Solution Notes for DNY-NTCONSTRUCT

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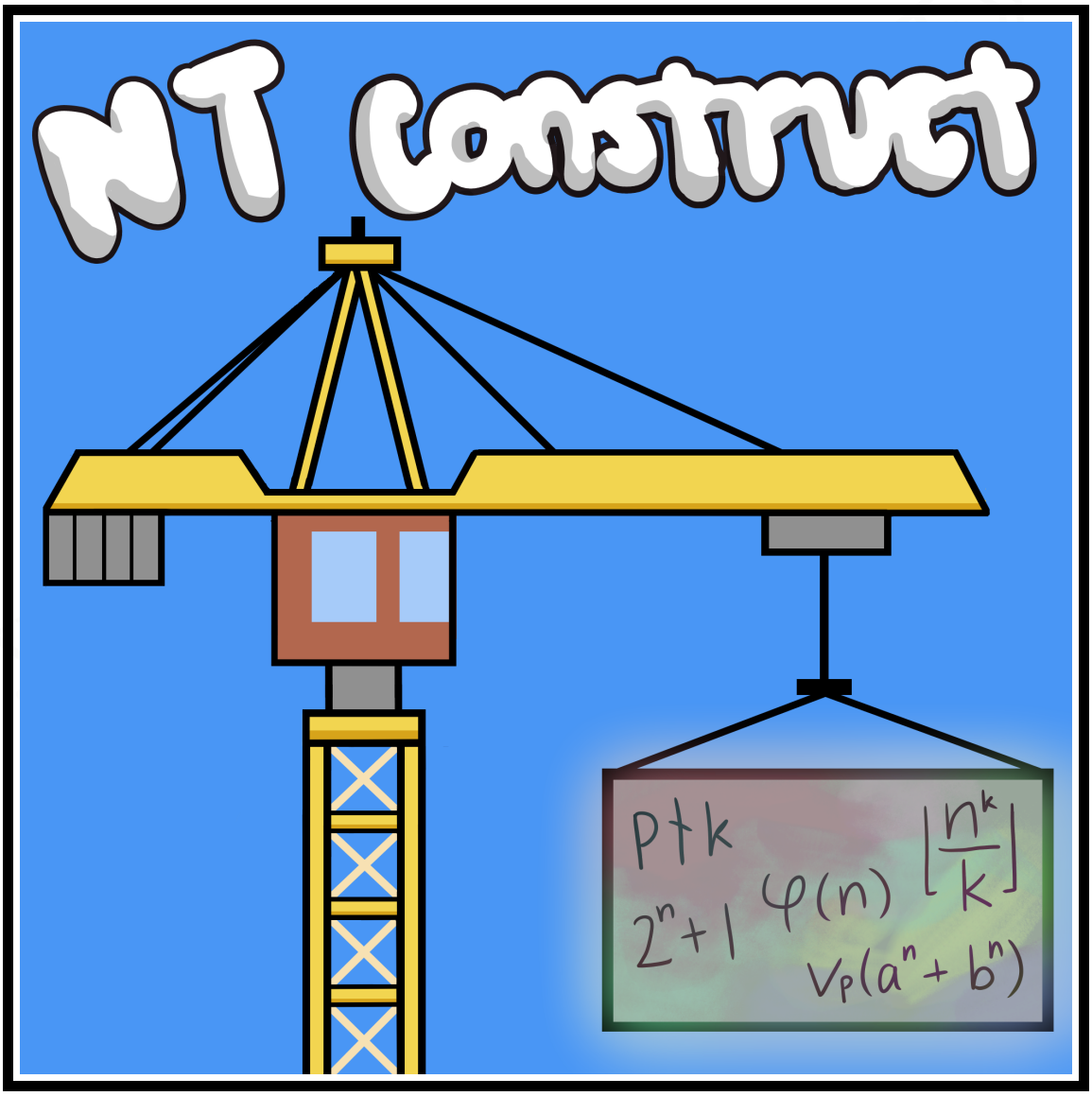
26 March 2023
DNY-SOL-NTCONSTRUCT



Contents

§1 USAMO 2017/1 (17AMO1)

Available online at https://aops.com/community/p8108366.

Problem statement

Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

One construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d+2^d}{d+2}$. Then set

$$a = \frac{x+d}{2}, \qquad b = \frac{x-d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$\begin{aligned} a+b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\ &\iff b^{a-b} \equiv 1 \pmod{a+b} \\ &\iff b^d \equiv 1 \pmod{d+2b} \\ &\iff (-2)^d \equiv d^d \pmod{d+2b} \\ &\iff d+2b \mid d^d + 2^d. \end{aligned}$$

So it would be enough that

$$d + 2b = \frac{d^d+2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d+2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$.

Remark. Ryan Kim points out that in fact, $(a, b) = (2n-1, 2n+1)$ is always a solution.

