Weird Geometry

Also known as: Revenge of the Pentagon

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§1 Lecture Notes

§1.1 Synopsis

Personal anecdote: in preparing this lecture, I looked through all previous examples of pentagon and hexagon problems I did involving some weird length condition. I found that 100% of these problems were susceptible to complex bash. Later I was told Zuming had similar results.

These "weird" geometry problems are a bit of a monkey wrench. The main advice I have is to **bash more**, *especially* if you have side conditions and an n-gon ($n \geq 5$). It is true that some of the problems require some synthetic observation before you are able to proceed. But the other direction holds equally often: many of these problems have no synthetic solution at all, and still more are basically impossible to solve synthetically in exam conditions even.

Consequently algebra will play a big role in many of these problems.

§1.2 Degrees of freedom

One thing that you'll see come up often is that even the algebra is interesting in some way: In particular, one often looks at **degrees of freedom**: the number of real numbers which are needed to specify the entire figure. (This is hard to make precise[1], but easy to get the hang of.) This quantity is especially interesting in "weird" geometry, since it's much harder to see at a glance.

Convention: In OTIS materials, we will consider figures only distinct up to translation and rotation, but *not* up to scaling. Thus for example,

- A triangle has three degrees of freedom[2]: it is uniquely determined by its side lengths. Or, it is uniquely determined by a side and two angles.
- A cyclic quadrilateral has four degrees of freedom; for example, it is uniquely determined by its side lengths.
- But a generic quadrilateral has *five* degrees of freedom. For example, it is uniquely determined by its four side lengths and the length of one of its diagonals.

As a reminder, every variable you set on the unit circle encodes one degree of freedom; so a typical *abc* triangle setup has three degrees of freedom in variables. Every variable you set which is *not* on the unit circle will encode *two* degrees of freedom: if x is a point, then your calculation will involve x and $\overline{x}$, so it effectively will have two variables in it. This makes sense: it takes two real numbers to specify a random point in $\mathbb{R}^2$.

If your variables encode N degrees of freedom, but the problem has only M degrees of freedom, there will be $N - M$ relations between them. It's nice to have $N = M$ (equivalently $N - M = 0$) but sometimes nothing can be done about it, especially in these "weird" geometry problems.

§1.3 Hidden geometric inequalities

Geometry problems which feel "overconstrained" (like USAMO 2002 below) end up being inequalities 90% of the time; the RMM 2017/6 problem below is the only exception I'm aware of. Still other times, degree-counting will lead us to a solution anyways (as in USAMO 2011).

[1] "Dimension of moduli space" is the way you make it precise.

[2] Fans of Cartesian and trig often prefer two degrees of freedom, because they use a different convention.

§1.4 One more tip

Lemma 1.1

If an n-gon is oriented, the sum of the vectors corresponding to the sides is zero.

This is obvious, but it will be a useful way to think about large n-gons in coordinate systems.

§1.5 Contest practice

Example 1.2 (USAMO 2002)

Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

Walkthrough. This is an example of a problem which has a strange result in the degrees of freedom.

Consider the set of triangles satisfying the condition: we would normally expect it to be a two-dimensional set, since a triangle has three degrees of freedom and we have imposed one constraint.

However, the problem statement is telling us that in fact the set of triangles has exactly one degree of freedom (determined up to scaling). Roughly, the fact that $3 - 1 \neq 1$ leads us to believe this should probably be an inequality.

(a) Let $x = s - a$, $y = s - b$, $z = s - c$ and rewrite the condition in terms of x, y, z.

(b) Decide whether to replace the $=$ sign with $\geq$ or $\leq$ to get an inequality that you think is true.

(c) Prove the inequality from (b) using Cauchy-Schwarz.

(d) Determine the equality case in (c), and use it to find the ratio $x : y : z$.

(e) Conclude that $BC : CA : AB = 13 : 40 : 45$.

Example 1.3 (USAMO 2011)

In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

Walkthrough. We begin by dimension-counting.

(a) Show that the space of hexagons satisfying the hypotheses of the problem should have three degrees of freedom.

(b) Find an example of a hexagon (even degenerate, meaning 180° angles allowed) that satisfies the result. (Hint: triforce shape.)

(c) Generalize the example in (b) to a three-dimensional space of hexagons. You should find that in all your examples, the three diagonals concur at the orthocenter of $\triangle ACE$.

The main idea now is that from (a) we know the hypothesis space has three dimensions, but from (c) we have already found a three-dimensional space. So we show they coincide. In other words, call a hexagon *satisfactory* if it satisfies the hypothesis, and *excellent* if it comes from (c) (and hence is satisfactory). We will show that all satisfactory hexagons are excellent, thereby solving the problem.

To do this, we are going to construct a *phantom hexagon*.

