Welcome to the 49th United States of America Mathematical Olympiad!

Congratulations on your invitation to the 2020 USA Math Olympiad (USAMO), the pinnacle event of the American Mathematics Competition. This is a wonderful accomplishment for which you can be very proud.

Preparing for the USAMO

If this is your first time taking the USAMO, welcome! Here are two actions you can take to prepare for the competition.

The first is to **look at some past USAMO problems**, so you know what to expect. Here are four that we picked to get you started, in increasing order of difficulty. We chose one problem from each of four major subject areas.

2012/1 (algebra) Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \ldots, a_n$ with

$$\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n),$$

there exist three that are the side lengths of an acute triangle.

2014/4 (combinatorics) Let $k > 0$ be an integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all grid cells are empty. Then the players alternately take turns with A moving first. In her move, A may choose two adjacent hexagons which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves.

2004/3 (geometry) For which real numbers $k > 0$ can one dissect a $1 \times k$ rectangle into two similar but noncongruent polygons?

2013/5 (number theory) Let $m, n > 0$ be integers. Prove there exists an integer $c > 0$ such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base 10.

(Note that 2012/1, 2014/4, 2004/3 are implicitly two-part problems; see item 3 on the next page.) The solutions to these four problems are included as a separate attachment. An exhaustive list of all past USAMO problems and solutions can be found online.[1]

The second is to get practice **reading and writing proofs**. The four solutions we included are intended to be models that you can base your own work off of. More resources can be found on the web, for example the article `aops.com/news/articles/how-to-write-a-solution`.

[1] Online, you may see harder problems using theorems you have never heard of. It's okay to try and learn what a few of these are if you're curious. But on the other hand we would advise against cramming too many advanced techniques in the coming weeks. Aiming for full scores is a multi-year process, and trying to squeeze this into a few weeks can end up doing more harm than good.

Some tips for game day

Here is a bit of advice on what to expect during the exam itself.

1. On each day, there will be three problems to solve in 4.5 hours. Each problem is worth 7 points.

2. For geometry problems, the first page of your submission should be a **large, in-scale diagram**, constructed with ruler and compass. This diagram is just as helpful to you as to the graders — an accurate diagram can help you make guesses about what conjectures might be true, and also prevent you from wasting time trying to prove a statement which is false.

3. Certain problems are **implicit two-part problems**, and you should recognize them. In addition to the previous problems, here are two more examples (which are actual research problems!).

   If a problem asks to find a *minimum* or *maximum*, then you need to show your answer is both achievable and optimal. For example, suppose you wanted to find the smallest integer n such that any positive integer is the sum of at most n fourth powers. If you think the answer is $n = 19$, then you would need to do two things:

   (a) Show that every integer is the sum of at most 19 fourth powers; and

   (b) Give an example of an integer which can't be written as the sum of 18 or fewer fourth powers.

   (This is called Waring's problem.)

   Similarly, if a problem asks you to *find all possible values*, then the problem also implicitly has two parts. For example, suppose you wanted to find all integers $n \geq 3$ for which a regular n-gon could be dissected into 50 triangles with equal area. If you think the answer is $n \in \{3, 4, 5, 10, 25, 50\}$, then you would need to again do two things:

   (a) give dissections for these values of n; and

   (b) also prove that a dissection is impossible for any other value of n.

   (This example comes from equidissection theory.)

   It's not uncommon that one half is much easier than the other half.

4. The problems are only loosely in increasing order of difficulty. There will be some students who find the second problem easier than the first, and there will be some students who find the third problem easier than the second. Therefore, we really encourage you to **briefly try all three problems each day, before deciding how to allocate your time**. This also gives you the chance to make partial progress on all three problems; for example, sometimes the last problem will be a two-part problem where one half is actually quite straightforward.

5. That said, you will usually earn more points by solving one problem completely than making minor progress on two or three problems. Plan your time accordingly.

6. **Actively double-check your work while writing up solutions**. Olympiad problems are very complex. Many people believe they have solved a problem and only later realize there was a mistake. An advantage of writing cleanly and carefully is that it will be much easier for you to notice errors.

For this reason, you should usually not postpone writing up solutions for too long, unless you are very confident in your solution; otherwise, you may not have time to fix any mistakes. Be especially wary of solutions that seem too good to be true, or that don't use all the conditions of the problem. If your solution is split into sub-lemmas, you can also try to double-check these lemmas individually, say by trying a few examples.

7. Finally, this might be obvious, but **please make sure you receive the right exam**! If you accidentally take the wrong exam, you will only receive credit for overlapping problems.

Closing remarks

The JMO and USAMO are extremely challenging exams. Median scores are often in the single-digits, and almost nobody will solve all six problems. Solving a single problem completely is already a great achievement.

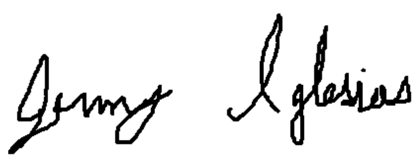
In fact, merely having the stamina to think deeply for the entire 9 hours is something to be proud of! The USAMO is a much greater test of endurance than the AMC or AIME, and you may not be used to thinking about a single problem for so long. Try to prepare yourself mentally for this, as much as you can.

Our editorial board has worked hard to choose six great problems we think are worthy of your sustained attention. The Putnam competition has the following quote in their letter to their problem committee:

> It used to be said that a Broadway musical was a success if the audience left the theater whistling the tunes. I want to see contestants leaving the Putnam whistling the problems. They should be vivid and striking enough to be shared with roommates and teachers.

We very much feel the same way about the USAMO, and we hope that our chosen problems are exciting enough that you will continue thinking about them even after the contest concludes. You may find that you can learn just as much, if not more, from working on the problems without a time limit.

Congratulations again on making it this far. Good luck and have fun!



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