§2 USMCA 2019/1 (19USMCA1)

Problem statement

Kelvin the Frog and Alex the Kat are playing a game on an initially empty blackboard. Kelvin begins by writing a digit. Then, the players alternate inserting a digit anywhere into the number currently on the blackboard, including possibly a leading zero (e.g. 12 can become 123, 142, 512, 012, etc.). Alex wins if the blackboard shows a perfect square at any time, and Kelvin's goal is prevent Alex from winning. Does Alex have a winning strategy?

The answer is no, Kelvin can prevent a perfect square from ever appearing. There are several strategies; here is one.

Claim — Kelvin wins by initially writing the number 7, and then always adding either 7 or 8 to the end.

Proof. Alex clearly can't win on his first turn. Now, suppose that Alex leaves the number $A > 1$ on the board on his nth turn; we contend that Kelvin can prevent Alex from leaving a square on his $(n+1)$st turn as well.

Indeed, if Kelvin writes 7 or 8 as advertised, then he gets either $10A + 7$ or $10A + 8$. As no square ends in 7 or 8, the only way Alex could win on his $(n+1)$st turn is if $100A + 70 + d_7$ was a square, or $100A + 80 + d_8$ was a square. But no two squares exceeding 100 can differ by less than 20, so one of these cases is winning for Kelvin. □

Remark. As $876^2 = 767376$, it is not possible to simply insert 7's in every other digit.

§3 Shortlist 2007 N2 (07SLN2)

Available online at https://aops.com/community/p1187198.

Problem statement

Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

Just let $k = b^2$, so $b \equiv C^n \pmod{b^2}$. Hence $C^n = b(bx+1)$, but $\gcd(b, bx+1) = 1$ so $b = A^n$ for some A.

§4 APMO 2009/4 (09APMO4)

Available online at https://aops.com/community/p1434408.

Problem statement

Prove that for any positive integer n, there exists an arithmetic progression

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \dots, \quad \frac{a_n}{b_n}$$

of rational numbers, such that the $2n$ numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are pairwise distinct, and moreover $\gcd(a_i, b_i) = 1$ for every i.

Let $d = p_1 \dots p_n$ be the product of n primes, each prime larger than n. Then select an x with $x \equiv -i \pmod{p_i}$, for $i = 1, \dots, n$, and with x large in terms of d.

Consider the progression

$$\frac{x+1}{d}, \quad \frac{x+2}{d}, \quad \dots, \quad \frac{x+n}{d}$$

We claim it works.

Then, in the first fraction p_1 cancels from both the numerator and denominator, and that is the only cancellation (since $p_1 > n$). In general, the reduced ith fraction has

$$a_i = \frac{x+i}{p_i}$$
$$b_i = \frac{d}{p_i} = p_1 \dots p_{i-1} p_{i+1} \dots p_n.$$

Obviously b_i and a_i are pairwise distinct. Moreover if x is large enough, then $a_i > d$ for all i while $b_i < d$ for all i. This completes the proof.

§5 Shortlist 2017 N2 (17SLN2)

Available online at https://aops.com/community/p10632294.

Problem statement

Let p be a fixed prime number. Ankan and Ryan play the following turn-based game, with Ankan moving first. On their turn, each player selects an index $i \in \{0, \dots, p-1\}$ not chosen on a previous turn, and a digit $a_i \in \{0, \dots, 9\}$. This continues until all indices have been chosen (hence for p turns). Then, Ankan wins if the number

$$N = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_{p-1}10^{p-1}$$

is divisible by p; otherwise Ryan wins. For each prime p, determine which player has the winning strategy.

The first player Ankan can always win.

Assume first $\gcd(p, 10) = 1$, and let e be the order of $10 \pmod p$. Ankan begins by choosing $a_{p-1} = 0$.

Now let $p - 1 = de$. We consider two cases:

- If e is even, then $10^{e/2} \equiv -1 \pmod p$. Ankan imagines pairing the indices $\{0, 1, \dots, p-2\}$ into pairs which differ by $e/2$ in the obvious way (there are d pairs of 2 each). Now whenever Ryan picks a number a_i, Ankan selects the corresponding index j and sets $a_j = a_i$. As $10^j a_j + 10^i a_i \equiv 0 \pmod p$ this strategy wins.
- If e is odd, then d must be even. Ankan imagines pairing the indices $\{0, 1, \dots, p-2\}$ into pairs which differ by e in the obvious way (there are $d/2$ pairs of 2 each). Now whenever Ryan picks a number a_i Ankan selects the corresponding index j and sets $a_j = 9 - a_i$. Thus $10^j a_j + 10^i a_i \equiv 9 \cdot 10^i \pmod p$. So the final number is a multiple of $9 \dots 9 = 10^e - 1$ which is divisible by p.

If $p = 2$ or $p = 5$, Ankan just picks $a_0 = 0$ and wins. Thus Ankan has the winning strategy.

> **Remark.** One can phrase this solution without the use of orders d and e; it's merely casework on the value of $10^{\frac{1}{2}(p-1)}$.

§6 China 2019/2 (19CHN2)

Available online at https://aops.com/community/p11293588.

