Solutions to the selected USAMO problems

The USA(J)MO Editorial Board

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The solutions to all four problems we mentioned are included in full below. In addition to showing how to solve the problems, we think they are good templates to show how we expect a correct solution to be written. (Actually, they are a little bit on the verbose side, and during the competition you could get away with being more succinct. However, if in doubt it is always better to include more details.)

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§1 Solution to 2012/1

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Lemma. For positive integers m, we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the original problem. Denote by (†) the standing assumption that $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$. The solution is divided into two parts.

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \geq F_i a_1^2$. In particular, $a_n^2 \geq F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \leq n \cdot a_1$. Therefore we have $F_n \leq n^2$, and so $n \leq 12$, contradiction!

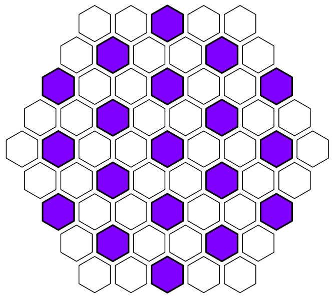
Proof that no $n \leq 12$ have the property. Assume that $n \leq 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \leq 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \geq F_j + F_i = a_j^2 + a_i^2$.

This problem was proposed by Titu Andreescu.

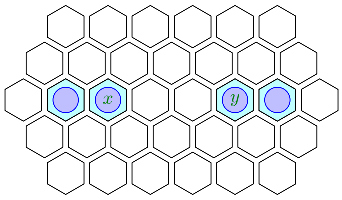
§2 Solution to 2014/4

The answer is $k = 6$. The solution is divided into two parts.

Proof that A cannot win if $k = 6$. We give a strategy for B to prevent A's victory. Shade in every third cell, as shown in the right figure. Then A can never cover two shaded cells simultaneously on her turn. Now suppose B always removes a counter on a shaded cell (and otherwise does whatever he wants). Then he can prevent A from ever getting six consecutive counters, because any six consecutive cells contain two shaded cells.

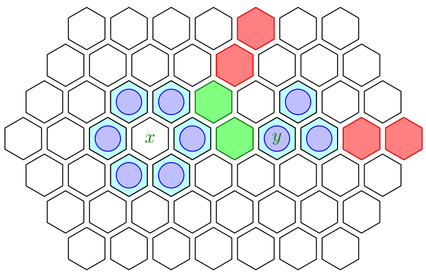


Example of a winning strategy for A when $k = 5$.
We describe a winning strategy for A explicitly. Note that after B's first turn there is one counter, so then A may create an equilateral triangle, and hence after B's second turn there are two consecutive counters. Then, on her third turn, A places a pair of counters two spaces away on the same line. Label the two inner cells x and y as shown below.



Now it is B's turn to move; in order to avoid losing immediately, he must remove either x or y. Then on any subsequent turn, A can replace x or y (whichever was removed) and add one more adjacent counter. This continues until either x or y has all its neighbors filled (we ask A to do so in such a way that she avoids filling in the two central cells between x and y as long as possible).

So, let's say without loss of generality (by symmetry) that x is completely surrounded by tokens. Again, B must choose to remove x (or A wins on her next turn). After x is removed by B, consider the following figure.



We let A play in the two marked green cells. Then, regardless of what move B plays, one of the two choices of moves marked in red lets A win. Thus, we have described a winning strategy when $k = 5$ for A.

This problem was proposed by Palmer Mebane.

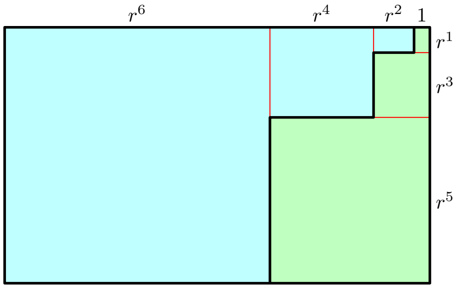
§3 Solution to 2004/3

Answer: the dissection is possible for every $k > 0$ except for $k = 1$.

Construction for $k > 1$. For every integer $n \geq 2$ and real number $r \geq 1$, we define a shape $\mathcal{R}(n, r)$ as follows.

- We start with a rectangle of width 1 and height r. To its left, we glue a rectangle of height r and width r^2 to its left.
- Then, we glue a rectangle of width $1 + r^2$ and height r^3 below our figure, followed by a rectangle of height $r + r^3$ and width r^4 to the left of our figure.
- Next, we glue a rectangle of width $1 + r^2 + r^4$ and height r^5 below our figure, followed by a rectangle of height $r + r^3 + r^5$ and width r^6 to the left of our figure.

...and so on, until we have $2n$ pieces. The picture $\mathcal{R}(3, r)$ is shown below.



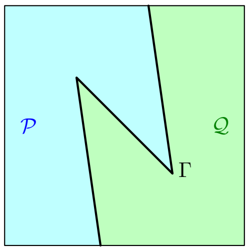
Observe that by construction, the entire shape $\mathcal{R}(n, r)$ is a rectangle which consists of two similar "staircase" polygons, with similarity ratio r. Note that $\mathcal{R}(n, r)$ is similar to a $1 \times f_n(r)$ rectangle where

$$f_n(r) = \frac{1 + r^2 + \cdots + r^{2n}}{r + r^3 + \cdots + r^{2n-1}}$$

is the aspect ratio of $\mathcal{R}(n, r)$.

