amounts to showing $a^2 - b^2$, $a - b$, i (corresponding to $\overline{A_2B_2}$, $\overline{A_1B_1}$, $\overline{PP}$) have their arguments an arithmetic progression, equivalently

$$\frac{(a-b)^2}{i(a^2-b^2)} \in \mathbb{R} \iff \frac{(a-b)^2}{i(a^2-b^2)} = \frac{\left(\frac{1}{a} - \frac{1}{b}\right)^2}{\frac{1}{i}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}$$

which is obvious.

Remark. One can use directed angle chasing for this last part too. Let $\overline{BC}$ meet ℓ at K and $\overline{B_2C_2}$ meet ℓ at L. Evidently

$$\begin{aligned} -\measuredangle B_2LP &= \measuredangle LPB_2 + \measuredangle PB_2L \\ &= 2\measuredangle KPB + \measuredangle PB_2C_2 \\ &= 2\measuredangle KPB + 2\measuredangle PBC \\ &= -2\measuredangle PKB \\ &= \measuredangle PKB_1 \end{aligned}$$

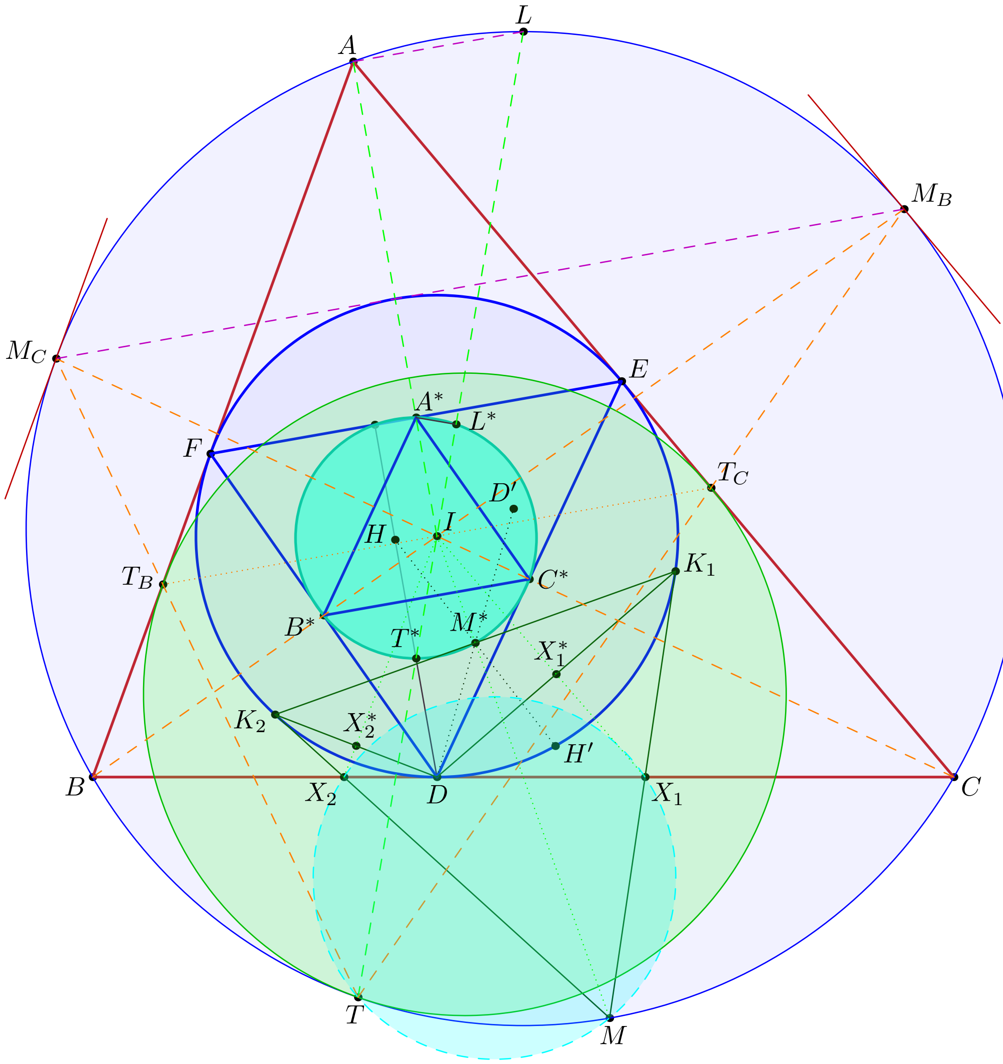
as required.

