Solutions to the selected JMO problems

The USA(J)MO Editorial Board

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The solutions to all four problems we mentioned are included in full below. In addition to showing how to solve the problems, we think they are good templates to show how we expect a correct solution to be written. (Actually, they are a little bit on the verbose side, and during the competition you could get away with being more succinct. However, if in doubt it is always better to include more details.)

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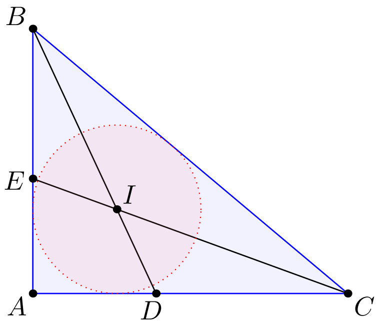
§1 Solution to 2010/J6

We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135^\circ$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90^\circ}{2} = 45^\circ.$$

So, $\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 135^\circ$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 + BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

However, $\sqrt{2}$ is irrational, and yet the above equations imply that $\sqrt{2} = \frac{(AB^2+AC^2)-(BI^2+CI^2)}{BI \cdot CI}$. This produces the desired contradiction.

This problem was proposed by Zuming Feng.

§2 Solution to 2012/J2

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Lemma. For positive integers m, we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the original problem. Denote by (†) the standing assumption that $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$. The solution is divided into two parts.

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \geq F_i a_1^2$. In particular, $a_n^2 \geq F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \leq n \cdot a_1$. Therefore we have $F_n \leq n^2$, and so $n \leq 12$, contradiction!

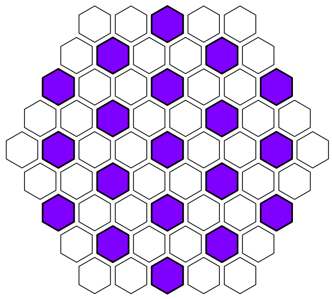
Proof that no $n \leq 12$ have the property. Assume that $n \leq 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \leq 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \geq F_j + F_i = a_j^2 + a_i^2$.

This problem was proposed by Titu Andreescu.

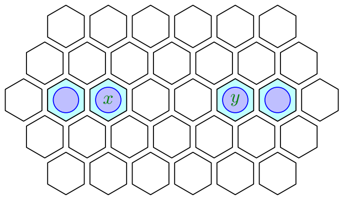
§3 Solution to 2014/J5

The answer is $k = 6$. The solution is divided into two parts.

Proof that A cannot win if $k = 6$. We give a strategy for B to prevent A's victory. Shade in every third cell, as shown in the right figure. Then A can never cover two shaded cells simultaneously on her turn. Now suppose B always removes a counter on a shaded cell (and otherwise does whatever he wants). Then he can prevent A from ever getting six consecutive counters, because any six consecutive cells contain two shaded cells.

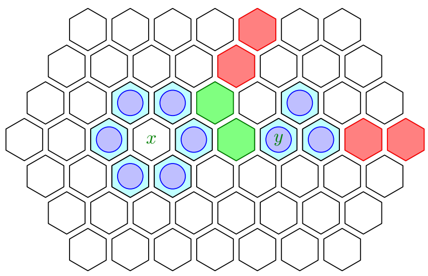


Example of a winning strategy for A when $k = 5$.
We describe a winning strategy for A explicitly. Note that after B's first turn there is one counter, so then A may create an equilateral triangle, and hence after B's second turn there are two consecutive counters. Then, on her third turn, A places a pair of counters two spaces away on the same line. Label the two inner cells x and y as shown below.



Now it is B's turn to move; in order to avoid losing immediately, he must remove either x or y. Then on any subsequent turn, A can replace x or y (whichever was removed) and add one more adjacent counter. This continues until either x or y has all its neighbors filled (we ask A to do so in such a way that she avoids filling in the two central cells between x and y as long as possible).

So, let's say without loss of generality (by symmetry) that x is completely surrounded by tokens. Again, B must choose to remove x (or A wins on her next turn). After x is removed by B, consider the following figure.



We let A play in the two marked green cells. Then, regardless of what move B plays, one of the two choices of moves marked in red lets A win. Thus, we have described a winning strategy when $k = 5$ for A.

This problem was proposed by Palmer Mebane.

§4 Solution to 2016/J2

We will prove that $\boxed{n = 20 + 2^{19} = 524308}$ fits the bill.

Claim. For this n we have $5^n \equiv 5^{20} \pmod{10^{20}}$.

Proof. Since 5^{20} divides $5^n - 5^{20}$, we just need to prove 2^{20} divides $5^n - 5^{20}$. This may be factored as

$$\begin{aligned} 5^{20+2^{19}} - 5^{20} &= 5^{20}\left(5^{2^{19}} - 1\right) \\ &= 5^{20}(5-1)(5+1)\left(5^2+1\right)\left(5^4+1\right)\dots\left(5^{2^{18}}+1\right) \end{aligned}$$

and each factor in parentheses is even, as needed.[1] □

Putting these two together now implies

$$5^n \equiv 5^{20} \pmod{10^{20}}.$$

In other words, the last 20 digits of 5^n will match the decimal representation of 5^{20}, with leading zeros.

However, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}$$

and hence the first 6 of those 20 digits will all be zero. This completes the proof! (To be concrete, it turns out that $5^{20} = 95367431640625$ and so the last 20 digits of 5^n will be 00000095367431640625.)

This problem was proposed by Evan Chen. Side note: despite the apparently simplicity of the solution, less than 10% of contestants solved it. It is not easy at all.

One way to think of this: the first time the digit 0 appears at all is $5^8 = 390625$. The fact that $625 = 5^4$ shows the key idea in miniature; $5^8 \equiv 625 \pmod{10000}$. The above problem uses the same approach with 4 replaced by the much larger 20, so that more zeros appear.

[1] An alternative approach for students who know Euler's theorem is to simply notice $\varphi(2^{20}) = 2^{19}$, where φ is the Euler phi function. Therefore $5^{2^{19}} \equiv 1 \pmod{2^{20}}$ and so $5^{2^{19}+20} \equiv 5^{20} \pmod{2^{20}}$. The hands-on proof gives a tad more; since $5-1 = 2^2$, in fact 2^{21} divides $5^{2^{19}} - 1$, not just 2^{20}.