(d) Let $ABCDEF$ be a satisfactory hexagon. Show that one can construct an excellent hexagon $A'B'C'D'E'F'$ which has the same angles as $ABCDEF$.

(e) Consider the unit complex numbers in the directions $\overrightarrow{BA}$ and $\overrightarrow{DE}$ respectively and let $\vec{x}$ denote their sum. Define $\vec{y}$, $\vec{z}$ similarly. Show that $AB \cdot \vec{x} + BC \cdot \vec{y} + CD \cdot \vec{z} = A'B' \cdot \vec{x} + B'C' \cdot \vec{y} + C'D' \cdot \vec{z} = 0$. (This is again using the idea that a hexagon is six vectors with vanishing sum.)

(f) Show that $\vec{x}$, $\vec{y}$, $\vec{z}$ are nonzero. (This is where the non-parallel hypothesis is used.)

(g) Prove that no two of $\vec{x}$, $\vec{y}$, $\vec{z}$ are scalar multiples of each other. (This is annoying and you can skip this part if you want. The non-parallel hypothesis is used here, too.)

(h) Use (e) and (g) to prove that $AB : BC : CD = A'B' : B'C' : C'D'$, and thus the hexagons $ABCDEF$ and $A'B'C'D'E'F'$ are similar.

(i) Come up with a counterexample if the condition on non-parallel sides is dropped.

Example 1.4 (USA TST 2016)

Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

Walkthrough. We will use complex numbers with a, b, c the unit circle.

(a) Find three distinct examples of points P that work.

(b) Show that the locus of P "should" have 0 degrees of freedom, i.e. it should be a finite set of points. (This is again dimension counting.)

(c) Write down an equation in $\mathbb{C}$ for p corresponding to the condition that $\angle PAB + \angle PBC + \angle PCA = 90^\circ$. You don't have to worry about configuration issues: just ensure that the equation you have is valid for all points P in the interior of ABC.

(d) Write down an equation in $\mathbb{C}$ for p corresponding to AA_1, BB_1, CC_1 are concurrent.

(e) Rewrite (or redo) both equations so that they are degree 3 polynomials in p and $\overline{p}$.

The curves you found in (c) and (d) are called the **McCay cubic** and **Darboux cubic**, respectively. You can take for granted they are distinct and nondegenerate (proving this is surprisingly obnoxious; I've been putting off doing it for a few years now).

Once you have this, we can view the curves as two equations in $\mathbb{C}^2$ in two variables p and $q = \overline{p}$. After this, it follows from **Bézout theorem** that the curves intersect in exactly $3 \cdot 3 = 9$ points, with multiplicity.

(f) If the incenter I lies on both curves, what other three points must also lie on both curves?

(g) Find three more points which satisfy the equations (algebraically; the geometric interpretation won't make much sense here).

(h) Conclude that the nine points you found in (a), (f), (g) are the only solutions, and identify the ones that do lie inside ABC.

Example 1.5 (RMM 2017)

Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points in the interior of segments AB, BC, CD, DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Walkthrough. This is one of the longest solutions to an example problem, so don't worry if this walkthrough seems intimidating. It is.

Let $PQRS$ be any quadrilateral (possibly concave or self-intersecting!), and let $O = \overline{PR} \cap \overline{QS}$. Let W, X, Y, Z be the feet from O to $\overline{SP}$, $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$.

We say that a quadruple of points

$$(A, B, C, D) \in \overline{OW} \times \overline{OX} \times \overline{OY} \times \overline{OZ}$$

is *okay* if P, Q, R, S lies on lines AB, BC, CD, DA, and none of A, B, C, D coincide with O.

The big surprise of the problem comes out at the beginning:

(a) Give a heuristic reason why you expect for most convex quadrilaterals $PQRS$ there should be exactly one okay quadruple. (Count degrees of freedom.)

(b) Rephrase the RMM problem in terms of okay quadruples.

(c) Prove that for *almost all* convex quadrilaterals $PQRS$, even if not cyclic, there exists an okay quadruple! (This is really tricky. Try looking at orthocenters. A good Geogebra diagram will be super helpful; you'll notice quickly that $ABCD$ looks like it should be a parallelogram.)

(d) Why doesn't this contradict the problem statement? Double check your answer to (a), remembering that the point P lie *inside* segment AB, not just on line AB.

(e) Show that the construction you found in (b) never satisfies the problem condition.

Okay, that gives us a plan: we'll show that in nearly all cases, there is exactly one okay quadruple, as we predicted in (a). Since in (e) we verified this okay quadruple is not actually valid, this will more or less solve the problem.

Well, the devil is in the details. Let's consider a quadrilateral $PQRS$ now. Use complex numbers with O the origin and w, x, y, z as free variables. Then $A = c_1 w$ for some real number c_1, and similarly $B = c_2 x$, $C = c_3 y$, $D = c_4 z$, and we want $P \in \overline{AB}$ et cetera.