§11s Taiwan TST 2014/3J/3

Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose the tangents from M to the incircle of ABC intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ.

(Available online at https://aops.com/community/p3551881.)

We know that the line TI passes through the midpoint of arc $\widehat{BC}$ containing A; call this point L.



Set DEF as the intouch triangle of ABC. Let K_1 and K_2 be the contact points of the tangents from M (so that X_1 lies on $\overline{MK_1}$ and X_2 lies on $\overline{MK_2}$) and perform an inversion around the incircle. As usual we denote the inverse with a star. Now A^*, B^*, C^* are respectively the midpoints of $\overline{EF}$, $\overline{FD}$, $\overline{DE}$, and as usual $\Gamma^* = (A^*B^*C^*)$ is the nine-point circle of $\triangle DEF$.

Clearly M^* is an arbitrary point on Γ^*; moreover, it is the midpoint of $\overline{K_1K_2}$. Now let us determine the location of T^*. Now we claim T^* is the point diametrically opposite A^* on Γ^*. We see that L^* is some point also on Γ^*. Moreover,

$$\measuredangle IL^*A^* = -\measuredangle IAL = 90^\circ.$$

But because L, I, T are collinear it follows that L^*, I^*, T^* are collinear, whence

$$\measuredangle TL^*A^* = \measuredangle I^*L^*A^* = 90^\circ$$

as desired. That means it is also the midpoint of $\overline{DH}$, where H is the orthocenter of triangle DEF.

It is now time to prove that M^*, X_1^*, X_2^*, T^* are concyclic. Dilating by a factor of 2 at D, it is equivalent to prove that D', K_1, K_2, and H are concyclic, where D' is the reflection of D over M^*. Reflecting around M^* it is equivalent to prove that D, K_2, K_1, and H' are concyclic.

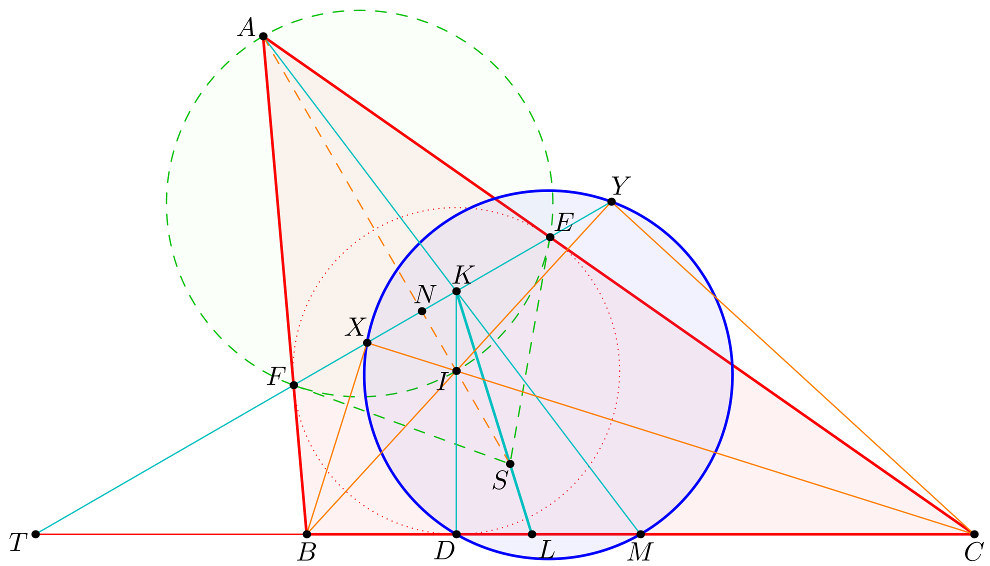
But the circumcircle of D, K_2 and K_1 is just Γ^* itself. Moreover our usual homothety between the nine-point circle Γ^* and the incircle implies that H' lies on Γ^* as well. So D, K_2, K_1, H' are concyclic on Γ^*. Thus M, X_1, X_2, and T are concyclic, which is what we wanted to show.