Problem statement

A *Pythagorean triple* is a set of three distinct positive integers $\{a, b, c\}$ which satisfies $a^2 + b^2 = c^2$. Prove that if P and Q are Pythagorean triples then there exists a finite sequence $P_0, \dots, P_n$ of Pythagorean triples satisfying $P = P_0$, $Q = P_n$, and $P_i \cap P_{i+1} \neq \varnothing$ for each $i = 0, \dots, n-1$.

Write $P \sim Q$ if $P \cap Q \neq \varnothing$. We say P and Q are *connected* if there exists a path as in the problem statement. Both these relations are obviously mutual.

We denote the triple $\{3n, 4n, 5n\}$ by $B(n)$. Note every Pythagorean triple has an element divisible by 4 (by looking modulo 8), hence intersects some $B(n)$. Thus it suffices to show that $B(n)$ is connected to $B(1)$ for every n.

Claim — The triples $B(n)$ and $B(2n)$ are connected for any integer $n > 0$.

Proof. We have

$$\begin{aligned} B(2n) = \{6n, 8n, 10n\} &\sim \{8n, 15n, 17n\} \sim \{9n, 12n, 15n\} \\ &\sim \{5n, 12n, 13n\} \sim \{3n, 4n, 5n\} = B(n). \square \end{aligned}$$

Claim — Let $p > 0$ be an odd integer, and $d > 0$ any integer. Then $B(dp)$ and $B\left(d \cdot \frac{p^2-1}{2}\right)$ are connected.

Proof. We have

$$\begin{aligned} B(dp) = \{d \cdot 3p, d \cdot 4p, d \cdot 5p\} &\sim \left\{d \cdot 4p, d \cdot 2(p^2-1), d \cdot 2(p^2+1)\right\} \\ &\sim \left\{d \cdot 3 \cdot \frac{p^2-1}{2}, d \cdot 4 \cdot \frac{p^2-1}{2}, d \cdot 5 \cdot \frac{p^2-1}{2}\right\} \\ &= B\left(d \cdot \frac{p^2-1}{2}\right). \square \end{aligned}$$

Indeed, let n be any integer. If n is even then $B(n)$ is connected to $B(n/2)$. Else if $p > 2$ is the *largest* prime factor of n, then $B(n)$ is connected to $B(n/p \cdot \frac{p^2-1}{2})$.

We claim that if we repeat this procedure, then eventually each $B(n)$ arrives at $B(1)$. Indeed, define the *complexity* of n to be the ordered pair $(p, \nu_p(n))$; then the complexity of n decreases lexicographically as we iterate the above procedure.

§7 Shortlist 2013 N3 (13SLN3)

Available online at https://aops.com/community/p3544101.

Problem statement

Prove that there exist infinitely many positive integers n such that the largest prime divisor of n^4+n^2+1 is equal to the largest prime divisor of $(n+1)^4+(n+1)^2+1$.

Define $f(n) = n^2+n+1$. Then

$$n^4+n^2+1 = (n^2+n+1)(n^2-n+1) = f(n)f(n-1).$$

So it suffices to show that $\operatorname{maxp} f(n)$ is at least the larger of $\operatorname{maxp} f(n-1)$ and $\operatorname{maxp} f(n+1)$ infinitely often, where $\operatorname{maxp}\bullet$ returns the largest prime divisor.

If not, either $\operatorname{maxp} f(1), \operatorname{maxp} f(2), \dots$ is eventually strictly increasing or strictly decreasing. Since the latter is impossible for integer sequences, we only need to show this sequence cannot increase monotonically. But $f(n^2) = f(n)f(n-1)$, so $\operatorname{maxp} f(n)^2$ is at most $\max(\operatorname{maxp} f(n), \operatorname{maxp} f(n-1))$, so the sequence cannot be strictly increasing at any time.

§8 BAMO 2011/5 (11BAMO5)

Available online at https://aops.com/community/p13035697.

Problem statement

Decide whether there exists a row of Pascal's triangle containing four pairwise distinct numbers a, b, c, d such that $a = 2b$ and $c = 2d$.

An example is $\binom{203}{68} = 2\binom{203}{67}$ and $\binom{203}{85} = 2\binom{203}{83}$.

To get this, the idea is to look for two adjacent entries and two entries off by one, and solving the corresponding equations. The first one is simple:

$$\binom{n}{j} = 2\binom{n}{j-1} \implies n = 3j - 1.$$

The second one is more involved:

$$\begin{aligned}
\binom{n}{k} &= 2\binom{n}{k-2} \\
\implies (n-k+1)(n-k+2) &= 2k(k-1) \\
\implies 4(n-k+1)(n-k+2) &= 8k(k-1) \\
\implies (2n-2k+3)^2 - 1 &= 2\left((2k-1)^2 - 1\right) \\
\implies (2n-2k+3)^2 - 2(2k-1)^2 &= -1
\end{aligned}$$

Using standard methods for the Pell equation:

- $(7+5\sqrt{2})(3+2\sqrt{2}) = 41+29\sqrt{2}$. So $k = 15$, $n = 34$, doesn't work.
- $(41+29\sqrt{2})(3+2\sqrt{2}) = 239+169\sqrt{2}$. Then $k = 85$, $n = 203$.