(f) Find p, q, r, s in terms of w, x, y, z.

(g) Write down the complex equations for $P \in \overline{AB}$, et cetera.

(h) The equations you got were quadratic, but $c_i \neq 0$. Use this to to get a new system of equations in $1/c_1$, et cetera, which is linear and for which you seek some solution.

(i) Prove that the determinant of the resulting system is zero if and only if

$$\prod_{\text{cyc}} (2w\overline{w} - \overline{w}x - w\overline{x}) = \prod_{\text{cyc}} (2x\overline{x} - \overline{w}x - w\overline{x}) .$$

Thus, for any quadrilateral not satisfying this property, we are already done.

(j) Let $\langle \bullet, \bullet \rangle$ denote the *dot product*. Show that (i) is equivalent to

$$\prod_{\text{cyc}} \langle w, w - x \rangle = \prod_{\text{cyc}} \langle x, w - x \rangle .$$

(k) Show that (j) is equivalent to

$$\prod_{\text{cyc}} \cos \angle RPQ = \prod_{\text{cyc}} \cos \angle RPS.$$

There is some angle chasing to do here.

Call a quadrilateral *extraordinary* if it satisfies this condition — so, which quadrilaterals are still left to deal with?

(l) Show that any cyclic quadrilateral is extraordinary (which we already expected, given the problem statement).

(m) Show that any quadrilateral with perpendicular diagonals is extraordinary.

(n) Show that any quadrilateral with parallel diagonals is extraordinary.

(o) Prove that (k) is equivalent to

$$\prod_{\text{cyc}} \langle q - p, p - r \rangle = \prod_{\text{cyc}} \langle s - p, p - r \rangle .$$

(This is a lot like part (k) in reverse.)

(p) Rewrite part (o) in terms of complex numbers p, q, r, s, and their conjugates. The resulting expression should be degree eight. (This is a lot like part (j) in reverse.)

(q) Rewrite the conditions in parts (l), (m), (n) using complex numbers too.

(r) Factor the expression you obtained in part (p). (You can actually argue this without any computation at all, using only the geometric observations and $4 + 2 + 2 = 8$).

Okay, almost done. We only have to eliminate the case that $PQRS$ has perpendicular diagonals, but is not cyclic.

(s) Coordinate bash this final case to show that it has no okay quadruples at all.

(t) Fun corollary (optional): four lines dissect a convex quadrilateral into nine smaller quadrangles to make it into a 3×3 array of quadrangular cells. Label these cells 1 through 9 from left to right and top to bottom. If the first eight cells have perpendicular diagonals, then so does the ninth.

§2 Practice Problems

Instructions: Solve [32♣]. If you have time, solve [42♣]. Problems with red weights are mandatory.

> Cheetahs can run.
> Eagles can fly.
> People can try.
> But that's about it.
>
> Natsuki's poem *Eagles Can Fly* in *Doki Doki Literature Club*

05IMO1 [3♣] **Required Problem 1** (IMO 2005/1). Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

(The solution to IMO 2005/1 is extremely clean if done correctly, and quite messy but eventually tractable otherwise. Even if you have a longer solution, be sure to check the solution notes afterwards.)

11SLG7 [5♣] **Required Problem 2** (Shortlist 2011). Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

13SLG3 [3♣] **Problem 3** (Shortlist 2013). In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \le \max\{\angle BAC, \angle ABC\}$.

98SLG6 [3♣] **Problem 4** (Shortlist 1998). Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

20CYBER6 [3♣] **Problem 5** (Cyberspace Competition 2020). Find all integers $n \ge 3$ such that the following statement is true: if $\mathcal{P}$ is a convex n-gon such that $n-1$ of its sides have equal length and $n-1$ of its angles have equal measure, then $\mathcal{P}$ is a regular polygon.

19SLG5 [3♣] **Problem 6** (Shortlist 2019 G5). Convex pentagon $ABCDE$ obeys $CD = DE$ and $\angle EDC \ne 2\angle ADB$. Point P is chosen inside the pentagon such that $AP = AE$ and $BP = BC$. Suppose that

$$[BCD] + [ADE] = [ABD] + [ABP]$$

where $[\Delta]$ is the area of triangle Δ. Show that P lies on diagonal $\overline{CE}$.

09USATST4 [5♣] **Problem 7** (USA TST 2009). Let ABP, BCQ, CAR be three non-overlapping triangles erected outside of acute triangle ABC. Let M be the midpoint of segment AP. Given that $\angle PAB = \angle CQB = 45^\circ$, $\angle ABP = \angle QBC = 75^\circ$, $\angle RAC = 105^\circ$, $RQ^2 = 6CM^2$, compute AC/AR.