§11t Taiwan Quiz 2015/3J/6

In scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of $\triangle AEF$ at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter $\overline{ST}$ is orthogonal to the nine-point circle of triangle BIC.

(Available online at https://aops.com/community/p5087419.)

Let D be the foot from I to $\overline{BC}$. Let X, Y denote the feet from B, C to CI and BI. We can show that $BIFX$, $CIEY$ are cyclic, so that X and Y lie on $\overline{EF}$. Now let M be the midpoint of $\overline{BC}$, and ω the circumcircle of $DMXY$. The problem reduces to showing that S lies on the polar of T to ω.



Let $K = \overline{AM} \cap \overline{EF}$. It's well known (say by SL 2005 G6) that points K, I, D are collinear. Let N be the midpoint of $\overline{EF}$, and $L = \overline{KS} \cap \overline{BC}$. From

$$-1 = (AI; NS) \overset{K}{=} (TL; MD)$$

and

$$-1 = (TD; BC) \overset{I}{=} (TK; YX)$$

we find that $T = \overline{MD} \cap \overline{YX}$ is the pole of line $\overline{KL}$ with respect to ω, completing the proof.

> **Remark.** August Chen notes that it's possible to prove $(TK; XY) = -1$ by constructing the orthocenter H of $\triangle BIC$, and using the Ceva/Menelaus lemma on $\triangle HXY$.

¶ **Authorship comments.** This problem was constructed backwards. The points X, Y, K were added because I knew already that they led to the nice configuration in question. I then tried to see if I could construct any nice harmonic quadrilaterals. I already had $(TK;XY)$, so I took the other harmonic conjugate and thus arrived at L. The construction of S followed after that; it was the result of projecting through K onto the angle bisector. Thus arrived the problem, which had an astonishingly short formulation.

A Generating Code

§A.1 Database dump script (Python)

```python
import sys
import yaml
from von import api
from typing import Any

with open('data.yaml') as f:
    data: list[dict[str, Any]] = yaml.load(f, Loader=yaml.SafeLoader)

print(r'''\documentclass[11pt]{scrreprt}
\usepackage[sexy]{evan}
\renewcommand{\thesection}{\thechapter\alph{section}}
\usepackage{epigraph}
\renewcommand{\epigraphsize}{\scriptsize}
\renewcommand{\epigraphwidth}{60ex}

\begin{document}
\title{Auto-Generated EGMO Solutions Treasury}
\maketitle
\tableofcontents
''')

for d in data:
    problems: list[str] = d['problems']
    chapter_name: str = d['name']
    print(r'\chapter{Solutions for %s}' % chapter_name)
    print(r'\epigraph{%s}{%s}' % (d['quote'], d['quote_source']))

    for key in problems:
        if not api.has(key):
            print("MISSING", key, 'from chapter', d['chapter'], file=sys.stderr)
        else:
            print(r'\section{%s}' % key)
            print(api.get_statement(key))
            if (url := api.get(key).url) is not None:
                print(
                    r'\par\medskip\noindent\textsf{\footnotesize (Available online at'
                    '\n'
                    r'\url{' + url + '}.)}')

            print('\n')
            print(r'\hrulebar')
            print('\n')
            print(api.get_solution(key))
            print('\n')

print(r'''\appendix
```

```
\renewcommand{\thesection}{\thechapter.\arabic{section}}
\chapter{Generating Code}
\section{Database dump script (Python)}
\lstinputlisting[language=Python]{compile.py}
\newpage
\section{Input data}
\lstinputlisting{data.yaml}

\end{document}''')
```