Now, given some $k > 1$, choose n such that $1 + \frac{1}{n} < k$. Note $f_n(1) = 1 + \frac{1}{n}$ but $f_n(k) > k$. Since f_n is continuous, by the intermediate value theorem there exists some value of $r > 1$ with $f_n(r) = k$, as needed.

Construction for $k < 1$. Reduces to the previous case since a $1 \times k$ rectangle and a $1 \times k^{-1}$ rectangle are similar.



Proof of impossibility for $k = 1$. Suppose we have a square dissected into two similar polygons $\mathcal{P} \sim \mathcal{Q}$. Let Γ be their common boundary. By counting the number of sides of $\mathcal{P}$ and $\mathcal{Q}$ we see Γ must run from one side of the square to an opposite side (possibly ending at a corner of the square). We orient the figure so Γ runs from north to south, with $\mathcal{P}$ to the west and $\mathcal{Q}$ to the east. Let s be the longest length of a segment in Γ.

> **Claim.** The longest side length of $\mathcal{P}$ is $\max(s, 1)$. Similarly, the longest side length of $\mathcal{Q}$ is $\max(s, 1)$ as well.
>
> *Proof.* The only edges of $\mathcal{P}$ not in Γ are the west edge of our original square, which has length 1, and the north/south edges of $\mathcal{P}$ (if any), which have length at most 1. An identical argument works for $\mathcal{Q}$. □

It follows the longest sides of $\mathcal{P}$ and $\mathcal{Q}$ have the same length! Hence the two polygons are in fact congruent, ending the proof.

This problem was proposed by Ricky Liu.

§4 Solution to 2013/5

Some optional motivation.[1] To give an example, consider the case $m = 4$ and $n = 23$. It happens to be true that

$$10^3 \cdot \frac{4}{41} = 97 + \frac{23}{41}.$$

This equation implies that the repeating decimal representations of $\frac{4}{41}$ and $\frac{23}{41}$ will be cyclic shifts of each other by 3 places. Indeed, a calculation gives that

$$\frac{1}{41} = 0.\overline{02439}, \qquad \frac{4}{41} = 0.\overline{09756}, \qquad \frac{23}{41} = 0.\overline{56097}$$

where the bar denotes the usual repeating decimal.

However, if the number under the bar of $1/41$ is denoted by c (that is, $c = 2439$), then the number under the bar of $\frac{4}{41}$ is $4c$ and the number under the bar of $\frac{23}{41}$ is $23c$. So in the case $(m, n) = (4, 23)$, the choice $c = 2439$ works great.

Solution. To get the above argument to work for bigger values of m and n, we'll need to replace 41 with some larger denominator D satisfying three ingredients:

- $\gcd(D, 10) = 1$, so the decimal representation of $1/D$ is indeed periodic;
- $D > \max(m, n)$, so $\frac{m}{D}$ and $\frac{n}{D}$ are both in in the interval $(0, 1)$; and
- for some exponent e, $10^e \cdot \frac{m}{D} - \frac{n}{D}$ is an integer; this gave us the shifting property.

We prove this is possible now for *any* m and n.

Claim. There exists a number D satisfies which satisfies all three conditions.

Proof. Let's describe a recipe to find D. Suppose we pick some exponent e and define the number

$$A = 10^e n - m.$$

Suppose 2 divides m exactly $r \geq 0$ times. Then if $e > r$, it follows that 2 divides A exactly r times too. Similarly, if 5 divides m exactly $s \geq 0$ times, then as long as $e > s$ it follows that 5 divides A exactly s times too.

Now choose any e larger than $\max(r, s)$, so the previous paragraph applies, and also big enough that $A > 2^r 5^s \max(m, n)$. Then the number $D = \frac{A}{2^r 5^s}$ obtained by deleting all factors of 2 and 5 from A should work. Indeed, by construction, $\gcd(D, 10) = 1$ and $D > \max(m, n)$. And the particular form of A gives us

$$10^e \cdot \frac{n}{D} - \frac{m}{D} = \frac{A}{D} = 2^r 5^s$$

which is an integer. □

[1] During the USAMO, it is not necessary to include motivation like this if the solution stands without it. However, we chose to supply it anyways so you can see how one might come up with the solution, and to make it easier for you to follow along.

Now we take c to be the number under the bar of $1/D$ (leading zeros removed). Then the decimal representation of $\frac{m}{D}$ is the decimal representation of cm repeated (possibly including leading zeros). Similarly, $\frac{n}{D}$ has the decimal representation of cm repeated (possibly including leading zeros). Finally, since

$$10^e \cdot \frac{m}{D} - \frac{n}{D} \text{ is an integer}$$

it follows that these repeating decimal representations are rotations of each other by e places, so in particular they have the same number of nonzero digits.

This problem was proposed by Richard Stong. Incidentally, for many students the number 142857 might ring a bell, and is quite related to the problem proposed here. It is possible (but not necessary) to find a D which is *prime*, although this requires more care.