§9 TSTST 2012/5 (12TSTST5)

Available online at https://aops.com/community/p2745867.

Problem statement

A rational number x is given. Prove that there exists a sequence $x_0, x_1, x_2, \dots$ of rational numbers with the following properties:

(a) $x_0 = x$;

(b) for every $n \geq 1$, either $x_n = 2x_{n-1}$ or $x_n = 2x_{n-1} + \frac{1}{n}$;

(c) x_n is an integer for some n.

Think of the sequence as a process over time. We'll show that:

> **Claim —** At any given time t, if the denominator of x_t has some odd prime power $q = p^e$, then we can delete a factor of p from the denominator, while only adding powers of two to the denominator.

(Thus we can just delete off all the odd primes one by one and then double appropriately many times.)

Proof. The idea is to add only fractions of the form $(2^k q)^{-1}$.

Indeed, let n be large, and suppose $t < 2^{r+1}q < 2^{r+2}q < \dots < 2^{r+m}q < n$. For some binary variables $\varepsilon_i \in \{0, 1\}$ we can have

$$x_n = 2^{n-t}x_t + c_1 \cdot \frac{\varepsilon_1}{q} + c_2 \cdot \frac{\varepsilon_2}{q} \cdots + c_s \cdot \frac{\varepsilon_m}{q}$$

where c_i is some power of 2 (to be exact, $c_i = \frac{2^{n-2^{r+i}q}}{2^{r+1}}$, but the exact value doesn't matter).

If m is large enough the set $\{0, c_1\} + \{0, c_2\} + \dots + \{0, c_m\}$ spans everything modulo p. (Actually, Cauchy-Davenport implies $m = p$ is enough, but one can also just use Pigeonhole to notice some residue appears more than p times, for $m = O(p^2)$.) Thus we can eliminate one factor of p from the denominator, as desired. □

§10 Shortlist 2014 N4 (14SLN4)

Available online at https://aops.com/community/p5083569.

Problem statement

Let $n > 1$ be an integer. Prove that there are infinitely many integers $k \geq 1$ such that

$$\left\lfloor \frac{n^k}{k} \right\rfloor$$

is odd.

If n is odd, then we can pick any prime p dividing n, and select $k = p^m$ for sufficiently large integers m.

Now suppose n is even. Choose any integer $e \geq 1$ and let p be an odd prime dividing $n^{2^e} - 2^e$ (since $n^{2^e} \neq 2^{e+1}$). Then

$$n^{2^e p} \equiv 2^e \pmod{2^e p}$$

since $2^e \mid n^{2^e p}$ holds, and also $(n^{2^e})^p \equiv n^{2^e} \equiv 2^e \pmod{p}$. So that is the remainder.

Then we can take $k = 2^e p$ and then

$$\left\lfloor \frac{n^k}{k} \right\rfloor = \frac{n^k - 2^e}{k}$$

is odd.

§11 USA TST 2007/4 (07USATST4)

Available online at https://aops.com/community/p982018.

Problem statement

Determine whether or not there exist positive integers a and b such that a does not divide $b^n - n$ for all positive integers n.

The answer is no.

In fact, for any fixed integer b, the sequence

$$b,\ b^b,\ b^{b^b},\ \ldots$$

is eventually constant modulo any integer. (This follows by induction on the exponent: for it to be eventually constant modulo a, it is enough to be eventually constant modulo $\varphi(a)$, hence modulo $\varphi(\varphi(a))$, etc.)

Therefore if n is a suitably tall power-tower of b's, then we will have $b^n \equiv n \pmod{a}$.

§12 China TST 2018/2/4 (18CHNTST24)

Available online at https://aops.com/community/p9659765.

Problem statement

Let k, M be positive integers such that $k-1$ is not squarefree. Prove that there exists a positive real number α such that $\lfloor \alpha \cdot k^n \rfloor$ and M are relatively prime for any positive integer n.

Let $p^2 \mid k-1$ be prime and let $d = \frac{k-1}{p}$. Consider the number

$$\alpha = N + \frac{1}{p} = N + 0.\overline{ddd\ldots}_k$$

in base k. We claim it works for a suitable integer N.

Indeed, we have

$$\lfloor \alpha k^n \rfloor = k^n N + d \cdot \frac{k^n - 1}{k-1} = \left(N + \frac{1}{p}\right) k^n - \frac{1}{p}.$$

If we pick N such that $p \nmid N$, then the middle expression is not divisible by p (since d is divisible by p). Moreover, we can select N such that $q \mid N + p^{-1}$ for every prime $q \mid M$ other than p. Thus the Chinese remainder theorem completes the problem.

> **Remark** (Example)**.** If $k = 10$, and $M = 2 \cdot 3 \cdot 5 \cdot 7$, then one could take $\alpha = 23.33333\ldots$.