§A.2 Input data


```yaml
- chapter: 1
  name: Angle Chasing
  quote: |
    I won't go easy on you, and I hope you won't go easy on me, either.
  quote_source: |
    Serral to Bunny before their semifinals match at
    \emph{DreamHack Starcraft 2 Masters} Atlanta 2022
  problems:
    - BAMO 1999/2
    - CGMO 2012/5
    - Canada 1991/3
    - Russia 1996/10.1
    - JMO 2011/5
    - Canada 1997/4
    - IMO 2006/1
    - USAMO 2010/1
    - IMO 2013/4
    - IMO 1985/1

- chapter: 2
  name: Circles
  quote: |
        \\


    \bigskip
    \emph{I've waited here every day \\
    But I ’dont know if I can tomorrow as well}
  quote_source: |
    \emph{Lullaby}, by Dreamcatcher
  problems:
    - USAMO 1990/5
    - BAMO 2012/4
    - JMO 2012/1
    - IMO 2008/1
    - USAMO 1997/2
    - IMO 1995/1
    - USAMO 1998/2
    - IMO 2000/1
    - Canada 1990/3
    - IMO 2009/2
    - Canada 2007/5
    - Iran TST 2011/1

- chapter: 3
  name: Lengths and Ratios
  quote: |
    I don't know what's weirder --- that you're fighting a stuffed animal,
    or that you seem to be losing.
  quote_source: |
    Susie Derkins, in \emph{Calvin and Hobbes}
  problems:
    - Shortlist 2006 G3
    - BAMO 2013/3
```

```yaml
    - USAMO 2003/4
    - USAMO 1993/2
    - EGMO 2013/1
    - APMO 2004/2
    - Shortlist 2001 G1
    - TSTST 2011/4
    - USAMO 2015/2

- chapter: 4
  name: Assorted Configurations
  quote: |
    We should switch from 5 answer choices to 6 answer choices
    so we can just bubble a lot of F's to express our feelings.
  quote_source: |
    Evan's reaction to the AMC edVistas website
  problems:
    - Hong Kong 1998
    - Shortlist 2003 G2
    - USAMO 1988/4
    - USAMO 1995/3
    - USA TST 2014/1
    - USA TST 2011/1
    - USAMO 2011/5
    - Japan 2009
    - Vietnam TST 2003/2
    - Sharygin 2013/16
    - APMO 2012/4
    - Shortlist 2002 G7

- chapter: 5
  name: Computational Geometry
  quote: |
    We both know we don't want to be here, so let's get this over with.
  quote_source: |
    Xiaoyu He, during a MOP 2013 test review
  problems:
    - APMO 2013/1
    - EGMO 2013/1
    - USAMO 2010/4
    - Iran 1999
    - CGMO 2002/4
    - IMO 2007/4
    - JMO 2013/5
    - CGMO 2007/5
    - Shortlist 2011 G1
    - IMO 2001/1
    - IMO 2001/5
    - IMO 2001/6

- chapter: 6
  name: Complex Numbers
  quote: |
    The real fun of living wisely is that you get to be smug about it.
  quote_source: |
    Hobbes, in \emph{Calvin and Hobbes}
  problems:
    - China TST 2011/2/1
```

```
    - USAMO 2015/2
    - China TST 2006/4/1
    - USA TST 2014/5
    - OMO 2013 F26
    - IMO 2009/2
    - APMO 2010/4
    - Shortlist 2006 G9
    - MOP 2006/4/1
    - Shortlist 1998 G6
    - ELMO SL 2013 G7

- chapter: 7
  name: Barycentric Coordinates
  quote: |
    I don't care if you're a devil in disguise!