16AMO5 [5♣] **Problem 8** (USAMO 2016/5). An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

13SLG5 [5♣] **Problem 9** (Shortlist 2013). Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

04AMO6 [5♣] **Problem 10** (USAMO 2004). A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

10SLG5 [5♣] **Problem 11** (Shortlist 2010). Let $ABCDE$ be a convex pentagon such that $\overline{BC} \parallel \overline{AE}$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of $\overline{CE}$, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

03IMO3 [5♣] **Required Problem 12** (IMO 2003). Each pair of opposite sides of convex hexagon has the property that the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that the hexagon is equiangular.

72PTNMB5 [5♣] **Problem 13** (Putnam 1972). Let $ABCD$ be a nondegenerate tetrahedron. Suppose that $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$. Prove that $AB = CD$ and $BC = DA$.

21USEMO3 [9♣] **Problem 14** (USEMO 2021/3). Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

[1♣] **Mini Survey.** Fill out feedback on the OTIS-WEB portal when submitting this problem set. Any thoughts on problems (e.g. especially nice, instructive, easy, etc.) or overall comments on the unit are welcome.

In addition, if you have any suggestions for problems to add, or want to write hints for one problem you really liked, please do so in the ARCH system!

The maximum number of [♣] for this unit is [65♣], including the mini-survey.

§3 Solutions to the walkthroughs

§3.1 Solution 1.2, USAMO 2002

Let $x = s - a$, $y = s - b$, $z = s - c$ in the usual fashion, then the equation reads

$$x^2 + 4y^2 + 9z^2 = \left(\frac{6}{7}(x + y + z)\right)^2.$$

However, by Cauchy-Schwarz, we have

$$\left(1 + \tfrac{1}{4} + \tfrac{1}{9}\right)\left(x^2 + 4y^2 + 9z^2\right) \ge (x + y + z)^2$$

with equality if and only if $1 : \frac{1}{2} : \frac{1}{3} = x : 2y : 3z$, id est $x : y : z = 1 : \frac{1}{4} : \frac{1}{9} = 36 : 9 : 4$. This is equivalent to $y + z : z + x : x + y = 13 : 40 : 45$.

> **Remark.** You can tell this is not a geometry problem because you eliminate the cotangents right away to get an algebra problem. . . and then you realize the problem claims that one equation can determine three variables up to scaling, at which point you realize it has to be an inequality (otherwise degrees of freedom don't work). So of course, Cauchy-Schwarz. . .

§3.2 Solution 1.3, USAMO 2011

We present the official solution. We say a hexagon is *satisfying* if it obeys the six conditions; note that intuitively we expect three degrees of freedom for satisfying hexagons.

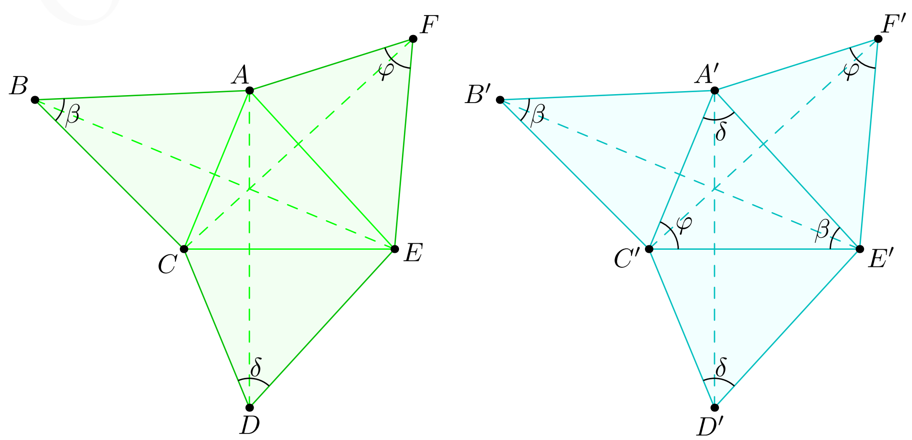
Main idea:

> **Claim** — In a satisfying hexagon, B, D, F are reflections of A, C, E across the sides of $\triangle ACE$.

(This claim looks plausible because every excellent hexagon is satisfying, and both configuration spaces are three-dimensional.) Call a hexagon of this shape "excellent"; in a excellent hexagon the diagonals clearly concur (at the orthocenter).

Set $\beta = \angle B$, $\delta = \angle D$, $\varphi = \angle F$.

Now given a satisfying hexagon $ABCDEF$, construct a "phantom hexagon" $A'B'C'D'E'F'$ with the same angles which is excellent (see figure). This is possible since $\beta + \delta + \varphi = 180°$.



Then it would suffice to prove that:

Lemma

A satisfying hexagon is uniquely determined by its angles up to similarity. That is, at most one hexagon (up to similarity) has angles β, δ, γ as above.