> **Remark** (Repeating base k mistake)**.** It is tempting to choose $\alpha = N + 0.\overline{ddd\ldots}_k$ in general, but one has to be careful in this case that $d \neq k-1$ because this would actually cause $\alpha = N + 1$.

§13 EGMO 2018/2 (18EGMO2)

Available online at https://aops.com/community/p10185417.

Problem statement

Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A (not necessarily distinct). Prove that there are infinitely many pairs of integers $x \geq 2$ and $y \geq 2$ for which

$$f(xy) < f(x) + f(y).$$

One of many constructions: let $n = 2^e + 1$ for $e \equiv 5 \pmod{10}$ and let $x = 11$, $y = n/11$ be our two integers.

We prove two lemmas:

Claim — For any $m \geq 2$ we have $f(m) \geq \lceil \log_2 m \rceil$.

Proof. This is obvious. □

It follows that $f(n) = e + 1$, since $n = \frac{n}{n-1} \cdot 2^e$.

Claim — $f(11) = 5$.

Proof. We have $11 = \frac{33}{32} \cdot \frac{4}{3} \cdot 2^3$. So it suffices to prove $f(11) > 4$.

Note that a decomposition of 11 must contain a fraction at most $\frac{11}{10} = 1.1$. But $2^3 \cdot 1.1 = 8.8 < 11$, contradiction. □

To finish, note that

$$f(11) + f(n/11) \geq 5 + \log_2(n/11) = 1 + \log_2(16n/11) > 1 + e = 1 + f(n).$$

Remark. Most solutions seem to involve picking n such that $f(n)$ is easy to compute. Indeed, it's hard to get nontrivial lower bounds other than the log, and even harder to actually come up with complicated constructions. It might be said the key to this problem is doing as little number theory as possible.

§14 USAMO 2006/5 (06AMO5)

Available online at https://aops.com/community/p490682.

Problem statement

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

We will think about the problem in terms of finite sequences of jumps $(s_1, s_2, \ldots, s_\ell)$, which we draw as

$$1 = x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \ldots \xrightarrow{s_\ell} x_\ell$$

where $s_k = x_k - x_{k-1}$ is the length of some hop. We say the sequence is *valid* if it has the property required by the problem: for each k, either $s_k = 1$ or $s_k = 2^{m_{x_{k-1}}+1}$.

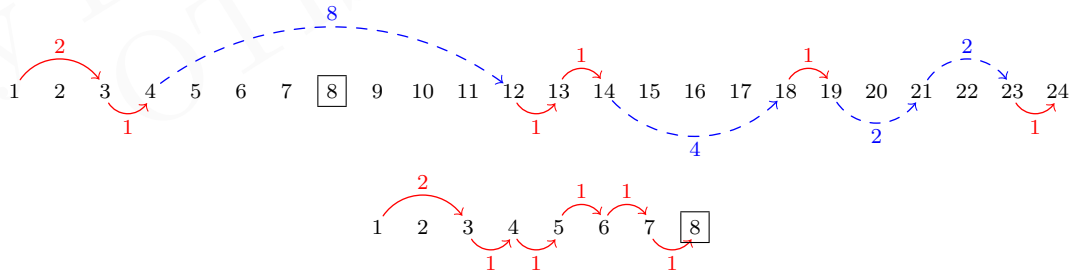
An example is shown below.

Lemma

Let $(s_1, \ldots, s_\ell)$ be a sequence of jumps. Suppose we delete pick an index k and exponent $e > 0$, and delete any jumps after the kth one which are divisible by 2^e. The resulting sequence is still valid.

Proof. We only have to look after the kth jump. The launching points of the remaining jumps after the kth one are now shifted by multiples of 2^e due to the deletions; so given a jump $x \xrightarrow{s} x+s$ we end up with a jump $x' \xrightarrow{s} x'+s$ where $x - x'$ is a multiple of 2^e.

But since $s < 2^e$, we have $\nu_2(x') < e$ and hence $\nu_2(x) = \nu_2(x')$ so the jump is valid. □



Now let's consider a valid path to $2^i k$ with ℓ steps, say

$$1 = x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \ldots \xrightarrow{s_\ell} x_\ell = 2^i \cdot k$$

where $s_i = x_i - x_{i-1}$ is the distance jumped.

We delete jumps in the following way: starting from the largest e and going downwards until $e = 0$, we delete all the jumps of length 2^e which end at a point exceeding the target 2^i.

By the lemma, at each stage, the path remains valid. We claim more:

Claim — Let $e \geq 0$. After the jumps of length greater than 2^e are deleted, the resulting end-point is at least 2^i, and divisible by $2^{\min(i,e)}$.

Proof. By downwards induction. Consider any step where *some* jump is deleted. Then, the end-point must be strictly greater than $x = 2^i - 2^e$ (i.e. we must be within 2^e of the target 2^i).