    I love you all the same!
  quote_source: |
    Misa Amane, in \emph{Death Note: The Last Name}
  problems:
    - IMO 2014/4
    - EGMO 2013/1
    - ELMO SL 2013 G3
    - IMO 2012/1
    - Shortlist 2001 G1
    - USA TST 2008/7
    - USAMO 2001/2
    - TSTST 2012/7
    - December TST 2012/1
    - Sharygin 2013/20
    - APMO 2013/5
    - USAMO 2005/3
    - Shortlist 2011 G2
    - Romania TST 2010/6/2
    - ELMO 2012/5
    - USA TST 2004/4
    - TSTST 2012/2
    - IMO 2004/5
    - Shortlist 2006 G4

- chapter: 8
  name: Inversion
  quote: |
    Humans are like high templar.
    They're fragile, weak, and cause storms when they're mad.
    And they love giving feedback to others
    despite being unable to receive feedback themselves.
  quote_source: ""
  problems:
    - BAMO 2011/4
    # Iran 1996 # wrong Iran problem
    - Shortlist 2003 G4
    - NIMO 2014
    - EGMO 2013/5
    - Russia 2009/10.2
    - Shortlist 1997/9
    - IMO 1993/2
    - IMO 1996/2
```

```yaml
    - IMO 2015/3
    - ELMO Shortlist 2013 # FGOB

- chapter: 9
  name: Projective Geometry
  quote: |
    I don't think Jane Street would appreciate
    all their thousands of dollars going to fruit snacks.
  quote_source: |
    Debbie Lee, at MOP 2022
  problems:
    - TSTST 2012/4
    - Singapore TST
    - Canada 1994/5
    - Bulgaria 2001
    - ELMO SL 2012 G3
    - IMO 2014/4
    - Shortlist 2004 G8
    - Sharygin 2013/16
    - Shortlist 2004 G2
    - January TST 2013/2
    - Brazil 2011/5
    - ELMO SL 2013 G3
    - APMO 2008/3
    - ELMO SL 2014 G2 # AC / BD / GH
    - ELMO Shortlist 2014 # GI, HJ, B-symmedian
    - Shortlist 2005 G6

- chapter: 10
  name: Complete Quadrilaterals
  quote: |
        \\


    \emph{Look at the sky, ’Ill leave a piece containing my heart there \\
    So, call me when the time comes}
  quote_source: |
    \emph{PLEASE PLEASE}, by EVERGLOW
  problems:
    - NIMO 2014
    - USAMO 2013/1
    - Shortlist 1995 G8
    - USA TST 2007/1
    - USAMO 2013/6
    - USA TST 2007/5
    - IMO 2005/5
    - USAMO 2006/6
    - Balkan 2009/2
    - TSTST 2012/7
    - TSTST 2012/2
    - USA TST 2009/2
    - Shortlist 2009 G4
    - Shortlist 2006 G9
    - Shortlist 2005 G5

- chapter: 11
  name: Personal Favorites
```

```
quote: |
  How do you \emph{accidentally} rob a bank??
quote_source: |
  \emph{RWBY Chibi}, Season 3, Episode 1
problems:
  - Canada 2000/4
  - EGMO 2012/1
  - ELMO 2013/4
  - Sharygin 2012
  - USAMTS 3/3/24
  - MOP 2012
  - Sharygin 2013/21
  - ELMO 2012/1
  - Sharygin 2013/14
  - Bulgaria 2012
  - Sharygin 2013/15
  - Sharygin 2013/18
  - USA TST 2015/1
  - EGMO 2014/2
  - OMO 2013 W49
  - USAMO 2007/6
  - Sharygin 2013/19
  - USA TST 2015/6
  - Iran TST 2009/9
  - IMO 2011/6
  - Taiwan TST 2014/3J/3
  - Taiwan Quiz 2015/3J/6
```