Proof. Consider any two satisfying hexagons $ABCDEF$ and $A'B'C'D'E'F'$ (not necessarily as constructed above!) with the same angles. We show they are similar.

To do this, consider the unit complex numbers in the directions $\overrightarrow{BA}$ and $\overrightarrow{DE}$ respectively and let $\vec{x}$ denote their sum. Define $\vec{y}$, $\vec{z}$ similarly. Note that the condition $\overline{AB} \nparallel \overline{DE}$ implies $\vec{x} \neq 0$, and similarly. Then we have the identities

$$AB \cdot \vec{x} + CD \cdot \vec{y} + EF \cdot \vec{z} = A'B' \cdot \vec{x} + C'D' \cdot \vec{y} + E'F' \cdot \vec{z} = 0.$$

So we would obtain $AB : CD : EF = A'B' : C'D' : E'F'$ if only we could show that $\vec{x}$, $\vec{y}$, $\vec{z}$ are not multiples of each other (linear dependency reasons). This is a tiresome computation with arguments, but here it is.

First note that none of β, δ, φ can be $90°$, since otherwise we get a pair of parallel sides. Now work in the complex plane, fix a reference such that $\vec{A} - \vec{B}$ has argument 0, and assume $ABCDEF$ are labelled counterclockwise. Then

- $\vec{B} - \vec{C}$ has argument $\pi - \beta$
- $\vec{C} - \vec{D}$ has argument $-(\beta + 3\varphi)$
- $\vec{D} - \vec{E}$ has argument $\pi - (\beta + 3\varphi + \delta)$
- $\vec{E} - \vec{F}$ has argument $-(4\beta + 3\varphi + \delta)$

So the argument of $\vec{x}$ has argument $\frac{\pi-(\beta+3\varphi+\delta)}{2} \pmod{\pi}$. The argument of $\vec{y}$ has argument $\frac{\pi-(5\beta+3\varphi+\delta)}{2} \pmod{\pi}$. Their difference is $2\beta \pmod{\pi}$, and since $\beta \neq 90°$, it follows that $\vec{x}$ and $\vec{y}$ are not multiples of each other; the other cases are similar. □

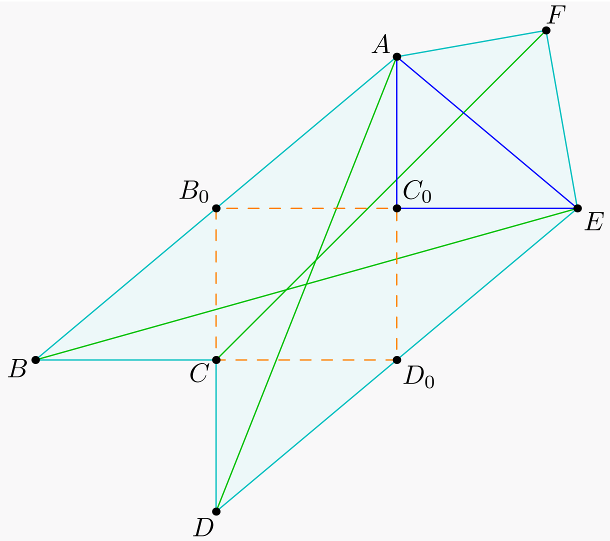
Then the lemma implies $ABCDEF \sim A'B'C'D'E'F$ and we're done.

Remark. This problem turned out to be known already. It appears in this reference:

Nikolai Beluhov, *Matematika*, 2008, issue 6, problem 3.

It was reprinted as Kvant, 2009, issue 2, problem M2130; the reprint is available at http://kvant.ras.ru/pdf/2009/2009-02.pdf.

Remark. The vector perspective also shows the condition about parallel sides cannot be dropped. Here is a counterexample from Ryan Kim in the event that it is.



By adjusting the figure above so that the triangles are right isosceles (instead of just right), one also finds an example of a hexagon which is satisfying and whose diagonals are concurrent, but which is *not* excellent.

§3.3 Solution 1.4, USA TST 2016

In complex numbers with ABC the unit circle, it is equivalent to solving the following two cubic equations in p and $q = \overline{p}$:

$$\begin{aligned}(p-a)(p-b)(p-c) &= (abc)^2(q-1/a)(q-1/b)(q-1/c) \\ 0 &= \prod_{\text{cyc}}(p+c-b-bcq) + \prod_{\text{cyc}}(p+b-c-bcq).\end{aligned}$$

Viewing this as two cubic curves in $(p,q) \in \mathbb{C}^2$, by *Bézout's Theorem* it follows there are at most nine solutions (unless both curves are not irreducible, but it's easy to check the first one cannot be factored). Moreover it is easy to name nine solutions (for ABC scalene): the three vertices, the three excenters, and I, O, H. Hence the answer is just those three triangle centers I, O and H.