It is also divisible by $2^{\min(i,e)}$ by induction hypothesis, since we are changing the end-point by multiples of 2^e. And the smallest multiple of $2^{\min(i,e)}$ exceeding x is 2^i. □

On the other hand by construction when the process ends the reduced path ends at a point at most 2^i, so it is 2^i as desired.

Therefore we have taken a path to $2^i k$ and reduced it to one to 2^i by deleting some jumps. This proves the result.

§15 Brazil 2015/3 (15BRA3)

Available online at https://aops.com/community/p5469253.

Problem statement

Given an integer $n > 1$ and its prime factorization $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, its *false derivative* is defined by

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} \ldots \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many integers $n > 2$ such that $f(n) = f(n-1) + 1$.

The idea behind the construction is as follows:

Claim — Let m be an integer and let

$$x = 169 \cdot 78m - 25$$
$$y = 27 \cdot 78m - 4.$$

If x and y are squarefree, then $27x = 169y + 1$ and

$$f(27x) = f(169y) + 1.$$

Proof. Note that $3 \nmid x$ and $13 \nmid y$. Then $f(27x) = 3 \cdot 3^2 = 27$ and $f(169y) = 2 \cdot 13 = 26$, as needed. □

Therefore, it is sufficient to show that there are infinitely many integers m for which x and y as defined above are squarefree.

Fix a large integer M and consider choices of $m \in \{1, \ldots, M\}$. For each prime p, the number of m for which $p^2 \mid x$ or $p^2 \mid y$ is at most $2 \left\lceil \frac{M}{p^2} \right\rceil$, and is zero if $p^2 > \max(x, y)$. So, the total number of invalid choices of $m \in \{1, \ldots, M\}$ is upper bounded by

$$\sum_{p=5}^{O(\sqrt{M})} 2 \left\lceil \frac{M}{p^2} \right\rceil < 2M \cdot \sum_{p \geq 5} \frac{1}{p^2} + O\left(\frac{\sqrt{M}}{\log M} \right) < 0.99M$$

for large enough M. This implies the result.

§16 RMM 2012/4 (12RMM4)

Available online at https://aops.com/community/p2617973.

Problem statement

Prove there are infinitely many integers n such that n does not divide 2^n+1, but divides $2^{2^n+1}+1$.

Zsig hammer! Define the sequence $n_0, n_1, \dots$ as follows. Set $n_0 = 3$, and then for $k \geq 1$ we let $n_k = pn_{k-1}$ where p is a primitive prime divisor of $2^{2^{n_{k-1}}+1}+1$ (by Zsigmondy). For example, $n_1 = 57$.

This sequence of n_k's works for $k \geq 1$, by construction.

It's very similar to IMO 2000 Problem 5.

§17 IMO 2004/6 (04IMO6)

Available online at https://aops.com/community/p99760.

Problem statement

We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n which have an alternating multiple.

If $20 \mid n$, then clearly n has no alternating multiple since the last two digits are both even. We will show the other values of n all work.

The construction is just rush-down do-it. The meat of the solution is the two following steps.

Claim (Tail construction) — For every even integer $w \geq 2$,

- there exists an even alternating multiple $g(w)$ of 2^{w+1} with exactly w digits, and
- there exists an even alternating multiple $h(w)$ of 5^w with exactly w digits.

(One might note this claim is implied by the problem, too.)

Proof. We prove the first point by induction on w. If $w = 2$, take $g(2) = 32$. In general, we can construct $g(w+2)$ from $g(w)$ by adding some element in

$$10^w \cdot \{10, 12, 14, 16, 18, 30, \ldots, 98\}$$

to $g(w)$, corresponding to the digits we want to append to the start. This multiple is automatically divisible by 2^{w+1}, and also can take any of the four possible values modulo 2^{w+3}.

The second point is a similar induction, with base case $h(2) = 50$. The same set above consists of numbers divisible by 5^w, and covers all residues modulo 5^{w+2}. Careful readers might recognize the second part as essentially USAMO 2003/1. □

Claim (Head construction) — If $\gcd(n, 10) = 1$, then for any b, there exists an even alternating number $f(b \bmod n)$ which is $b \pmod n$.

Proof. A standard argument shows that

$$10 \cdot \frac{100^m - 1}{99} = \underbrace{1010\ldots10}_{m \text{ 10's}} \equiv 0 \pmod n$$

for any m divisible by $\varphi(99n)$. Take a very large such m, and then add on b distinct numbers of the form $10^{\varphi(n)r}$ for various even values of r; these all are $1 \pmod n$ and change some of the 1's to 3's. □

Now, we can solve the problem. Consider three cases:

- If $n = 2^k m$ where $\gcd(m, 10) = 1$ and $k \geq 2$ is even, then the concatenated number
$$10^k f\left(-\frac{g(k)}{10^k} \bmod m\right) + g(k)$$
works fine.
- If $n = 5^k m$ where $\gcd(m, 10) = 1$ and $k \geq 2$ is even, then the concatenated number
$$10^k f\left(-\frac{h(k)}{10^k} \bmod m\right) + h(k)$$
works fine.
- If $n = 50m$ where $\gcd(m, 10) = 1$, then the concatenated number
$$100 f\left(-\frac{1}{2} \bmod m\right) + 50$$
works fine.

Since every non-multiple of 20 divides such a number, we are done.

§18 RMM 2021/2 (21RMM2)

Available online at https://aops.com/community/p23374854.

Problem statement

Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers a_1, a_2, ..., a_{20} such that, for each $k = 1, 2, \cdots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k : k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Two moves is sufficient. An example of a two-move strategy is to ask

$$S_1 = \{17, 20\}, \qquad S_2 = \{19, 20\}$$

which determines N modulo $17 \cdot 19 \cdot 20 = 6460 > 5000$.

We proceed to show no single move S is sufficient. Let us say that the numbers 11, 13, 16, 17, 19 are *big*, and the other fifteen numbers are *small*. The lcm of the small numbers is exactly $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$.

We consider two cases:

- If S has a single big number, then the task is clearly impossible.
- Then suppose $S = \{s_1, \dots, s_n\}$ where $n > 1$. Then, Xenia constructs t_1, ..., t_n such that
$$t_1 \equiv t_2 \equiv \cdots \equiv t_n \equiv 1 \pmod{2520}$$
and such that, whenever s_i is big (indices modulo n),
$$t_i \equiv 1 \pmod{s_i}, \qquad t_{i+1} \equiv 2521 \pmod{s_i}.$$
Then the set $\{t_1, \dots, t_n\}$ is a possible response corresponding to both $N = 1$ and $N = 2521$. Hence Xenia wins.

§19 USAMO 2012/3 (12AMO3)

Available online at https://aops.com/community/p2669119.

Problem statement

Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, ...of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

Answer: all $n > 2$.

For $n = 2$, we have $a_k + 2a_{2k} = 0$, which is clearly not possible, since it implies $a_{2^k} = \frac{a_1}{2^{k-1}}$ for all $k \geq 1$.

For $n \geq 3$ we will construct a *completely multiplicative* sequence (meaning $a_{ij} = a_i a_j$ for all i and j). Thus (a_i) is determined by its value on primes, and satisfies the condition as long as $a_1 + 2a_2 + \cdots + na_n = 0$. The idea is to take two large primes and use Bezout's theorem, but the details require significant care.

We start by solving the case where $n \geq 9$. In that case, by Bertrand postulate there exists primes p and q such that

$$\lceil n/2 \rceil < q < 2\lceil n/2 \rceil \quad \text{and} \quad \frac{1}{2}(q-1) < p < q - 1.$$

Clearly $p \neq q$, and $q \geq 7$, so $p > 3$. Also, $p < q < n$ but $2q > n$, and $4p \geq 4\left(\frac{1}{2}(q+1)\right) > n$. We now stipulate that $a_r = 1$ for any prime $r \neq p, q$ (in particular including $r = 2$ and $r = 3$). There are now three cases, identical in substance.

- If $p, 2p, 3p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$6p \cdot a_p + q \cdot a_q = 6p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(6p, q) = 1$.
- Else if $p, 2p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$3p \cdot a_p + q \cdot a_q = 3p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(3p, q) = 1$.
- Else if $p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that
$$p \cdot a_p + q \cdot a_q = p + q - \frac{1}{2}n(n+1)$$
which is possible by Bézout lemma, since $\gcd(p, q) = 1$. (This case is actually possible in a few edge cases, for example when $n = 9$, $q = 7$, $p = 5$.)

It remains to resolve the cases where $3 \leq n \leq 8$. We enumerate these cases manually:

- For $n = 3$, let $a_n = (-1)^{\nu_3(n)}$.

- For $n = 4$, let $a_n = (-1)^{\nu_2(n)+\nu_3(n)}$.
- For $n = 5$, let $a_n = (-2)^{\nu_5(n)}$.
- For $n = 6$, let $a_n = 5^{\nu_2(n)} \cdot 3^{\nu_3(n)} \cdot (-42)^{\nu_5(n)}$.
- For $n = 7$, let $a_n = (-3)^{\nu_7(n)}$.
- For $n = 8$, we can choose $(p, q) = (5, 7)$ in the prior construction.

This completes the constructions for all $n > 2$.

§20 TSTST 2016/3 (16TSTST3)

Available online at https://aops.com/community/p6575217.

Problem statement

Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

We claim that

$$Q(x) = 420(x^2-1)^2$$

works. Clearly, it suffices to prove the result when $n = 4$ and when n is an odd prime p. The case $n = 4$ is trivial, so assume now $n = p$ is an odd prime.

First, we prove the following easy claim.

Claim — For any odd prime p, there are at least $\frac{1}{2}(p-3)$ values of a for which $\left(\frac{1-a^2}{p}\right) = +1$.