Remark. On the other hand it is not easy to solve the cubics by hand; I tried for an hour without success. So I think this solution is only feasible with knowledge of algebraic geometry.

Remark. These two cubics have names:

- The locus of $\angle PAB + \angle PBC + \angle PCA = 90^\circ$ is the **McCay cubic**, which is the locus of points P for which P, P^*, O are collinear.
- The locus of the pedal condition is the **Darboux cubic**, which is the locus of points P for which P, P^*, L are collinear, L denoting the de Longchamps point.

Assuming $P \neq P^*$, this implies P and P^* both lie on the Euler line of $\triangle ABC$, which is possible only if $P = O$ or $P = H$.

Some other synthetic solutions are posted at https://aops.com/community/c6h1243902p6368189.

§3.4 Solution 1.5, RMM 2017

We present a solution by degree-counting complex numbers, then a solution by moving points.

(Unimportant remark: here is a fun corollary mentioned by the problem author. Four lines dissect a convex quadrilateral into nine smaller quadrangles to make it into a 3×3 array of quadrangular cells. Label these cells 1 through 9 from left to right and top to bottom. If the first eight cells have perpendicular diagonals, then so does the ninth.)

¶ First solution by complex numbers (Evan Chen) Suppose $PQRS$ is any quadrilateral (possibly concave or self-intersecting!). Let $O = \overline{PR} \cap \overline{QS}$. Let W, X, Y, Z be the feet from O to $\overline{SP}$, $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$.

We say that a quadruple of points

$$(A, B, C, D) \in \overline{OW} \times \overline{OX} \times \overline{OY} \times \overline{OZ}$$

is *okay* if P, Q, R, S lies on lines AB, BC, CD, DA, and $A, B, C, D \neq O$. If additionally $ABCD$ is convex and P, Q, R, S lie on the segments AB, BC, CD, DA, we say the quadruple is *excellent*. Thus the problem asks us to show any quadrilateral with an excellent quadruple is cyclic.

We say $PQRS$ is *ordinary* if there exists exactly one okay quadruple, *exceptional* otherwise.

We show now that "most quadrilaterals are ordinary".

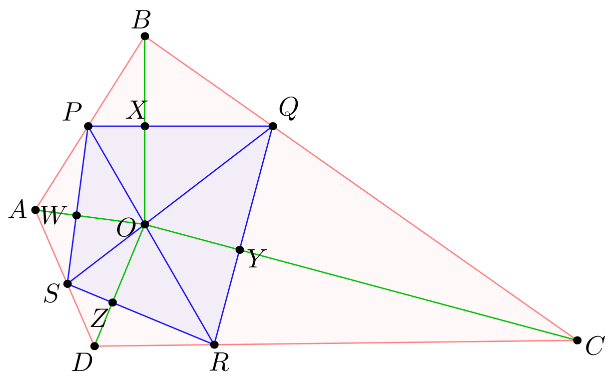
Claim — A quadrilateral is exceptional if and only if

$$\prod_{\text{cyc}} \cos \angle RPQ = \prod_{\text{cyc}} \cos \angle RPS.$$

Proof. We apply complex numbers with variables w, x, y, z and with O at the origin. To compute P, note that the foot O to $\overline{WX}$ is $\frac{\overline{w}x - w\overline{x}}{2(\overline{w}-\overline{x})}$, so we conclude

$$p = \frac{2wx(\overline{w} - \overline{x})}{\overline{w}x - w\overline{x}}.$$

Now, let $a = c_1 w$, $b = c_2 x$, $c = c_3 y$, $d = c_4 z$ where c_1, c_2, c_3, c_4 are real numbers.



Then P, A, B are collinear exactly if

$$0 = \det \begin{bmatrix} w & \overline{w} & \frac{1}{c_1} \\ x & \overline{x} & \frac{1}{c_2} \\ p & \overline{p} & 1 \end{bmatrix} \iff 0 = \det \begin{bmatrix} w & \overline{w} & 1/c_1 \\ x & \overline{x} & 1/c_2 \\ wx(\overline{w} - \overline{x}) & -\overline{wx}(w - x) & \frac{\overline{w}x - w\overline{x}}{2} \end{bmatrix}$$

Let $\frac{1}{c_1} = w\overline{w}\gamma_1$ (since $c_1 \neq 0$), and so on. Thus we obtain

$$(\overline{w}(w-x) + w(\overline{w}-\overline{x}))\,\gamma_1 - (\overline{x}(w-x) + x(\overline{w}-\overline{x}))\,\gamma_2 = \frac{(\overline{w}x - w\overline{x})^2}{2w\overline{w}x\overline{x}}.$$

and the three cyclic equations.