Proof. Note that if $k \neq 0$, $k \neq \pm 1$, $k^2 \neq -1$, then $a = 2(k+k^{-1})^{-1}$ works. Also $a = 0$ works. □

Let $F(x) = (x^2-1)^2$. The range of F modulo p is contained within the $\frac{1}{2}(p+1)$ quadratic residues modulo p. On the other hand, if for some t neither of $1 \pm t$ is a quadratic residue, then t^2 is omitted from the range of F as well. Call such a value of t *useful*, and let N be the number of useful residues. We aim to show $N \geq \frac{1}{4}p - 2$.

We compute a lower bound on the number N of useful t by writing

$$\begin{aligned}
N &= \frac{1}{4}\left(\sum_t \left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right] - \left(1-\left(\frac{2}{p}\right)\right) - \left(1-\left(\frac{-2}{p}\right)\right)\right) \\
&\geq \frac{1}{4}\sum_t \left[\left(1-\left(\frac{1-t}{p}\right)\right)\left(1-\left(\frac{1+t}{p}\right)\right)\right] - 1 \\
&= \frac{1}{4}\left(p + \sum_t \left(\frac{1-t^2}{p}\right)\right) - 1 \\
&\geq \frac{1}{4}\left(p + (+1)\cdot\tfrac{1}{2}(p-3) + 0\cdot 2 + (-1)\cdot((p-2) - \tfrac{1}{2}(p-3))\right) - 1 \\
&\geq \frac{1}{4}(p-5).
\end{aligned}$$

Thus, the range of F has size at most

$$\frac{1}{2}(p+1) - \frac{1}{2}N \leq \frac{3}{8}(p+3).$$

This is less than $0.499p$ for any $p \geq 11$.

Remark. In fact, the computation above is essentially an equality. There are only two points where terms are dropped: one, when $p \equiv 3 \pmod 4$ there are no $k^2 = -1$ in the lemma, and secondly, the terms $1-(2/p)$ and $1-(-2/p)$ are dropped in the initial estimate for N. With suitable modifications, one can show that in fact, the range of F is exactly equal to

$$\frac{1}{2}(p+1) - \frac{1}{2}N = \begin{cases} \frac{1}{8}(3p+5) & p \equiv 1 \pmod 8 \\ \frac{1}{8}(3p+7) & p \equiv 3 \pmod 8 \\ \frac{1}{8}(3p+9) & p \equiv 5 \pmod 8 \\ \frac{1}{8}(3p+3) & p \equiv 7 \pmod 8. \end{cases}$$

§21 Shortlist 2013 N4 (13SLN4)

Available online at https://aops.com/community/p3544103.

Problem statement

Determine whether there exists an infinite sequence of nonzero digits a_0, a_1, a_2, a_3, ...such that the number $\overline{a_k a_{k-1} \dots a_1 a_0}$ is a perfect square for all sufficiently large k.

The answer is no.

Assume for contradiction such a sequence exists, and let $x_k = \sqrt{\overline{a_k a_{k-1} \dots a_1 a_0}}$ for k large enough. Difference of squares gives

$$A_k \cdot B_k \coloneqq (x_{k+1} - x_k)(x_{k+1} + x_k) = a_{k+1} \cdot 10^k$$

with $\gcd(A_k, B_k) = 2\gcd(x_k, x_{k-1})$ since x_k and x_{k-1} have the same parity.

We now split the proof in two cases:

- First assume $\nu_5(x_k^2) = 2e < k$ for some index k. Then we actually need to have
$$2e = \nu_5(x_k^2) = \nu_5(x_{k+1}^2) = \cdots.$$
Thus in this situation, we need to have $\min(\nu_5(A_k), \nu_5(B_k)) = e$, and thus
$$\max(\nu_5(A_k), \nu_5(B_k)) = k - e.$$
So
$$\min(A_k, B_k) \geq 5^{k-e}.$$
- Otherwise, assume $\nu_5(x_k^2) \geq k$ for all k. Note in particular that $a_0 = 5$, thus all x_k are always odd. So one of A_k and B_k is divisible by 2^{k-1}. Hence for each k,
$$\min(A_k, B_k) \geq 2^{k-1} \cdot 5^{k/2}.$$

However, since $A_k B_k = a_{k+1} \cdot 10^k$ we also obviously have $\min(A_k, B_k) < \sqrt{9 \cdot 10^k}$ which is incompatible with both cases above, for sufficiently large k.

§22 Australia 2020/8 (20AUS8)

Problem statement

Prove that for each integer k satisfying $2 \le k \le 100$, there are positive integers b_2, b_3, ..., b_{101} such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

For an integer M to be chosen later, we will choose

$$b_2^2 = 69696M^{100!}$$
$$b_3^3 = \cdots = b_{100}^{100} = M^{100!}.$$

(Note that $69696 = 264^2$.) Then the desired equation becomes

$$b_{101}^{101} = (69696 + (k-2) - (100-k)) \cdot M^{100!}$$

and so we can let $M = 69696 + (k-2) - (100-k)$ and we're OK, since $M^{100!+1}$ is obviously a 100th power by Wilson's theorem.