Now the quadrilateral is exceptional if and only if this equation does not have exactly one solution. This is equivalent to the 4×4 determinant of this system vanishing, id est

$$\prod_{\text{cyc}} (2w\overline{w} - \overline{w}x - w\overline{x}) = \prod_{\text{cyc}} (2x\overline{x} - \overline{w}x - w\overline{x}) \iff \prod_{\text{cyc}} \langle w, w-x \rangle = \prod_{\text{cyc}} \langle x, x-w \rangle$$

where we use the notation $\langle z_1, z_2 \rangle = \frac{1}{2}(\overline{z_1}z_2 + z_1\overline{z_2})$ for the dot product $|z_1||z_2|\cos\angle(z_1,z_2)$. Then cancelling the product $\prod_{\text{cyc}} |OW||WX| = \prod_{\text{cyc}} |OX||WX|$ and using $\angle OWX = \angle RPQ$ et cetera, it's equivalent to

$$\prod_{\text{cyc}} \cos\angle RPQ = \prod_{\text{cyc}} \cos\angle RPS$$

as desired. □

We now follow up by classifying all exceptional quadrilaterals.

> **Claim** — A quadrilateral $PQRS$ (possibly concave or self-intersecting) is exceptional if and only if it is cyclic, or $\overline{PR} \perp \overline{QS}$ or $\overline{PR} \parallel \overline{QS}$ (which means $PQRS$ is self-intersecting).

Proof. First it's easy to check that all quadrilaterals mentioned are exceptional. We show they are the only ones by complex numbers. Let p, q, r, s be variables. Then in the same fashion as before:

$$\prod_{\text{cyc}} \cos\angle OPQ = \prod_{\text{cyc}} \cos\angle OPS \iff \prod_{\text{cyc}} \langle q-p, p-r \rangle - \prod_{\text{cyc}} \langle s-p, p-r \rangle = 0.$$

Now, the above is a polynomial of degree eight in p, q, r, s, $\overline{p}$, $\overline{q}$, $\overline{r}$, $\overline{s}$. However, it must have the following factors:

- $(p-q)(r-q)(\overline{p}-\overline{s})(\overline{r}-\overline{s}) - (\overline{p}-\overline{q})(\overline{r}-\overline{q})(p-s)(r-s)$, corresponding to $PQRS$ cyclic.
- $(p-r)(\overline{q}-\overline{s}) - (\overline{p}-\overline{r})(q-s)$, corresponding to $\overline{PR} \parallel \overline{QS}$.
- $(p-r)(\overline{q}-\overline{s}) + (\overline{p}-\overline{r})(q-s)$, corresponding to $\overline{PR} \perp \overline{QS}$.

Since the total degree is $4+2+2=8$, we are done unless the cosine expression is identically zero. This latter case is dispelled by e.g. taking $PQRS$ to be a parallelogram which is not a rhombus or rectangle. □

Thus nearly all quadrilaterals have a unique okay quadruple. The surprise is that this quadruple is never excellent:

> **Claim** — The unique okay quadruple of an ordinary quadrilateral $PQRS$ is not excellent.

Proof. Select A, B, C, D to be the *orthocenters* of $\triangle SOP$, $\triangle POQ$, $\triangle QOR$ and $\triangle ROS$ (noting that we have $A, B, C, D \neq O$ since $\overline{PR} \not\perp \overline{QS}$). Thus we've constructed a okay quadruple which is not excellent. $\square$

Therefore all that remains is to check that if a quadrilateral has perpendicular diagonals but is not cyclic, then it has no okay quadruples at all. This can be checked by Cartesian coordinates, which we outline before: set $O = (0, 0)$, $P = (0, p)$, $Q = (-q, 0)$, $R = (0, -r)$, $S = (s, 0)$. We may pick A and B such that $\overline{AB} \parallel \overline{QS}$, hence $A = (p^2/s, s)$ and then $D = \overline{AS} \cap \overline{OZ} = \frac{ps}{p^2+pr-s^2}(r, -s)$. Similarly, $C = \overline{BQ} \cap \overline{OX} = \frac{-pq}{p^2+pr-q^2}(r, q)$, (by replacing s with $-q$ everywhere). Now the points C, D, $R = (0, -r)$ are collinear if and only if

$$0 = \det \begin{bmatrix} prs & -ps^2 & p^2 + pr - s^2 \\ -pqr & -pq^2 & p^2 + pr - q^2 \\ 0 & -r & 1 \end{bmatrix} = pq(p + r)(q + s)(pr - qs)$$

which amounts to $PQRS$ being cyclic, as desired.

¶ Second solution by moving points (Anant Mudgal, un-edited) First we prove the following result.

Lemma

Let POQ be a non-isosceles triangle with circumcenter O_1 and assume $\angle POQ \neq 90^\circ$. Let ℓ_P, ℓ_Q be lines through P and Q respectively, perpendicular to $\overline{PQ}$. Suppose R, S lie on $\overline{OQ}, \overline{OP}$ respectively; such that $\overline{RS} \perp \overline{OO_1}$. Let $X \in \ell_P, Y \in \ell_Q$ with $\overline{OX} \perp \overline{PS}$ and $\overline{OY} \perp \overline{QR}$. Let $Z_P = \overline{XS} \cap \overline{OO_1}, Z_Q = \overline{YR} \cap \overline{OO_1}$. Then $Z_P = Z_Q$.

Proof. Move R with parameter r; $R \mapsto S$ is projective. Also $S \mapsto X$ and $R \mapsto Y$ are projective. Thus, Z_P moves on $\overline{OO_1}$ with parameter $\frac{p(r)}{q(r)}$ where p, q are polynomials of degree no more than 2. Thus, $Z_P = Z_Q$ is a polynomial equation in r of degree no more than 4. In order to show it is an identity, we only need to check five cases: $R = R'$ for any of $R' = O, R' = \infty, R' = P, \angle OR'Q = \angle OQP, \angle OR'Q = 90^\circ$ is clearly a solution. Thus, we are done! $\square$

Let $O = \overline{PR} \cap \overline{QS}$. Let ℓ_{PQ} be the line through O perpendicular to $\overline{PQ}$. Define $\ell_{QR}, \ell_{RS}, \ell SP$ similarly. For now, suppose $\overline{PO} \perp \overline{OQ}$ is not the case.

Suppose $A \in \ell_{PQ}$; define $B \overset{\text{def}}{\coloneqq} \overline{PA} \cap \ell_{PQ}$, $C \overset{\text{def}}{\coloneqq} \overline{QB} \cap \ell_{QR}$, $D \overset{\text{def}}{\coloneqq} \overline{RC} \cap \ell_{RS}$ and $E \overset{\text{def}}{\coloneqq} \overline{SD} \cap \ell_{SP}$. Define $f : \ell_{PQ} \mapsto \ell_{PQ}$ and $f(A) = E$ for all A is a projective map.

Claim — If $PQRS$ is cyclic then f is the identity map.

Proof. If $PQRS$ is a rectangle then we are done by symmetry. WLOG $OP \neq OQ$ (else we could define f with respect to other lines). Note that $A = E$ when $A = O$ and when A coincides with the orthocenter of $\triangle POQ$. By the lemma, $A = E$ also when A lies on the line at infinity. Thus, f is indeed the identity map. $\square$

Onto the problem. Note that f has three fixed points; the vertex A of the convex quadrilateral $ABCD$, the orthocenter of triangle POS and the point O. Thus, f is an identity. We show that the only quadrilaterals for which f is an identity are cyclic. Indeed, fix P, Q, S, O and move point R on line $\overline{PO}$ with parameter r; fix $\overline{AB} \parallel \overline{QS}$. Then $R \mapsto C$ and $R \mapsto D$ are projective maps (redefining D to lie on $\overline{AS}$).

Now R, C, D collinear is a cubic (at best) equation in r. This fails at $r = \infty$ so it is not an identity and at most three values of r work. Now $R = P, R = O$ and $PQRS$ cyclic work and so nor more roots exist. Thus, $PQRS$ is cyclic.

Remark. If $\overline{PR} \perp \overline{QS}$ then $PQRS$ must be cyclic in order for the condition to hold. We can prove this by a simple coordinate bash. Note the following results.

- Let ℓ_1, ℓ_2, ℓ_3 be lines concurrent at O and $P_1(t), P_2(t), P_3(t)$ be rational-linear functions (projective maps on a point moving with parameter t) denoting positions of points P_1, P_2, P_3 on these lines. Then P_1, P_2, P_3 collinear is a polynomial equation in t of degree ≤ 3. Indeed, take a homography mapping O to a point at infinity. Then these lines become parallel. Now we just wish to solve $\overrightarrow{P_3} = \lambda\overrightarrow{P_1} + (1-\lambda)\overrightarrow{P_2}$ for fixed λ (expressing ℓ_3 by section formula in ℓ_1, ℓ_2). This is clearly of degree ≤ 3 in t.
- Let ℓ_1, ℓ_2, ℓ_3 be lines concurrent at O and $P_1(t), P_2(t)$ be rational-linear functions (projective maps on a point moving with parameter t) denoting positions of points P_1, P_2 on their locii. Then $P_3 = \ell_3 \cap \overline{P_1P_2}$ is parametrised by $\frac{p(r)}{q(r)}$ where p, q have degree at most 2. Indeed, take a homography mapping O to a point at infinity. Then these lines become parallel. Now $\overrightarrow{P_3} = \lambda\overrightarrow{P_1} + (1-\lambda)\overrightarrow{P_2}$ for fixed λ (expressing ℓ_3 by section formula in ℓ_1, ℓ_2). Thus, P_3 has the desired parametric form.